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Stabilizing feedback design for time delayed polynomial systems using kinetic realizations

György Lipták[c,*], Katalin M. Hangos[a,b], Gábor Szederkényi[a,c]

[a] Process Control Research Group, Systems and Control Laboratory, Institute for Computer Science and Control, Kende u. 13-17, Budapest H-1111, Hungary
[b] Department of Electrical Engineering and Information Systems, University of Pannonia, Egyetem u. 10, Veszprém H-8200, Hungary
[c] Faculty of Information Technology and Bionics, Pázmány Péter Catholic University, Práter u. 50/a, Budapest H-1083, Hungary

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ABSTRACT

A new stabilizing feedback design method is proposed in this paper for time delayed polynomial systems with a linear input structure. The task is to transform the open loop system to a time delayed complex balanced kinetic system by using a polynomial state feedback structure which guarantees stability with arbitrary time delays.

It is shown that the required computations can be performed by simple linear programming, when the only goal is the semistability of the chosen equilibrium point. If one wants to achieve additionally the uniqueness of the closed loop equilibrium point to ensure local asymptotic stability, then the extended optimization problem requires the application of semidefinite programming. The existence of the solution and computability of the feedback do not depend on the magnitude of the delays.

It is shown that involving additional monomials into the feedback beyond the ones contained in the open-loop model does not improve the solvability of the semistabilization problem, but it may ensure the uniqueness of the prescribed complex balanced equilibrium point. Thus, two variants of a systematic method are proposed to find appropriate extra monomials for the feedback. One of these requires to solve a linear programming optimization problem even in this extended case.

1. Introduction

It is known that important dynamical phenomena occurring in real world can often be explained in a satisfactory way by using time delays in the equations [8,10,12]. At the same time, differential equations with time delay can produce much more complicated dynamics than ordinary differential equations in the general case. For example, a time delay may cause a stable equilibrium to become unstable, even if the dynamics is linear. Therefore, there is a tendency to avoid the presence of time delays in dynamic system models, by using e.g. approximations, see [9,16]. The above mentioned problems of the presence of time delays in linear systems indicate that the task of dynamic analysis and controller design for time delayed nonlinear systems present challenges for nowadays advanced control theory, too.

In the general case of *nonlinear systems with time delays*, a few sophisticated methods have appeared for dynamic analysis and control design. For example, it is reported that the stabilization of polynomial time delay systems can be solved by using a Lyapunov–Krasovskii functional and sum of squares decomposition [27,28].

It is widely known, that the special structure or properties of a nonlinear system may enable to develop efficient methods for their dynamic analysis and control, therefore special nonlinear system classes are considered in the time delayed case. The class of *nonnnegative systems*, that are dynamical systems with nonnegative state variables, is one of such special class of nonlinear systems, by which several kinds of dynamical phenomena in nature or technology can be modelled [10]. These include e.g. biochemical reaction networks, population dynamics, a wide range of process systems, and certain economical or transportation processes [13,33,35]. Nonnegativity can be successfully exploited in the stability analysis of time delay models [10,11]. However, majority of the control design methods for nonnegative delayed systems have been developed for linear models [4,22,24].

The class of *kinetic systems* is a subclass of nonnegative systems, for which important and useful results are known on the relation between the dynamical properties and the associated graph structure, is useful for modeling a wide range of processes in var-

* Corresponding author.
E-mail addresses: lipgyorgy@scl.sztaki.hu (G. Lipták), hangos@scl.sztaki.hu (K.M. Hangos), szeder@itk.ppke.hu (G. Szederkényi).

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ious application fields [3]. Beside the ability to describe complex nonlinear phenomena, kinetic systems have a simple mathematical structure facilitating the kinetic realization of nonlinear models and the application of computational methods for model analysis and controller design. The idea of applying time delays in kinetic models is not new [29]. Delayed reactions can be used to describe the phenomenon when the consumption of reactants is immediate, while the formation of products starts only after a certain time. Possible motivation for the application of delayed reactions can be the approximation of reaction cascades with the omission of some intermediate complexes and reactions, or the modeling of explicit (e.g., transport) delays in compartmental models which are known to be formally kinetic [10,35].

A key property of kinetic systems is called *complex balance* guaranteeing stability, which is global with a known parameter-independent logarithmic Lyapunov function in several special cases, and possibly, even generally [5]. The complex balance property has been defined and analysed recently for kinetic systems containing constant time delays in [18], where it was shown using an appropriate Lyapunov–Krasovskii functional that complex balanced kinetic systems with arbitrary constant time-delays are semistable.

There are a few papers dealing with feedback controller design of complex balanced kinetic systems (or more generally, kinetic systems with stable equilibria) that are able to shift the equilibrium point of the closed loop system to a given value using linear state or output feedback [32]. A similar problem is solved in the paper [25] using passivity theory in a special case of complex balanced kinetic systems with time delayed connections.

For general (not necessarily kinetic and not necessarily complex balanced) polynomial systems an optimization based feedback design method was proposed in [21] that transforms a non-delayed nonlinear polynomial model to a closed loop system which has a complex balanced realization. The design is based on the fact that the directed graph structure and parameterization (called realization) of a polynomial kinetic system is generally non-unique, and complex balance (among other important features) is a realization property [20].

Motivated by the above mentioned facts and results, we have proposed the first attempt to solve the stabilizing controller design problem in the kinetic framework for delayed nonnegative polynomial models that are not necessarily kinetic or complex balanced in our recent conference paper [17]. However, this method used a fixed, pre-defined monomial feedback structure, and the optimization problem has been solved by semidefinite programming. Therefore, the purpose of this paper is twofold. Firstly, our goal is to solve the feedback structure design problem, i.e. to find a method of new monomial selection in the feedback such that the closed loop system has a unique complex balanced equilibrium point. A further aim is to simplify the optimization problem related to the feedback structure selection and design such that the solution needs only a computationally easy method (e.g. linear programming).

2. Basic notions on time delayed kinetic systems

Throughout the paper, the following notations are used. If n is a positive integer, $\mathbb{R}^n$ and $\mathbb{Z}^n$ denote the n-dimensional space of real and integer column vectors, respectively. The symbols $\mathbb{R}^n_+$ ($\mathbb{Z}^n_+$) and $\overline{\mathbb{R}}^n_+$ ($\overline{\mathbb{Z}}^n_+$) denote the sets of n-dimensional element-wise positive and nonnegative real (integer) vectors, respectively. For every $\tau \geq 0$, the symbol $\mathcal{C} = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-\tau \leq s \leq 0} |\phi(s)|$ for $\phi \in \mathcal{C}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. Let $\mathcal{C}_+ = C([-\tau, 0], \mathbb{R}^n_+)$ and $\overline{\mathcal{C}}_+ = C([-\tau, 0], \overline{\mathbb{R}}^n_+)$ denote the set of positive and nonnegative functions in $\mathcal{C}$. For an n-dimensional column vector v, $\mathrm{diag}(v)$ is the $n \times n$ diagonal matrix with v_i for $1 \leq i \leq n$ in its diagonal. Finally, $\mathbf{1}$ denotes a column vector with all entries being 1, and $\mathbf{0}$ denotes the zero vector.

2.1. Nonnegative polynomial systems with time delay

We consider a continuous-time polynomial autonomous system with time delay in the following form

$$\dot{x}(t) = F(x_t) = M_0\, \psi(x(t)) + \sum_{i=1}^{p} M_i\, \psi(x(t-\tau_i)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $M_i \in \mathbb{R}^{n\times m}$, $0 \leq i \leq p$ are constant coefficient matrices, and $\tau_i > 0$, $1 \leq i \leq p$ are the time delays. The *monomial* mapping ψ is defined as

$$\psi_j(x) = \prod_{i=1}^{n} x_i^{Y_{i,j}}, \quad 1 \leq j \leq m, \tag{2}$$

where the exponents $Y_{i,j}$ are nonnegative integers forming the *monomial composition matrix* $Y \in \overline{\mathbb{Z}}^{n\times m}_+$. Solutions of (1) are generated by initial data $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$, where τ is the largest delay and $\phi \in \overline{\mathcal{C}}_+$ is a nonnegative continuous vector valued initial function. For every $t \geq 0$, a segment of the solution is defined by $x_t(s) = x(t+s)$ for $-\tau \leq s \leq 0$.

Nonnegativity. A delayed differential equation system is called nonnegative when its solutions are nonnegative for every nonnegative initial function. The time delayed polynomial system (1) is nonnegative if and only if the following condition is fulfilled [31]: if $\phi \in \overline{\mathcal{C}}_+$ and $\phi_i(0) = 0$ for some $i \in \{1, \ldots, n\}$, then $F_i(\phi) \geq 0$. This condition is equivalent to the following one: the matrices M_i for $1 \leq i \leq p$ are nonnegative and $M_0\, \psi(x)$ is essentially nonnegative, i.e. when $x \in \overline{\mathbb{R}}^n_+$ with $x_i = 0$ then $[M_0\, \psi(x)]_i \geq 0$ for $1 \leq i \leq n$.

2.2. Time delayed kinetic systems

Kinetic systems [3] form a special subclass of nonnegative polynomial systems. The description of a kinetic system is based upon the notion of *species* X_i, $1 \leq i \leq n$, *complexes* C_j, $1 \leq j \leq m$ and *reactions* between the complexes. The complexes that correspond to the monomials above, are defined by the linear nonnegative integer combination of the species, i.e $C_j = \sum_{i=1}^{n} Y_{i,j} X_i$ for $1 \leq j \leq m$. The dynamic model of a kinetic system describes the transformation of the complexes, into each other through p reactions of the form $C_i \mapsto C_j$. A positive constant is associated to each reaction as a reaction rate coefficient, and each reaction has also a nonnegative real number $\tau_i \geq 0$ for $1 \leq i \leq p$ associated to it that represents the time delay of the reaction. Then the DDE model of time delayed autonomous kinetic systems is as follows

$$\dot{x}(t) = Y\left[A_0 - \sum_{i=0}^{p} \mathrm{diag}(\mathbf{1}^T A_i)\right]\psi(x(t)) + \sum_{i=1}^{p} Y A_i \psi(x(t-\tau_i)), \tag{3}$$

where $Y \in \overline{\mathbb{Z}}^{n\times m}_+$ is the complex composition matrix, ψ is the monomial mapping defined by Eq. (2), and $A_i \in \overline{\mathbb{R}}^{n\times n}_+$ for $0 \leq i \leq p$. Here $[A_0]_{j,k}$ is the rate coefficient of the non-delayed reaction from complex C_k to C_j. Moreover, $[A_i]_{j,k}$, $1 \leq i \leq p$ is the rate coefficient of the reaction from complex C_k to C_j with delay τ_i. If there is no reaction between C_k and C_j (with time delay τ_i) then $[A_i]_{j,k} = 0$. Solutions of (3) are generated by initial data $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$, where τ is the largest delay and $\phi \in \overline{\mathcal{C}}_+$ is a nonnegative continuous vector valued initial function.

Kinetic systems with the DDE model (3) are also called *delayed chemical reaction networks obeying the mass-action law, that is abbreviated by delayed CRN in this paper.*

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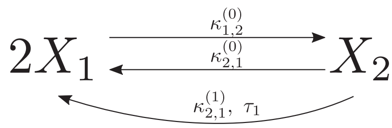

Fig. 1. Reaction graph of the example model (4).

Reaction graph of delayed CRNs. Generalizing the concept of reaction graphs by e.g. [7], we can represent the structure of a delayed CRN by a directed and labeled multigraph, where the label of an edge is not only the reaction rate constant, but also the time delay. Reactions with the same source and product complexes, but with different time delays occur as parallel edges in the reaction graph. Fig. 1 depicts an example of a reaction graph of a delayed CRN. Moreover, delayed loop reactions having the same complex as both the source and product may also appear in delayed CRNs contrary to the case of classical, undelayed CRNs.

Important dynamic properties of a CRN depend on some of the structural properties of the reaction graph, most notably on its connectivity and on its strong components. A CRN is called *weakly reversible* if whenever there exists a directed path from C_i to C_j in its reaction graph, then there exists a directed path from C_j to C_i, too. In graph theoretic terms, this means that all components of the reaction graph are strongly connected components.

Delayed example. Consider two species, X_1 and X_2 that react in a reversible reaction $2X_1 \leftrightarrows X_2$, and let us have a third reaction converting X_2 to $2X_1$ with a delay. Then the elementary reaction steps are

$$2X_1 \xrightarrow{\kappa_{1,2}^{(0)}} X_2;\; X_2 \xrightarrow{\kappa_{2,1}^{(0)}} 2X_1;\; X_2 \xrightarrow{\kappa_{2,1}^{(1)},\,\tau_1} 2X_1$$

The corresponding reaction graph is depicted in Fig. 1. The DDEs (3) have the following parameters

$$Y = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \psi(x) = \begin{bmatrix} x_1^2 \\ x_2 \end{bmatrix},$$
$$A_0 = \begin{bmatrix} 0 & \kappa_{2,1}^{(0)} \\ \kappa_{1,2}^{(0)} & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \kappa_{2,1}^{(1)} \\ 0 & 0 \end{bmatrix}. \tag{4}$$

Time delayed kinetic realizations. We say that the time delayed system (1) is *kinetically realizable* (or shortly, *kinetic*) with the monomial composition matrix Y if it can be represented in the form (3), i.e. there exist $p+1$ nonnegative matrices A_i, $0 \le i \le p$ such that

$$M_0 = Y\left[A_0 - \sum_{i=0}^{p} \mathrm{diag}(\mathbf{1}^T A_i)\right], \tag{5}$$
$$M_i = YA_i, \quad 1 \le i \le p.$$

The tuple of matrices $(Y, A_0, \dots, A_p)$ is called a *time delayed kinetic realization*.

It was shown in [34] that a delayed polynomial system of the form (1) is kinetic if and only if the non-delayed part characterized by M_0 does not contain negative cross-effects and all the entries of matrices M_i, $1 \le i \le p$ are nonnegative. In this case, it is also possible to algorithmically construct a delayed CRN that realizes the dynamics of (1). It is important to remark that generally, several different CRN structures may realize exactly the same dynamics, and this non-uniqueness will be exploited for the feedback design in the delayed case, too.

It is easy to verify that the existence of the factorization in Eq. (5) allows the direct construction of the reaction graph of a delayed nonnegative system as it is defined in [18]. Moreover, if Y is fixed then (5) is a linear constraint and thus it can be solved for A_i, $0 \le i \le p$, e.g. in the framework of linear programming (LP).

Positive stoichiometric compatibility classes. We define the stoichiometric subspace for the delayed kinetic model (3) as follows

$$\mathcal{S} = \mathrm{span}\left\{Y_{\cdot,j} - Y_{\cdot,k} \;\middle|\; [A_i]_{j,k} > 0\right\}.$$

We can introduce the time delayed positive stoichiometric compatibility classes in the form [18]

$$\mathcal{S}_\phi = \left\{\theta \in \overline{\mathcal{C}}_+ \;\middle|\; h(\theta) - h(\phi) \in \mathcal{S}\right\}, \tag{6}$$

where the functional $h : \overline{\mathcal{C}}_+ \to \mathbb{R}^n$ is defined by

$$h(\phi) = \phi(0) + \sum_{i=1}^{p}\sum_{j=1}^{m}\sum_{k=1}^{m}\left([A_i]_{j,k}\int_{-\tau_i}^{0}[\psi_k(\phi(s)))]\,ds\right)Y_{\cdot,k}.$$

It was shown in [18] that the delayed positive stoichiometric compatibility classes $\mathcal{S}_\phi$ are positively invariant under the mass-action kinetic system (3), i.e. $x_0 \in \mathcal{S}_\phi$ implies $x_t \in \mathcal{S}_\phi$ for all $t \ge 0$.

Equilibria and complex balance. By a *positive equilibrium* of (3), we mean a positive vector $x^* \in \mathbb{R}_+^n$ such that $x(t) \equiv x^*$ is a solution of (3). We denote the set of all positive equilibria with $\mathcal{E}_+$.

It is easy to see that Eq. (3) and its undelayed counterpart share the same equilibria. To show this we define the matrices $\overline{M} = \sum_{i=0}^{p} M_i$ and $\overline{A} = \sum_{i=0}^{p} A_i$. Then we obtain the undelayed kinetic system corresponding to (3) as follows

$$\dot{x}(t) = \overline{M}\,\psi(x(t)) = Y\left[\overline{A} - \mathrm{diag}(\mathbf{1}^T\overline{A})\right]\psi(x(t)), \tag{7}$$

where the matrix $\overline{A} - \mathrm{diag}(\mathbf{1}^T\overline{A})$ is a column conservation matrix (or a Kirchhoff matrix). Clearly, the positive vector $x^* \in \mathbb{R}_+^n$ is an *equilibrium point* of the time delayed kinetic system if and only if it is an equilibrium point of the corresponding undelayed kinetic system (7), i.e.

$$Y\left[\overline{A} - \mathrm{diag}(\mathbf{1}^T\overline{A})\right]\psi(x^*) = \mathbf{0}.$$

The positive equilibrium point x^* is called *complex balanced* if it fulfills the equation [15]

$$\left[\overline{A} - \mathrm{diag}(\mathbf{1}^T\overline{A})\right]\psi(x^*) = \mathbf{0}. \tag{8}$$

It is well-known [14] that if Eq. (7) and hence (3) has a positive complex balanced equilibrium x^*, then any other positive equilibrium is complex balanced and the set of all positive equilibria $\mathcal{E}_+$ can be characterized by

$$\mathcal{E}_+ = \left\{x \in \mathbf{R}_+^n \mid \mathrm{Ln}(x) - \mathrm{Ln}(x^*) \in \mathcal{S}^\perp\right\}, \tag{9}$$

where $[\mathrm{Ln}(x)]_i = \ln(x_i)$ for $1 \le i \le n$. Therefore, we call a kinetic system complex balanced with or without time delay, if the complex balanced property holds for any equilibrium point x^*. It is known from [15] that complex balance implies weak reversibility.

Unique complex balanced equilibrium. The structure of complex balanced equilibria depends on the dimension of stoichiometric subspace $\mathcal{S}$. In the special case, when $\mathcal{S} = \mathbb{R}^n$ then $\mathcal{S}^\perp = \{\mathbf{0}\}$, so there exists only one positive equilibrium $\mathcal{E}_+ = \{x^*\}$. In the next, we give an equivalent convex condition for the uniqueness of the complex balanced equilibrium.

First, we introduce the *Laplacian matrix* $\mathcal{L}(x^*)$ at a positive equilibrium point [30] as

$$\mathcal{L}(x^*) = \left[\mathrm{diag}(\mathbf{1}^T\overline{A}) - \overline{A}\right]\mathrm{diag}(\psi(x^*)). \tag{10}$$

Then, the positive equilibrium point x^* is complex balanced if and only if the Laplacian $\mathcal{L}(x^*)$ is balanced, i.e $\mathcal{L}(x^*)\,\mathbf{1} = \mathbf{0}$. This gives rise to the uniqueness result below.

Theorem 2.1 [19]. *Let the equilibrium x^* be complex balanced. Then, x^* is a unique positive equilibrium if and only if the matrix*

$$Y\left[\mathcal{L}(x^*) + \mathcal{L}(x^*)^T\right]Y^T$$

is positive definite.

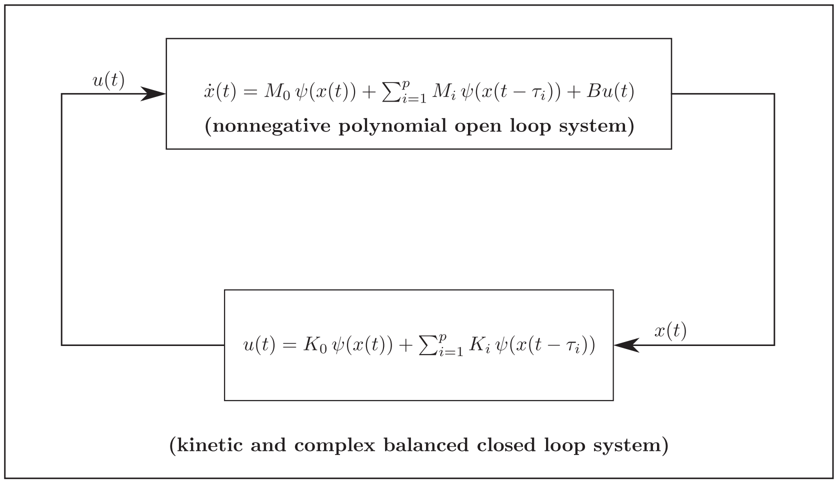

Fig. 2. Structure and properties of the kinetic feedback loop.

2.3. Stability of complex balanced kinetic systems with time delay

A kinetic system may have a connected set of positive equilibria $\mathcal{E}_+$ (see Eq. (9)), therefore we introduce the following definition. An equilibrium x^* of Eq. (3) is called *semistable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $\phi \in \mathcal{C}$ and $\|\phi - x^*\| \leq \delta$, then the solution $x(t)$ with initial data $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$ converges to a Lyapunov stable equilibrium of (3) as $t \to \infty$. For more details on the notion and application of semistability, see [10]. We will use the following recent stability result for complex balanced time delayed kinetic systems.

Theorem 2.2 [18]**.** *Every complex balanced equilibrium x^* of the delayed kinetic system (3) is semistable. Moreover, when the system has only one complex balanced equilibrium point, then it is locally asymptotically stable with respect to the positive orthant.*

3. Formulation and solution of the feedback design problem

In this section, we consider a stabilizing feedback design method for the open loop system

$$\dot{x}(t) = M_0\,\psi(x(t)) + \sum_{i=1}^{p} M_i\,\psi(x(t-\tau_i)) + Bu(t), \tag{11}$$

where $u(t) \in \mathbb{R}^r$ is the input of the system, and $B \in \mathbb{R}^{n \times r}$.

We remark that the above open loop system (11) is not necessarily kinetic, but it is a delayed polynomial system with linear input structure. In the special case when the open system (11) is kinetic, its model corresponds to a DDE model of an open delayed CRN where the inputs are chosen as the concentrations of the species in the inlet flows, see the discussion in [21] of the non-delayed case.

The aim of the feedback is to stabilize the open loop system (11) at the given positive equilibrium point x^*. For this, *we transform the open loop system (11) into a complex balanced kinetic closed loop system with a desired equilibrium point x^**. The problem will be formulated as a semidefinite programming (SDP) problem.

3.1. The state feedback law

We assume a polynomial feedback of the form

$$u(t) = K_0\,\psi(x(t)) + \sum_{i=1}^{p} K_i\,\psi(x(t-\tau_i)), \tag{12}$$

where the matrices $K_i \in \mathbb{R}^{r \times m}$, $0 \leq i \leq p$ are the parameters of the feedback. The control structure is illustrated in Fig. 2

With the above feedback (12), the closed loop system has the form

$$\dot{x}(t) = [M_0 + BK_0]\psi(x(t)) + \sum_{i=1}^{p}[M_i + BK_i]\psi(x(t-\tau_i)). \tag{13}$$

3.2. Feedback design for closed loop semistability using linear programming

We are looking for the feedback parameters K_i, $0 \leq i \leq p$ such that the closed-loop system (13) has a kinetic realization $(Y, A_0, \ldots, A_p)$ which is complex balanced with an equilibrium point x^* (that may not be unique). Therefore, the *decision variables* are $K_i \in \mathbb{R}^{r \times m}$, $0 \leq i \leq p$, and $A_i \in \overline{\mathbb{R}}_+^{m \times m}$, $0 \leq i \leq p$. The optimization problem for the feedback design can be written as

$$\min \ell(K_0, \ldots, K_p, A_0, \ldots, A_p), \tag{14}$$

$$\text{s.t.} \quad M_0 + BK_0 = Y\left[A_0 - \sum_{i=0}^{p} \mathrm{diag}(\mathbf{1}^T A_i)\right], \tag{15}$$

$$M_i + BK_i = YA_i, \quad 1 \leq i \leq p, \tag{16}$$

$$\mathcal{L}(x^*)\mathbf{1} = \mathbf{0}, \tag{17}$$

where $x^* > 0$ is given, ℓ is a linear objective function and $\mathcal{L}(x^*)$ is defined in (10). The constraints (15) and (16) guarantee that the solution will be a *kinetic realization of the closed-loop system*. Eq. (17) ensures that the *closed loop system is complex balanced* with equilibrium x^*. It is clear that all the constraints are linear in the decision variables, therefore, the *feasibility of (14)–(17) can be easily decided and solution(s) can be found (if they exist) in the framework of linear programming.* Comments on the selection of the objective function ℓ are given at the end of Section 4 in remark (R1).

3.3. Ensuring uniqueness of the equilibrium by semidefinite programming

Let us recall Theorem 2.1 that gives a necessary and sufficient condition for the uniqueness of complex balanced equilibria. Therefore, *adding the constraint*

$$Y\left[\mathcal{L}(x^*) + \mathcal{L}(x^*)^T\right]Y^T > 0, \tag{18}$$

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to (15)–(17) *ensures that the closed loop system has a unique complex balanced equilibrium point* x^*. However, (18) is not a linear constraint any more, but it can still be handled efficiently together with (15)–(17) using semidefinite programming.

4. The effect of additional feedback monomials on the closed loop dynamics

The results in this subsection show that *involving new monomials into the feedback law (12) does not improve the solvability of the feedback problem from the point of view of finding realizations with weak reversibility or complex balance, but can ensure the uniqueness of the equilibrium point of the closed loop system.*

In the following, we call the kth monomial ψ_k *additional* if $[M_i]_{\cdot,k} = \mathbf{0}$ for $0 \le i \le p$, i.e. it does not occur on the right hand side of the open loop system (11). Therefore, we call a controlled system of the form (13) a *closed loop system without additional monomials*, if for all monomials ψ_k such that $[M_i]_{\cdot,k} = \mathbf{0}$ for $0 \le i \le p$, then $[K_i]_{\cdot,k} = \mathbf{0}$ for $0 \le i \le p$.

4.1. The effect of additional monomials on complex balance

The following three lemmas and a theorem show that additional complexes in the feedback loop do not improve the complex balanced property of the realization and the equilibrium point.

Lemma 4.1. *Consider the open loop system (11) and the feedback law (12). Assume that there exists a closed loop system with feedback parameters K_i, $0 \le i \le p$ and it has a realization $(Y, A_0, \dots, A_p)$ where the kth additional complex occur in delayed reactions as a source complex, i.e. $\mathbf{1}^T[A_i]_{\cdot,k} > 0$ for some $1 \le i \le p$. Then, there exist other feedback parameters K_i', $0 \le i \le p$ such that the corresponding closed loop system has a realization $(Y, A_0', \dots, A_p')$, where $[A_i']_{\cdot,j} = [A_i]_{\cdot,j}$ for $j \ne k$, $0 \le i \le p$, $[A_i']_{\cdot,k} = \mathbf{0}$ for $1 \le i \le p$, and*

$$[A_0']_{\cdot,k} = \sum_{i=0}^{p} [A_i]_{\cdot,k}.$$

Proof. *Proof of* Lemma 4.1 To prove this lemma, we can construct the feedback parameters K_i', $0 \le i \le p$ as $[K_i']_{\cdot,j} = [K_i]_{\cdot,j}$ for $j \ne k$, $0 \le i \le p$, $[K_i']_{\cdot,k} = \mathbf{0}$ for $1 \le i \le p$, and

$$[K_0']_{\cdot,k} = \sum_{i=0}^{p} [K_i]_{\cdot,k},$$

which gives a closed loop system with the realization $(Y, A_0', \dots, A_p')$. □

Lemma 4.2. *Consider the open loop system (11) and the feedback law (12). Assume that there exists a closed loop system with feedback parameters K_i, $0 \le i \le p$ and it has a realization $(Y, A_0, \dots, A_p)$ where the kth additional complex occur only in undelayed reactions as a source complex, i.e. $\mathbf{1}^T[A_0]_{\cdot,k} > 0$ and $\mathbf{1}^T[A_i]_{\cdot,k} = 0$ for $1 \le i \le p$. Then, there exist other feedback parameters K_i', $0 \le i \le p$ such that the corresponding closed loop system has a realization $(Y, A_0', \dots, A_p')$, where*

$$A_0' = A_0 + \left(\mathbf{1}^T [A_0]_{\cdot,k}\right)^{-1} [A_0]_{\cdot,k} [A_0]_{k,\cdot} - \mathbf{e_k} [A_0]_{k,\cdot} - [A_0]_{\cdot,k} \mathbf{e_k}^T$$
$$A_i' = A_i + \left(\mathbf{1}^T [A_0]_{\cdot,k}\right)^{-1} [A_0]_{\cdot,k} [A_i]_{k,\cdot} - \mathbf{e_k} [A_i]_{k,\cdot}, \quad \text{for } 1 \le i \le p, \tag{19}$$

and

$$\overline{A}' = \overline{A} + \left(\mathbf{1}^T \overline{A}_{\cdot,k}\right)^{-1} \overline{A}_{\cdot,k} \overline{A}_{k,\cdot} - \mathbf{e_k} \overline{A}_{k,\cdot} - \overline{A}_{\cdot,k} \mathbf{e_k}^T.$$

Proof. *Proof of* Lemma 4.2 The matrices A_i', $0 \le i \le p$ remain nonnegative, because $[A_i]_{k,k} = 0$, so $(Y, A_0', \dots, A_p')$ is a valid realization. Then, we can construct the corresponding feedback gains such that

$$Y\left[A_0' - \text{diag}\left(\mathbf{1}^T \overline{A} + \mathbf{1}^T [A_0]_{\cdot,k} \mathbf{e_k}^T\right)\right]$$
$$= M_0 + B\left[K_0 + \left(\mathbf{1}^T [A_0]_{\cdot,k}\right)^{-1} [K_0]_{\cdot,k} [A_0]_{k,\cdot} - [K_0]_{\cdot,k} \mathbf{e_k}^T\right],$$

and therefore the feedback parameter K_0' is

$$K_0' = K_0 + \left(\mathbf{1}^T [A_0]_{\cdot,k}\right)^{-1} [K_0]_{\cdot,k} [A_0]_{k,\cdot} - [K_0]_{\cdot,k} \mathbf{e_k}^T.$$

To construct the delayed feedback gains K_i', we consider

$$YA_i' = M_i + B\left[K_i + \left(\mathbf{1}^T [A_0]_{\cdot,k}\right)^{-1} [K_0]_{\cdot,k} [A_i]_{k,\cdot}\right], \quad \text{for } 1 \le i \le p,$$

so the feedback gains have the form

$$K_i' = K_i + \left(\mathbf{1}^T [A_0]_{\cdot,k}\right)^{-1} [K_0]_{\cdot,k} [A_i]_{k,\cdot}, \quad \text{for } 1 \le i \le p.$$

□

Lemma 4.3. *Let A_i and A_i' be nonnegative matrices where the matrices A_i' are constructed by (19), for $0 \le i \le p$. Then,*

$$\text{Ker}\left(\overline{A} - \text{diag}(\mathbf{1}^T \overline{A})\right) \subseteq \text{Ker}\left(\overline{A}' - \text{diag}(\mathbf{1}^T \overline{A}')\right).$$

Proof. Let us take an element q of $\text{Ker}\left(\overline{A} - \text{diag}(\mathbf{1}^T \overline{A})\right)$, then

$$q_k = \left(\mathbf{1}^T \overline{A}_{\cdot,k}\right)^{-1} \overline{A}_{k,\cdot} q. \tag{20}$$

Let us consider the product $(\overline{A}' - \text{diag}(\mathbf{1}^T \overline{A}'))q$ as follows

$$\left(\overline{A}' - \text{diag}(\mathbf{1}^T \overline{A}')\right) q$$
$$= \left(\overline{A} - \text{diag}(\mathbf{1}^T \overline{A}) + \mathbf{1}^T \overline{A}_{\cdot,k}^{-1} \overline{A}_{\cdot,k} \overline{A}_{k,\cdot} - \mathbf{e_k} \overline{A}_{k,\cdot}\right) q$$
$$- (\overline{A}_{\cdot,k} - \mathbf{1}^T \overline{A}_{\cdot,k} \mathbf{e_k}) q_k. \tag{21}$$

We can now substitute (20) into (21) to obtain

$$\left(\overline{A}' p - \text{diag}(\mathbf{1}^T \overline{A}')\right) q$$
$$= \left(\overline{A} - \text{diag}(\mathbf{1}^T \overline{A}) + \left(\mathbf{1}^T \overline{A}_{\cdot,k}\right)^{-1} \overline{A}_{\cdot,k} \overline{A}_{k,\cdot} - \mathbf{e_k} \overline{A}_{k,\cdot}\right.$$
$$\left. - \left(\mathbf{1}^T \overline{A}_{\cdot,k}\right)^{-1} \overline{A}_{\cdot,k} \overline{A}_{k,\cdot} + \mathbf{e_k} \overline{A}_{k,\cdot}\right) q$$
$$= \left(\overline{A} - \text{diag}(\mathbf{1}^T \overline{A})\right) q = 0.$$

that means $q \in \text{Ker}\left(\overline{A}' - \text{diag}(\mathbf{1}^T \overline{A}')\right)$. □

Theorem 4.1. *Consider the open loop system (11) and the feedback law (12). Suppose there exists a complex balanced closed loop system (13) with additional complexes and equilibrium points in the set $\mathcal{E}_+$ of Eq. (9). Then there exists another complex balanced closed loop system without additional complexes that has equilibrium points $\mathcal{E}_+'$ such that $\mathcal{E}_+ \subseteq \mathcal{E}_+'$.*

Proof. Now we make use of the previously proven Lemmas. If there exists a closed loop system with a complex balanced realization $(Y, A_0, \dots, A_p)$, then there exists another one with realization $(Y, A_0', \dots, A_p')$ where the additional complexes are isolated (see Lemma 4.2). Furthermore, $\text{Ker}\left(\overline{A} - \text{diag}(\mathbf{1}^T \overline{A})\right) \subseteq \text{Ker}\left(\overline{A}' - \text{diag}(\mathbf{1}^T \overline{A}')\right)$ (according to Lemma 4.3.). From the assumption it follows that $(Y, A_0, \dots, A_p)$ is complex balanced, therefore the set of equilibrium points of (13) can be described as

$$\mathcal{E}_+ = \left\{x \in \mathbf{R}_+^n \mid \psi(x) \in \text{Ker}\left(\overline{A} - \text{diag}(\mathbf{1}^T \overline{A})\right)\right\}.$$

Therefore, $(Y, A_0', \dots, A_p')$ is also complex balanced and $\mathcal{E}_+ \subseteq \mathcal{E}_+'$. □

4.2. The effect of additional monomials on the uniqueness of the equilibrium point

Finally we show that considering additional monomials in the feedback can ensure the uniqueness of the complex balanced equilibrium point of the closed loop system.

The following lemma is the immediate consequence of Corollary 4.6 in [7].

Lemma 4.4. *If $(Y, A_0, \ldots, A_p)$ is weakly reversible then*

$$\mathcal{S} = \mathrm{Im}\Big(Y\big[\overline{A} - \mathrm{diag}(\mathbf{1}^T\overline{A})\big]\Big).$$

Using this result, we can prove the following theorem.

Theorem 4.2. *Consider the open loop system (11) and the feedback law (12). Suppose there exists a complex balanced closed loop system (13) with the equilibrium points $\mathcal{E}_+$. Furthermore, assume that $\dim\big(\mathrm{Im}(\overline{M}) + \mathrm{Im}(B)\big) = n$ and B has a rational basis. Then, there exists another feedback law with additional monomials such that the new closed loop system has only one positive equilibrium point, i.e. $\mathcal{E}'_+ = \{x^*\}$, where $x^* \in \mathcal{E}_+$ and it is complex balanced.*

Proof. Consider the new feedback in the form

$$u'(t) = u(t) + K_+\psi_+(x(t)) + K_-\psi_-(x(t)),$$

where ψ_+ and ψ_- contain the new monomials. The new monomials are generated by Y_+, and Y_- such that $\mathrm{Im}(Y_+ - Y_-) = \mathrm{Im}(B)$, and Y_+, and Y_- are nonnegative integer matrices. The feedback parameters fulfill the following conditions

$$BK_+ = (Y_- - Y_+)\cdot \mathrm{diag}(\psi_+(x^*))^{-1},$$

and

$$BK_- = (Y_+ - Y_-)\cdot \mathrm{diag}(\psi_-(x^*))^{-1}.$$

The new closed loop system has a kinetic realization in the form

$$Y' = \begin{bmatrix} Y & Y_+ & Y_- \end{bmatrix},$$

$$A'_0 = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{diag}(\psi_-(x^*))^{-1} \\ \mathbf{0} & \mathrm{diag}(\psi_+(x^*))^{-1} & \mathbf{0} \end{bmatrix},$$

and

$$A'_i = \begin{bmatrix} A_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ for } 1 \le i \le p.$$

To show that x^* is a complex balanced equilibrium of the new closed loop system, we rewrite the expression $\Big[\overline{A}' - \mathrm{diag}(\mathbf{1}^T\overline{A}')\Big]\psi'(x^*)$ as follows

$$\begin{bmatrix} \overline{A}\,\psi(x^*) - \mathrm{diag}(\mathbf{1}^T\overline{A})\,\psi(x^*) \\ \mathrm{diag}(\psi_-(x^*))^{-1}\psi_-(x^*) - \mathrm{diag}(\psi_+(x^*))^{-1}\psi_+(x^*) \\ \mathrm{diag}(\psi_+(x^*))^{-1}\psi_+(x^*) - \mathrm{diag}(\psi_-(x^*))^{-1}\psi_-(x^*) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

By using the Lemma 4.4, we get the corresponding stoichiometric subspace as follows

$$\mathcal{S}' = \mathrm{Im}\Big(\begin{bmatrix} \overline{M} + B\overline{K} & BK_- & BK_+ \end{bmatrix}\Big),$$

which contains the subspace $\mathrm{Im}(\overline{M})$ and $\mathrm{Im}(B)$. Therefore, the dimension of the stoichiometric subspace is n which implies that the new closed loop system has only one positive equilibrium x^* which is complex balanced. □

Remarks.

(R1) In the optimization problem (14)–(17), we use an arbitrary linear objective function, since the fundamental control goals are achieved through satisfying the constraints. Defining the goal of the optimization as minimizing the L_1 norm of the vector composed of the feedback gain entries, i.e.

$$\min_{K_i, A_i} \quad \sum_{i=0}^{p}\sum_{j=1}^{m}\sum_{k=1}^{m} |[K_i]_{j,k}|$$

results in sparse feedback matrices, and can be easily fit into both linear programming and semidefinite programming. There are other possible choices, e.g. in [21], we proposed an objective function which minimizes the eigenvalue of the linearized closed-loop system with the largest real part.

(R2) It is straightforward to introduce additional linear constraints for the feedback structure, e.g. we can prescribe a distributed control structure, or exclude certain delays from the feedback.

(R3) The *existence of a solution* to the feedback design problem *can be decided* by checking the feasibility of the optimization problem (14)–(17) *in the framework of linear programming*. If it is infeasible with the given input structure then there is no feedback that could make the closed loop system complex balanced kinetic with the given equilibrium point.

5. Illustrative examples

In the following, we present the applicability of the proposed design technique on illustrative examples. The algorithms were implemented in [26] using the YALMIP modeling language [23]. MOSEK [1] was used to solve the SDP problems.

5.1. A simple low dimensional computational example

5.1.1. System description

Let us consider the nonnegative open loop system in the form

$$\begin{aligned} \dot{x}_1(t) &= 2x_1^2(t) - 2x_1(t)x_2(t) + u(t) \\ \dot{x}_2(t) &= -x_2(t) + x_1^2(t - \tau_1), \end{aligned} \tag{22}$$

where x_1, x_2 are the states, u is the input and $\tau_1 > 0$ is an arbitrary constant time delay. The open loop system is characterized by the following matrices

$$M_0 = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 0 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $u(t) = 0$, the system has an unstable positive equilibrium point in $x_1^* = x_2^* = 1$.

5.1.2. Stabilizing feedback design

The aim of the feedback is to stabilize x^*. In this case, we extend the original optimization problem (14) by an additional constraint to exclude the delay element from the feedback. Then the optimization gives the following feedback parameters

$$K_0 = \begin{bmatrix} 3.2089 & -4 & 0.7911 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

and therefore, the input is computed as

$$u(t) = 3.2089x_2(t) - 4x_1^2(t) + 0.7911x_1(t)x_2(t). \tag{23}$$

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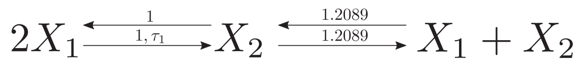

Fig. 3. Reaction graph of the closed loop realization (24).

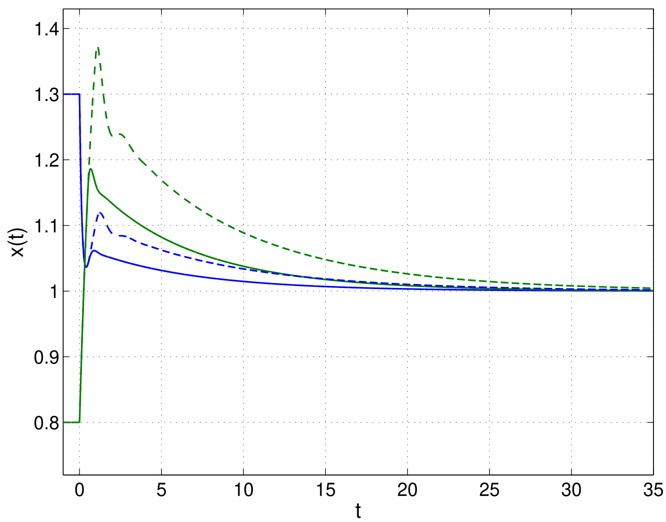

Fig. 4. Simulation results for the system (22) with the stabilizing feedback (23). The simulation is started from two different constant initial functions, $\phi_1(s) = 1.3$, and $\phi_2(s) = 0.8$ with two different time delays $\tau_1 = 0.5$, and $\tau_1 = 1$. The blue and green curves show the simulated states x_1 and x_2, respectively. The dashed-curves correspond to the larger delay $\tau_1 = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The computed feedback (23) does not contain delay. The computed kinetic realization of the closed loop system is given by

$$A_0 = \begin{bmatrix} 0 & 0 & 1.209 \\ 1 & 0 & 0 \\ 1.209 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

which is complex balanced at the equilibrium point $x_1^* = x_2^* = 1$, so the closed loop system is stable. The stability is guaranteed by the following Lyapunov–Krasovskii functional [18]

$$V(x_t) = x_1(t)[\ln(x_1(t)) - 1] + x_2(t)[\ln(x_2(t)) - 1] + 2 + \int_{t-\tau_1}^{t} \left\{ x_1^2(s)\left[\ln(x_1^2(s)) - 1\right] + 1 \right\} ds.$$

Fig. 3 shows the reaction graph of the closed loop system and Fig. 4 illustrates the time domain behavior.

5.2. Illustrative example on choosing the additional monomials in the feedback structure

In this subsection, we show a simple example, where we can achieve a unique complex balanced equilibrium only with additional monomials. First, we design a feedback which transforms the open loop system to a complex balanced closed loop. After that, we choose additional monomials according to Theorem 4.2. Finally, we compute a new feedback with these additional monomials which transforms the system such that it has a unique complex balanced equilibrium.

5.2.1. System description

Consider the three dimensional time delay system with one input as follows

$$\dot{x}_1(t) = 5x_1(t-\tau_1)x_2(t-\tau_1)x_3(t-\tau_1) - 5x_1^2(t)x_2(t) - 3x_1(t)x_2^2(t) + u(t)$$

$$\dot{x}_2(t) = 4x_1(t-\tau_1)x_2(t-\tau_1)x_3(t-\tau_1) + 3x_1^2(t)x_2(t) - 5x_1(t)x_2^2(t) + u(t)$$

$$\dot{x}_3(t) = -x_1(t)x_2(t)x_3(t) + 2x_1(t)x_2^2(t),$$

where x_1, x_2, and x_3 are the states, u is the input, and τ_1 denotes the state delay. Using the notations of this paper, we can construct the following matrices

$$M_0 = \begin{bmatrix} 0 & -5 & -3 \\ 0 & 3 & -5 \\ -1 & 0 & 2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The goal of this feedback design is transform the open-loop system such that it has a unique complex balanced equilibrium $x^* = [1\ 2\ 4]^T$.

5.2.2. Stabilizing feedback design without additional monomials

Without additional complex the optimization problem (14)–(18) is infeasible, there exists no complex balanced closed loop system with only one equilibrium. Without the constraint (18), however, the problem is feasible. Then we get the following feedback matrices

$$K_0 = \begin{bmatrix} -1 & 0.2387 & 3 \end{bmatrix}, \; K_1 = \begin{bmatrix} -3 & 0.7613 & 0 \end{bmatrix},$$

and closed loop realization matrices

$$A_0 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 3.7462 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0.2538 & 0 \\ 0 & 0.2538 & 0 \end{bmatrix}.$$

The set of positive equilibria is

$$\mathcal{E}_+ = \left\{ x \in \mathbb{R}_+^n \mid x \in \mathrm{Im}(x^*) \right\}.$$

Fig. 5 shows the simulation result of this case with different constant initial functions. It is visible that different initial functions lead to different equilibria with this constrained type of feedback.

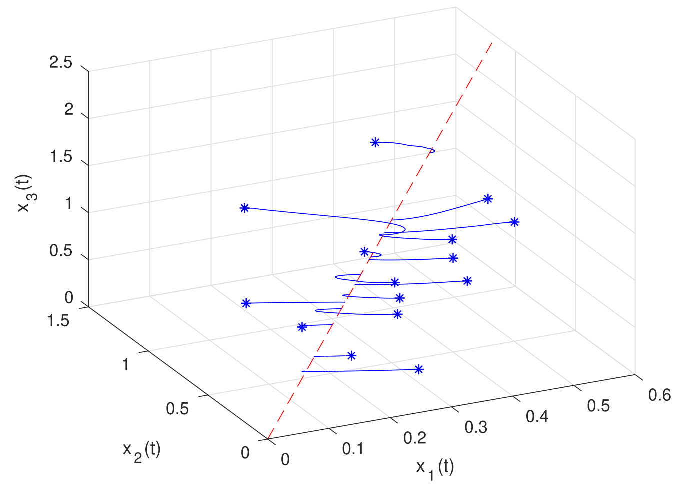

Fig. 5. Simulation results for the system described in Section 5.2.1 with the delay $\tau_1 = 0.5$. The loop was closed with the stabilizing feedback described in Section 5.2.2. The different initial functions are randomly sampled constant functions (blue asterisks). In this case, we have a set of complex balanced equilibria (red dashed line), so the trajectories (solid blue curves) converge to different equilibria. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

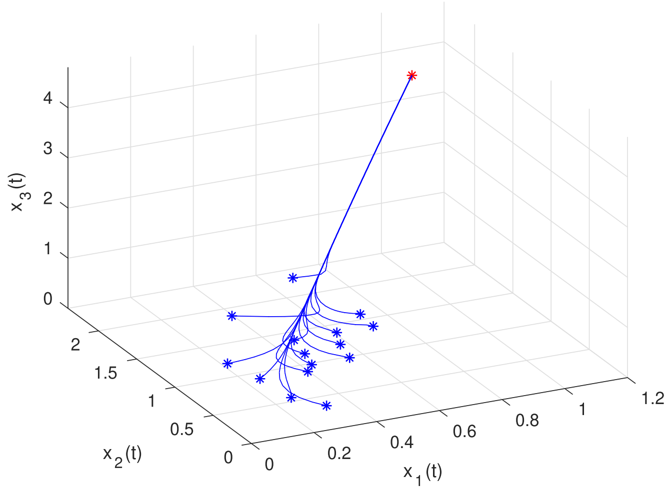

Fig. 6. Simulation results for the system described in Section 5.2.1 with the delay $\tau_1 = 0.5$. The loop was closed with the stabilizing feedback constructed by Theorem 4.2. The different initial functions are randomly sampled constant functions (blue asterisks). In this case, we have a unique complex balanced equilibrium (red asterisk). Trajectories (solid blue curves) converge to the desired equilibrium $x^* = [1\ 2\ 4]^T$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5.2.3. Stabilizing feedback design with additional monomials

As the condition $\dim\big(\mathrm{Im}(\overline{M}) + \mathrm{Im}(B)\big) = n$ is fulfilled, and B has a rational basis, we can design a new feedback with additional monomials which guarantees the uniqueness of x^*. We use the construction of Theorem 4.2 to get a sufficient set of new monomials as follows

$$Y_+ = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \ Y_- = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then the extended complex composition matrix is

$$Y' = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with the additional feedback monomials in the last two columns.

There are two ways to obtain the new feedback: by using the construction of Theorem 4.2, or by using the result of the optimization problem (14)–(18). Next we show the results of both methods.

Feedback constructed by using Theorem 4.2. In this case, the gains of the additional monomials are $K_+ = -0.25$ and $K_- = 0.5$. Then the corresponding feedback gains are

$$K_0' = \begin{bmatrix} -1 & 0.2387 & 3 & -0.25 & 0.5 \end{bmatrix},$$

$$K_1' = \begin{bmatrix} -3 & 0.7613 & 0 & 0 & 0 \end{bmatrix},$$

and the complex balanced realization is

$$A_0' = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3.7462 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.25 & 0 \end{bmatrix},$$

$$A_1' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0.2538 & 0 & 0 & 0 \\ 0 & 0.2538 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Fig. 6 shows the simulation result of this case using different constant initial functions. The figure shows that now all trajectories converge to the desired unique equilibrium point.

Feedback designed by optimization. We may also consider the additional monomials in the optimization problem (14)–(18) to get the new feedback. In that case, the new feedback gains are

$$K_0'' = \begin{bmatrix} -1 & 0.0716 & 3 & 0 & 0.1363 \end{bmatrix},$$

$$K_1'' = \begin{bmatrix} -3 & 0.2954 & 0 & 0 & 0.4967 \end{bmatrix}.$$

The corresponding realization has the form

$$A_0'' = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5267 \\ 0 & 3.3 & 0 & 0 & 0.5267 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1.1715 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1'' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0.0670 & 0 & 0 & 0.1063 \\ 0 & 0.0670 & 0 & 0 & 0.1063 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0946 & 0 & 0 & 0.1778 \end{bmatrix}.$$

Comparison of the two feedback design methods. It is important to note that both feedback result in a closed loop system which is complex balanced and has the unique equilibrium $x^* = [1\ 2\ 4]^T$, but their feedback gains and complex balanced realizations are different.

The optimization based method offers the possibility to specify other design goals (e.g. no delay in the feedback), too. In this case, one computes the L_1 minimization of the feedback coefficients that gives $\sum_{i,j,k} |[K_i']_{j,k}| = 8.75$ and $\sum_{i,j,k} |[K_i'']_{j,k}| = 8$.

The other difference between the two feedback gains is that the optimization based feedback uses less monomials, namely ψ_+ has zero coefficient in K_0'' and K_1''. On the other hand, the method based on Theorem 4.2 is computationally simpler, it requires only to solve an LP optimization problem.

6. Conclusions and future work

A feedback design method was proposed in this paper to transform a delayed polynomial system with linear input structure to a delayed complex balanced kinetic system which guarantees semistability or local asymptotic stability.

The existence and computability of the feedback do not depend on the magnitude of the delays. Moreover, the applied optimization-based computation framework allows the introduction of additional design requirements by appropriately selecting the objective function or by introducing further linear constraints such as the exclusion of delayed monomials from the feedback. All the required computations can be performed by convex optimization. The problem can be solved by linear programming, when only semistability of the chosen equilibrium point is needed. If one wants to ensure the uniqueness of the chosen closed loop equilibrium point, then the extended optimization problem requires the application of semidefinite programming.

Furthermore, we have shown that involving new monomials into the feedback law does not improve the solvability of the feedback design problem from the point of view of the existence of complex balanced closed loop realizations. However, assuming mild conditions, two alternative systematic methods have been proposed to extend the initial feedback with additional monomials to ensure the uniqueness of the selected complex balanced equilibrium. One of the design methods requires to solve an LP optimization problem even in this extended case. This result can also be applied for non-delayed polynomial models and therefore, improves the methodology described in [21] as well.

Further work will be focused on generalizing our method to handle parametric uncertainty, similarly to our previous results in [21]. Furthermore, we plan to relate our stability conditions to other existing results such as [2] and [6] that were derived for different delayed model classes.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejcon.2020.06.007.

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