Chapter 2
Stability and the Kleinian View of Geometry

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Abstract Youla parametrization of stabilizing controllers is a fundamental result of control theory: starting from a special, double coprime, factorization of the plant provides a formula for the stabilizing controllers as a function of the elements of the set of stable systems. In this case the set of parameters is universal, i.e., does not depend on the plant but only the dimension of the signal spaces. Based on the geometric techniques introduced in our previous work this paper provides an alternative, geometry based parametrization. In contrast to the Youla case, this parametrization is coordinate free: it is based only on the knowledge of the plant and a single stabilizing controller. While the parameter set itself is not universal, its elements can be generated by a universal algorithm. Moreover, it is shown that on the parameters of the strongly stabilizing controllers a simple group structure can be defined. Besides its theoretical and educative value the presentation also provides a possible tool for the algorithmic development.

2.1 Introduction and Motivation

In many of Euclid's theorems, he moves parts of figures on top of other figures. Felix Klein, in the late 1800s, developed an axiomatic basis for Euclidean geometry that started with the notion of an existing set of transformations and he proposed that geometry should be defined as the study of transformations (symmetries) and of the objects that transformations leave unchanged, or invariant. This view has come to be known as the Erlanger Program. The set of symmetries of an object has a very nice algebraic structure: they form a group. By studying this algebraic structure, we can

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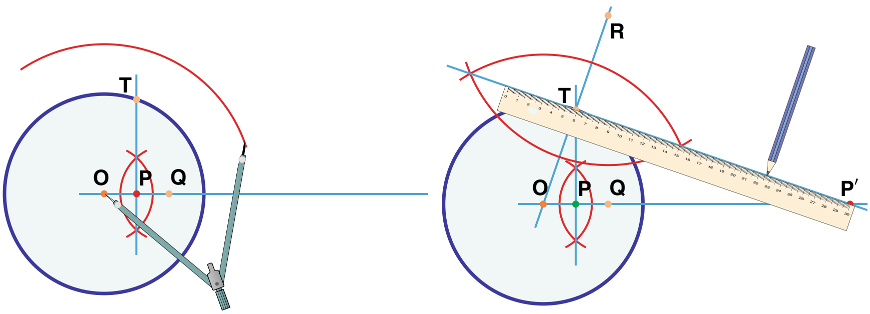
gain deeper insight into the geometry of the figures under consideration. Another advantage of Klein's approach is that it allows us to relate different geometries.

Klein proposed group theory as a mean of formulating and understanding geometrical constructions. In [36] the authors emphasise Klein's approach to geometry and demonstrate that a natural framework to formulate various control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms. The observation that any geometric property of a configuration, which is invariant under an euclidean or hyperbolic motion, may be reliably investigated after the data has been moved into a convenient position in the model, facilitates considerably the solution of the problems. In this work we put an emphasize on this concept of the geometry and its direct applicability to control problems.

The branches of mathematics that are useful in dealing with engineering problems are analysis, algebra, and geometry. Although engineers favour graphic representations, geometry seems to have been applied to a limited extent and elementary geometrical treatment is often considered difficult to understand. Thus, in order to put geometry and geometrical thought in a position to become a reliable engineering tool, a certain mechanism is needed that translates geometrical facts into a more accessible form for everyday algorithms. The compass and ruler should be changed to something else, possibly some series of numbers that can be manipulated more easily and the results can be interpreted more directly in terms of the given engineering problem (Fig. 2.1).



(a) Compass (b) Straightedge

Fig. 2.1 Euclidean constructions Klein proposed group theory as a mean of formulating and understanding geometrical constructions. The idea of constructions comes from a need to create certain objects in the proofs. Geometric constructions were restricted to the use of only a straightedge and compass and are related to Euclid's first three axioms: to draw a straight line from any point to any point, to produce a finite straight line continuously in a straight line and to draw a circle with any center and radius. The idealized ruler, known as a straightedge, is assumed to be infinite in length, and has no markings on it because none of the postulates provides us with the ability to measure lengths. While modern geometry has advanced well beyond the graphical constructions that can be performed with ruler and compass, it is important to stress that visualization might facilitate our understanding and might open the door for our intuition even on fields where, due to an increased complexity, a direct approach would be less appropriate

The link between algebra and geometry goes back to the introduction of real coordinates in the Euclidean plane by Descartes. By fixing a unit and defining the product of two line segments as another segment, Descartes gave a geometric justification of algebraic manipulations of symbols. The axiomatic approach to the Euclidean plane is seldom used because a truly rigorous development is very demanding while the Cartesian product of the reals provides an easy-to-use model. Descartes has managed to solve a lot of ancient problems by algebrizing geometry, and thus by finding a way to express geometrical facts in terms of other entities, in this case, numbers. Note that being a one-to-one mapping, this "naming" preserves information, so that we can study the corresponding group operations simply by looking at these operations' effect on the coordinates ("names"), even though the group elements themselves might be any kind of weird creatures.

The invention of Cartesian coordinates revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra, and provides enlightening geometric interpretations for many other branches of mathematics. Thus, coordinates, in general, are the most essential tools for the applied disciplines that deal with geometry. Descartes justifies algebra by interpreting it in geometry, but this is not the only choice: Hilbert will go the other way, using algebra to produce models of his geometric axioms. Actually this interplay between geometry, its group theoretical manifestation, algebra and control theory is what we are interested in.

The standard way to define the Euclidean plane is a two-dimensional real vector space equipped with an inner product: vectors correspond to the points of the Euclidean plane, the addition operation corresponds to translation, and the inner product implies notions of angle and distance. Since there is no canonical choice of where the origin should go in the space, technically an Euclidean space is not a vector space but rather an affine space on which a vector space acts by translations.

2.1.1 Invariants

In contrast to traditional geometric control theory, see, e.g., [2, 9, 39] for the linear and [1, 18, 19, 25] for the nonlinear theory, which is centered on a local view, our approach revolves around a global view. While the former uses tools from differential geometry, Lie algebra, algebraic geometry, and treats system concepts like controllability, as geometric properties of the state space or its subspaces the latter focuses on an input-output—coordinate free—framework where different transformation groups which leave a given global property invariant play a fundamental role.

In the first case the invariants are the so-called invariant or controlled invariant subspaces, and the suitable change of coordinates and system transforms (diffeomorphisms), see, e.g., the Kalman decomposition, reveal these properties. In contrast, our interest is in the transformation groups that leave a given global property, e.g., stability or $\mathscr{H}_\infty$ norm, invariant. One of the most important consequences of the approach is that through the analogous of the classical geometric constructions it not only might give hints for efficient algorithms but the underlaying algebraic structure, i.e., the given group operation, also provides tools for controller manipulations that preserves the property at hand, called controller blending.

There are a lot of applications for controller blending: both in the LTI system framework, [26, 32] and in the framework using gain-scheduling, LPV techniques, see [8, 15, 16, 31]. While these approaches exploit the so called Youla parametrization of stabilizing controllers, they do not provide an exhaustive characterization of the topic. The approach presented in this book does not only provide a general approach to this problem but, as an interesting side effect of these investigations, also shows that the proposed operation leaves invariant the strongly stabilizing controllers and defines a group structure on them. Moreover, one can define a blending that preserves stability and it is defined directly in terms of the plant and controller, without the necessity to use any factorization.

2.1.2 A Projective View

As a starting point of Euclidean and non-Euclidean worlds the most fundamental geometries are the projective and affine-ones. Perhaps it is not very surprising that feedback stability is related to such geometries. Following the Kleinian project we have to identify the proper mathematical objects and the groups associated to these objects that are related to the concept of stability and stabilizing controllers.

The determination of the stability of dynamical feedback systems from open loop characteristics is of crucial importance in control system design, and its study has attracted considerable research effort during the past fifty years. Until the early 1960s almost all these methods were for scalar input-output feedback systems; however, the rapid developments in the state-space representation of dynamical systems and their realizations from transfer functions led to an equally important development in stability criteria for multivariable feedback systems.

Much of the early work attempted to establish generalizations of the Nyquist, Popov and circle criteria by utilizing an extended version of the mathematical structures used for establishing scalar results. Later it became clear that such system representations are inadequate for the analysis of generalized multivariable operators in feedback systems. It turns out that an approach based upon the systems input-output spaces is required: the only systems representation admissible a priori is the input-output map which defines the system while the existence of every other representations are deduced from these properties. Thus the concept of input-output stability is essentially based upon the theory of operators defined on Hilbert (Banach) spaces.

Control theory should study also stability of feedback systems in which the open-loop operator is unstable or at least oscillatory. Such maps are clearly not contained in Banach spaces and some mathematical description is necessary if feedback stability is to be interpreted from open loop system descriptions. This is achieved by ruling out from the model class those unbounded operators that might “explode” and establishing the stability problem in an extended space which contains well-behaved as well as asymptotically unbounded functions, see [12]. The generalized extended space contains all functions which are integrable or summable over finite intervals. A disadvantage of the method is that the resulting space is a Banach space while we would prefer to work in a Hilbert space context for signals, and the set of stable operators for plants.

Since unbounded operators on a given space do not form an algebra—nor even a linear space, because each one is defined on its own domain—the association of the operator with a linear space, its graph subspace, turns to be fruitful. This leads us to the study of the generalized projective geometries that copy the constructions of the projective plane into a more complex mathematical setting while maintaining the original relations between the main entities and the original ideas. In doing this our main tools are algebraic: group theory, see [33], and the framework of the so called Jordan pairs will help us to obtain the proper interpretations and to achieve new results, see [34].

All these topics involves an advanced mathematical machinery in which often the underlying geometrical ideas remain hidden. Our aim is to highlight some of these geometric governing principles that facilitate the solution of these problems. We try to avoid, wherever is possible, the technical details which can be found in the cited references. We assume, however, some background knowledge from the reader concerning basic mathematical constructions and control theory. Therefore the style of the book is informal where the statements are rather meta-mathematical than mathematical. Throughout the presentation we always assume a reasonable algebraic structure in which our plants and controllers reside: as an example, the set of matrices, MIMO plants form $\mathscr{RL}_\infty$ ($\mathscr{RH}_\infty$), the set of finite dimensional LTV (LPV) plants. In a strictly formal presentation the details would be overwhelming that would distract the reader from the main message of the book. Concerning the possible details that one should complement to the statements of the work in order to construct a formal framework for robust LTV stability see, e.g., [22].

The main concern of this work is to highlight the deep relation that exists between the seemingly different fields of geometry, algebra and control, see Fig. 2.2. While the Kleinian view makes the link between geometry and group theory, through dif-

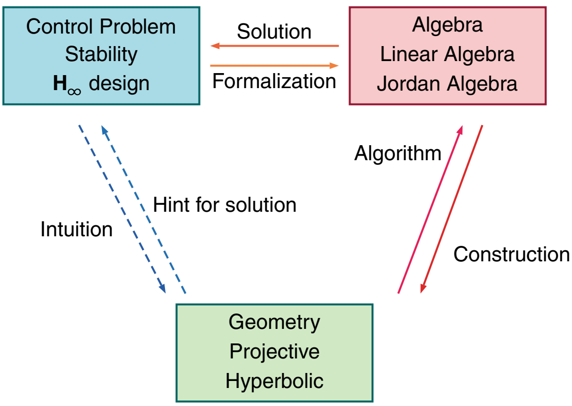


Fig. 2.2 Interplay: geometry, algebra and control

ferent representations and homomorphism the abstract group theoretical facts obtain an algebraic (linear algebraic) formulation that opens the way to engineering applications. We would like to stress that it is a very fruitful strategy to try to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction; finally the solution of the original control problem can be formulated in an algorithmic way by transposing the geometric ideas into the proper algebraic terms.

The main contribution of this work relative to the previous efforts is the following: it is shown that, in contrast to the classical Youla approach, there is a parametrisation of the entire controller set which can be described entirely in a coordinate free way, i.e., just by using the knowledge of the plant P and of the given stabilizing controller K_0. The corresponding parameter set is given in geometric terms, i.e., by providing an associated algebraic (semigroup, group) structure. It turns out that the geometry of stable controllers is surprisingly simple.

2.2 A Glimpse on Modern Geometry—The Kleinian View

Geometry ranges from the very concrete and visual to the very abstract and fundamental: it deals and studies the interrelations between very concrete objects such as points, lines, circles, and planes while on the other side, geometry is a benchmark for logical rigour. Algebraic structures form a parallel world, in which each geometric object and relation has an algebraic manifestation. In this algebraic world the considerations may be also very concrete and algorithmic or very abstract and fundamental.

While it is relatively easy to transform geometric objects into algebraic ones the "naive" approaches to representing geometric objects are very often not the right ones. Introducing more sophisticated algebraic methods often proves to be ultimately more powerful and elegant. Finding the right algebraic structure may open new perspectives on and deep insights into matters that seemed to be elementary at first sight and help to generalize, interpret and understand.

There is a rich interplay of geometric structures and their algebraic counterparts. In this section we will study very simple objects, such as points, lines, circles, conics, angles, distances, and their relations. Also the operations will be quite elementary, e.g., intersecting two lines, intersecting a line and a conic, etc. The emphasis are on structures: the algebraic representation of an object is always related to the operations that should be performed with the object. These advanced representations may lead to new insights and broaden our understanding of the seemingly well-known objects. Moreover these findings will be also useful in our control oriented investigations.

In the plane very elementary operations such as computing the line through two points and computing the intersection of two lines can be very elegantly expressed if lines as well as points are represented by three-dimensional homogeneous coordinates (where nonzero scalar multiples are identified). Taking a closer look at the relation of planar points and their three-dimensional representing vectors, it is apparent that certain vectors do not represent points in the real Euclidean plane. These nonexistent points may be interpreted as points that are infinitely far away; extending the usual plane by these new points at infinity a richer geometric system can be obtained: the system of projective geometry, which turned out to be one of the most fundamental structures having the most elegant algebraic representation.

Projective geometry was viewed as a relatively insignificant area within the domain of Euclidean geometry until in 1859 Cayley demonstrated that projective geometry was actually the most general and that Euclidean geometry was merely a specialization. Later, Klein demonstrated how non-Euclidean geometries could be included. In the spirit of the Erlangen program projective geometry is characterized by invariants under transformations of the projective group. It turns out that the incidence structure and the cross-ratio are the fundamental invariants under projective transformations.

Projective geometry become a fundamental area of modern mathematics with far reaching applications both in the mathematical theory, as algebraic geometry, and also in different applications fields, such as art, computer vision or even control theory, see, e.g., [11]. For a thorough treatment of the subject the interested reader might consult [10] or [4, 6]. In elaborating this chapter we mostly follow the approach of the more recent enlightening account of [30] to the topic.

2.2.1 Elements of Projective Geometry

Following Hilbert's approach a projective plane is a triple $(\mathscr{P}, L, I)$ where $\mathscr{P}$ is a set, called the set of points, L is a set called the set of lines, and I is a subset of $\mathscr{P} \times L$, called the incidence relation ($(P, l) \in I$ means: P is contained in l). The axioms of this geometry are: every two distinct points are contained in a unique line, every two distinct lines contain a unique point and there are four distinct points of which no three are collinear, i.e., lie on a single line. We will denote by $l = A \vee B$ the line passing through two points and by $L = a \wedge b$ the intersection of two lines.

A complete quadrangle is a set of four points A, B, C and D, no three collinear, and the six lines determined by these four points: AB and CD, AC and BD, and AD and BC are said to be pairs of opposite sides. The points at which pairs of opposite sides intersect are called diagonal points of the quadrangle.

A fourth axiom for a projective plane is Fano's Axiom: the three diagonal points of a complete quadrangle are never collinear. A projective plane that does not satisfy this axiom is the Fano plane determined by the seven-point and seven-line geometry.

In the ordinary plane parallel lines do not meet. In contrast, projective geometry formalizes one of the central principles of perspective, i.e., parallel lines meet at infinity. In essence it may be thought of as an extension of Euclidean geometry in which the direction of each line is subsumed within the line as an extra point, and in which a horizon of directions corresponding to coplanar lines is regarded as a line. Thus, two parallel lines meet on a horizon line in virtue of their possessing the same direction (Fig. 2.3).

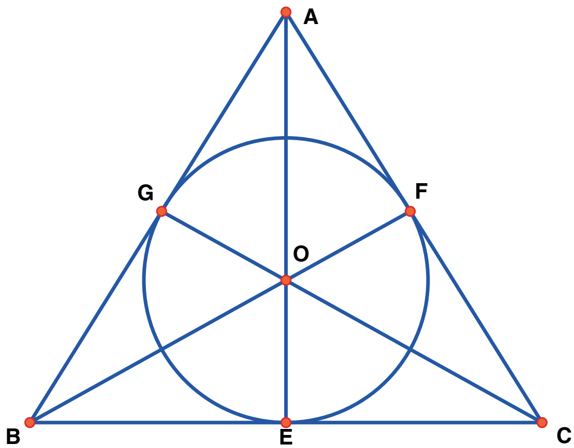


Fig. 2.3 Fano plane: the corresponding projective geometry consists of exactly seven points and seven lines with the incidence relation described by the attached figure. The circle together with the six segments represent the seven lines

Thus we can introduce a special hyperplane, the hyperplane at infinity or ideal hyperplane, and the points at infinity will be those on this hyperplane. Idealized directions are referred to as points at infinity, while idealized horizons are referred to as lines at infinity.

We say that two subspaces are parallel if they have the same intersection with this special hyperplane. Parallelism is an equivalence relation, however, infinity is a metric concept. A purely projective geometry does not single out any points, lines or plane and in this regard parallel and nonparallel lines are not treated as separate cases. In contrast, an affine space can be regarded as a projective space with a distinguished hyperplane.

Coordinates are important in the analytical development of projective geometry as an essential tool for calculations which may be used to verify and illustrate relations unambiguously. However, coordinates are typically based upon metrical considerations and an important question arose: how could such coordinates be logically applied to projective relations? Klein supplied an answer to this by suggesting the use of von Staudt's projective constructions which are employed to define the algebra of points. It is important to emphasize that in projective geometry coordinates are not understand in the ordinary metrical sense; they are a set of numbers, arbitrarily but systematically assigned to different points.

In order to assign coordinates to points on a line m it is required to select three distinct points P_0, P_1 and P_∞ which, by the special nature of the constructions, are endowed with the properties of 0, 1 and ∞.

As an illustration the addition of points on a line is defined using two special projective constructions, see Fig. 2.4. It can be shown that this algebra of points is isomorphic to the field of real numbers and can be extended to include the concept of infinity: a unique real number is associated with each point on the line with the exception of a single point which assumes a correspondence with infinity. The unique real number associated with each point is the non-homogeneous coordinate of the

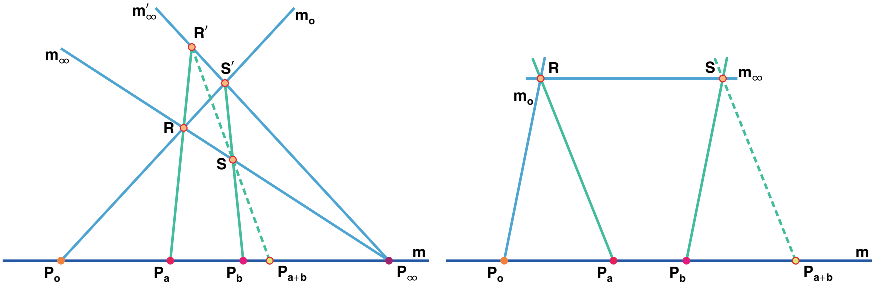


Fig. 2.4 Projective addition: for the addition of two points let us fix the points P_0 and P_∞. Then a fixed line m_0 through P_0 meets the two distinct fixed lines m_∞ and m'_∞ in the points R and S', respectively, while the lines $P_a R$ and $P_b S'$ meet m'_∞ and m_∞ at R' and S. The line $R'S$ meets m at $P_a + P_b = P_{a+b}$. By reversing the latter steps, subtraction can be analogously constructed, e.g., $P_a = P_{a+b} - P_b$. Observe that by sending point P_∞ to infinity we obtain the special configuration based on the "Euclidean" parallels and the common addition on the real line

point on the line. The exceptional role of the point associated with infinity can be removed upon the introduction of homogeneous coordinates.

The cross-ratio plays a fundamental role in the development of projective geometry. It was already known to Pappus of Alexandria and was used by Karl von Staudt to present the first entirely synthetic treatment of the projective geometry by introducing the notion of a throw a pair of ordered pairs of points on a line. Throws are separated into equivalence classes by the projectivities of the line, relative to its situation in a plane.

As a synthetic definition consider a line m embedded in a projective plane and use complete quadrilaterals to define addition and multiplication. Given any throw $\{[A, B], [C, D]\}$ and any fifth point E, there exist many complete quadrilaterals for which each of the pairs of the throws lie on the intersections of opposing lines of the quadrilateral, and such that one of the other lines passes through E. However, for each of these complete quadrilaterals the remaining line cuts m at the same point.

This defines a quinary operator cr on the points of m. One fixes three distinct points of m, calling them 0, 1 and ∞ and then places them in a certain way in three of the arguments of cr to obtain a binary operator. One of these ways defines addition, and another way defines multiplication such that the complement of ∞ in m becomes a field.

In order to obtain coordinates for the points of the projective plane $\mathbb{P}^2(\mathbb{R})$ we should chose a projective basis consisting of four distinct points 0, ∞_x, ∞_y and 1, i.e., the origin, an infinite point on the x-axis m_x, an infinite point on the y-axis m_y and a point with coordinates $(1, 1)^T$, respectively. We can also define points $1_x = (0 \vee \infty_x) \wedge (1 \vee \infty_y)$ and $1_y = (0 \vee \infty_y) \wedge (1 \vee \infty_x)$. A point X on m_x is uniquely determined by the cross-ratio $\mathrm{cr}(0, \infty_x, X, 1_x) = x$ and analogously for a point Y on m_y we have $\mathrm{cr}(0, \infty_y, Y, 1_y) = y$. Any point P of $\mathbb{P}^2(\mathbb{R})$ that does not lie on the line $m_\infty = \infty_x \vee \infty_y$ defines uniquely two points $P_x = m_x \wedge (P \vee \infty_y)$ and $P_y = m_y \wedge (P \vee \infty_x)$ from which it can be reconstructed according to $P = (P_x \vee \infty_y) \wedge (P_y \vee \infty_x)$. For an illustration of this construction see Fig. 2.5.

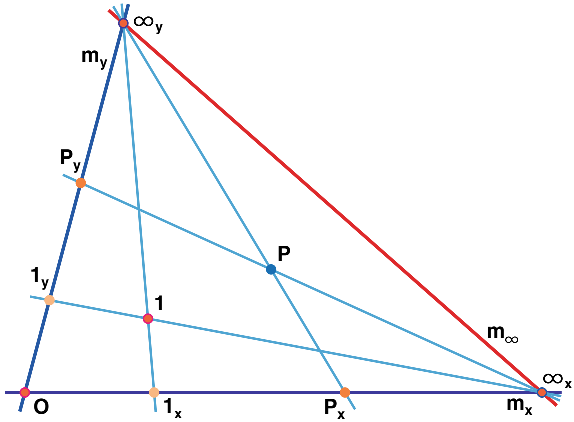


Fig. 2.5 Projective coordinates

Although the point triple (P_0, P_1, P_∞) (called scale) is selected arbitrarily, the addition and multiplication constructions impart them with the special properties associated with $(0, 1, \infty)$. From a projective point of view, however, all points have identical properties. Three distinct new points may be chosen as another scale and all other points relabeled in terms of it. By way of projective transformations, all scales and subsequently all coordinates, are projectively equivalent.

An algebraic model for doing projective geometry in the style of analytic geometry is given by homogeneous coordinates. When the vector space V is coordinatized by fixing a basis, a projective point is a 1-space $\{\lambda(x_0, x_1, \ldots, x_n) \mid \lambda \in \mathbb{F}\}$, i.e., an equivalence class $X \sim [x]$ of all vectors that differ by a nonzero multiple, and we can say that this point has coordinates $(x_0, x_1, \ldots, x_n)$. Note that $(x_0, x_1, \ldots, x_n)$ and $\lambda(x_0, x_1, \ldots, x_n)$ denotes the same point for $\lambda \neq 0$. Such coordinates are called homogeneous coordinates. By using homogeneous coordinates we can introduce a special hyperplane, e.g., the one defined by $x_n = 0$, the so called finite points will be the ones with $x_n \neq 0$, while the points at infinity will be those on the hyperplane.

A central concept in projective geometry is that of duality. The simplest illustration of duality is in the projective plane, where the statements "two distinct points determine a unique line" and "two distinct lines determine a unique point" show the same structure as propositions.

A line l passing through two points A and B may be described as the join of the two points, i.e., $l = A \vee B$ and dually, the intersection L point of two lines a and b may be described as the meet of the two lines, i.e., $L = a \wedge b$.

The principle of duality in the plane is that incidence relations remain valid when the roles of points and lines are interchanged, where the point P and line p are (projectively) dual objects.

The dualistic properties of projective geometry may be elegantly expressed in an analytic manner by employing homogeneous coordinates: the condition for a point $X \sim [x]$ with $x = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}^T$ and a line $m \sim [M]$ with $M = \begin{pmatrix} M_0 & M_1 & M_2 \end{pmatrix}$ to be incident may be expressed as the linear relation

$$M_0x_0 + M_1x_1 + M_2x_2 = 0, \text{ i.e., } Mx = 0.$$

Since condition $x_2 \neq 0$ selects out the finite points the line at infinity will corresponds to $m_\infty \sim [(0, 0, 1)]$. Here we assume that all homogeneous coordinates of a point are represented by column vectors while those that corresponds to lines are row vectors. However, it is more convenient to identify the lines with column vectors, too. This can be done through the pairing $\langle \cdot, \cdot \rangle$ as $m \sim \langle m, \cdot \rangle$. Thus the set of all points on the line r through the given points P, Q can be expressed with the condition $\langle r, \lambda p + \mu q \rangle = 0$ for all $\lambda, \mu \in \mathbb{R}$.

Assuming that the coordinates M are fixed while the coordinates x are free to vary, then this equation ($x \in \mathrm{Ker}(m^T)$) represents the locus of points which are incident to the line m. Dually, if the coordinates x are fixed and m is free to vary, then the equation ($m \in \mathrm{Im}(x)^\perp$) represents the pencil of lines which are incident to the point x.

Thus we extend the Euclidean plane by introducing elements at infinity: one point at infinity for each direction and one global line at infinity that contains all these points. We also have a coordinate representation of these objects. Actually the incidence relation $(X, m) \in I$ is expressed as $([x], [m]) \in I_{\mathbb{R}^2}$ defined by the condition $x \perp m$. Thus, by the identification determined by the homogeneous coordinates of the points and lines with equivalence classes of vectors, we have that $(\mathscr{P}_{\mathbb{R}^2}, L_{\mathbb{R}^2}, I_{\mathbb{R}^2})$ is a projective plane: $\mathbb{P}^2(\mathbb{R})$. While this is a simple observation it has an important consequence: it consists the link between geometry and algebra.

From the projective viewpoint the distinction of infinite and finite elements is completely unnatural: it is only a kind of artefact that arises when we interpret the Euclidean plane in a projective setup. Often it is fruitful to interpret Euclidean theorems in a projective framework and vice-versa. To do it we have to model the drawing of a parallel to a line through a point on the projective plane: set the line at infinity (m_∞) and define the operator parallel $(P, m) = P \vee (m \wedge m_\infty)$.

2.2.2 Projective Transformations

Klein stated that a geometry is defined as the properties of a space which remain invariant under all transformations of space (or the coordinate system) by a group of transformations. Thus Euclidean geometry is the theory of objects invariant with respect to Euclidean congruence transformations. For projective geometry, the group of transformations is characterized by those which preserve relations of incidence. An analysis of projective transformations not only identifies important invariant relations but also forms a foundation for developing metrical geometries.

The group of automorphisms of n-dimensional projective space $\mathbb{P}^n(\mathbb{R})$ are induced by the linear automorphisms of $\mathbb{R}^{n+1}$. These can be projective automorphisms, projective collineations or regular projective maps. The group of projective automorphisms of $\mathbb{P}^n(\mathbb{R})$ is denoted by $\mathrm{PGL}(n)$, and is called the projective linear group. Thus the action of projective automorphisms on points can be expressed as $[Ax]$ and, accordingly, on the hyperplanes $[A^{-T}m]$.

The fixed points of the projective automorphisms are given by the (right) eigenvector of the matrix A. It follows that every projective transformation has at least one invariant point and one invariant line. Moreover there is exactly one projective isomorphism which transforms a given fundamental set into another one.

The restriction of a projective mapping in $\mathbb{P}^n(\mathbb{R})$ to a line l is called a projectivity, which is uniquely defined by the images of three distinct points of the line. A projective automorphism of a line, if it is not the identity mapping, has 0, 1, or 2 fixed points. Then the corresponding projective automorphism is called elliptic, parabolic or hyperbolic, respectively. In the complex projective plane there are no elliptic projectivities.

A collineation is a one–to–one linear transformation preserving the incidence relation in which each element is mapped into a corresponding element of the same type (e.g., point to point) whereas a correlation differs in that each element is mapped into a corresponding dual element (e.g., point to line).

It is often useful to consider singular linear mappings, whose domain is a projective space of dimension n and whose image space has a different dimension. Singular projective mapping means a linear mapping which is not quadratic and regular, i.e., it is not a projective isomorphism. Such mappings are generalizations of the concept of central projection from projective three-space onto a plane. A central projection from $\mathbb{P}^n(\mathbb{R})$ onto a subspace V via a center W is given by $\pi(P) = (O \vee P) \wedge V$, where it is required that W and V are complementary subspaces. For all linear mappings $\lambda : \mathbb{P}^n \mapsto \mathbb{P}^m$ there is a central projection π from onto a subspace V and a projective isomorphism α of V onto $\mathbb{P}^m$ such that $\lambda = \pi\alpha$. A linear mapping has a kernel (center or exceptional subspace) Z which is independent of the decomposition. The points $Q \in P \vee Z$ have the property that $\pi(P) = \pi(Q)$.

2.2.3 A Trapezoidal Addition

We conclude this section by reviewing a specific configuration of the projective plane, and its associated special addition law, which bears relevance to the study of feedback stability from a projective point of view.

First, let us list some facts important to us concerning the case $d = 1$, i.e., the projective line $\mathbb{P}^1(\mathbb{R})$. If V is a one dimensional subspaces (line) of a vector space, by choosing a basis of V gives an identification of V with $\mathbb{P}^1(\mathbb{R})$. But another choice of basis of V gives another identification of V with $\mathbb{P}^1(\mathbb{R})$, leading to the group of projective transformations of $\mathbb{P}^1(\mathbb{R})$. As it is shown by this case, groups (isomorphisms) occur in the description of the differences between parametrizations that preserve a certain structure.

The projective line $\mathbb{P}^1(\mathbb{R})$ is the set of lines through 0 in $\mathbb{R}^2$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(\mathbb{R}^2)$ we have the map: $\mathbb{R}^2 \mapsto \mathbb{R}^2,\ x \mapsto Mx$ that sends lines through 0 to lines through 0, and hence gives us a map from $\mathbb{P}^1(\mathbb{R})$ to $\mathbb{P}^1(\mathbb{R})$. Written out in detail

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \begin{pmatrix} ax_0 + bx_1 \\ cx_0 + dx_1 \end{pmatrix},$$

i.e., in inhomogeneous coordinates

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} ax + b \\ cx + d \end{pmatrix}, \quad x \mapsto \frac{ax + b}{cx + d},$$

if $cx + d \neq 0$. Thus the fractional linear transformations from $\mathbb{R}$ to $\mathbb{R}$ is linear in homogeneous coordinates.

It is obvious that M and M' in $\mathrm{GL}(\mathbb{R}^2)$ give the same projective transformation on $\mathbb{P}^1(\mathbb{R})$ precisely when there is a k in $\mathbb{R}^*$ with $M' = kM$. Thus the group of projective transformations (projectivities) of $\mathbb{P}^1(\mathbb{R})$ is the quotient group $\mathrm{PGL}(\mathbb{R}^2)$ of $\mathrm{GL}(\mathbb{R}^2)$ by the subgroup of scalar matrices.

In other words, Möbius transformations can be seen as the restriction of the projective transformation to the set of finite points. Note that while the projective transformation M is linear, and it is defined everywhere, the Möbius transformation is nonlinear (rational) and it is defined only on the domain $cx + d \neq 0$. If $(x, 1)^T$ and $(y = \frac{ax+b}{cx+d}, 1)^T$ are considered as specific (normalized) homogeneous coordinates of the finite points, then we can say that the Möbius transformation acts on a coordinate level while the projectivity M acts on the geometric, projective level.

Projective transformations leave the cross ratio

$$\mathrm{cr}(p,\ q,\ r,\ s) = \frac{(r-p)/(r-q)}{(s-p)/(s-q)}, \quad p,\ q,\ r,\ s \in \mathbb{P}^1(\mathbb{R})$$

invariant, i.e., if g is a projective transformation then

$$\mathrm{cr}(g(p),\ g(q),\ g(r),\ g(s)) = \mathrm{cr}(p,\ q,\ r,\ s).$$

Since Möbius transformations are only a restriction of projective transformations on finite points, invariance holds.

We have already seen that in terms of homogeneous coordinates the Euclidean plane $\mathbb{R}^2$ can be embedded into $\mathbb{P}^2(\mathbb{R})$ by taking its finite points, i.e., by the map

$$\mathbb{R}^2 \mapsto \mathbb{P}^2(\mathbb{R}),\ (x,\ y)^T \mapsto (x, y, 1)^T.$$

The points of $\mathbb{P}^2(\mathbb{R})$ that are not in the image of this map are the ideal points $(x, y, 0)^T$. Thus the set of ideal points is in bijection with the set of points $(x, y)^T$ of the projective line $\mathbb{P}^1(\mathbb{R})$. This is the set of directions in $\mathbb{R}^2$, that correspond to the points on the horizon in $\mathbb{P}^2(\mathbb{R})$.

An example for this embedding in terms of projective coordinates is depicted on Fig. 2.5. Recall that in order to obtain coordinates we should choose a projective basis consisting of four distinct points 0, ∞_x, ∞_y and 1. In an obvious way the construction defines an addition operation on the plane, see Fig. 2.6.

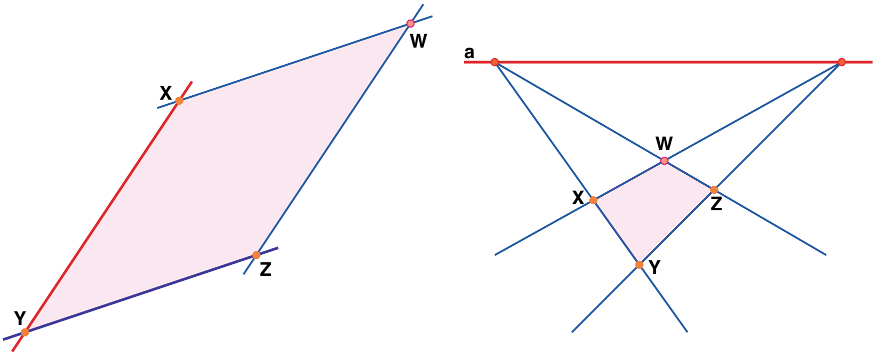


Fig. 2.6 Parallel addition: set the origin to the point Y and let m_x and m_z the directions determined by the points X and Z. If the points ∞_x, ∞_y are set to infinity we obtain a usual setting for parallel vector addition: the coordinates of the point W are constructed by taking parallels to m_x and m_z through W. For a "projective" vector addition we can set the points ∞_x, ∞_y on a given line a of $\mathbb{R}^2$ intersecting m_x and m_z. The point W is provided as $W = \Big[\big((X \vee Y) \wedge a\big) \vee Z\Big] \wedge \Big[\big((Z \vee Y) \wedge a\big) \vee X\Big]$

In [5] these constructions were generalized in order to provide a friendly introduction to Jordan triplets and to illustrate algebraic concepts through elementary constructions performed in the plain geometry. We reproduce here those constructions from [5] that are relevant for our control oriented view.

Parallel addition: recall that we imagine a point "infinitely far" on the line l, given by intersecting l with an ideal line i; then the parallel to X is the line joining this infinitely far point $l \wedge i$ with X: $k = X \vee (l \wedge i)$ is the unique parallel of l through the point X.

In the usual, parallel, view given three non-collinear points X, Y, Z we can construct a fourth point according to

$$W = \Big(\big((X \vee Y) \wedge i\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge i\big) \vee X\Big),$$

which is the intersection of the parallel of $X \vee Y$ through Z with the one of $Z \vee Y$ through X.

Note that the initial construction works well if we assume that X, Y, Z are not collinear, but it is not defined if X, Y, Z are on a common line l. Nevertheless, the map associating to the triple X, Y, Z the fourth point W admits a continuous extension from its initial domain of definition (non-collinear triples) to the bigger set of all triples.

If we choose a line a in the plane and three non-collinear points X, Y, Z in the plane such that the lines $X \vee Y$ and $Z \vee Y$ are not parallel to a then it is possible to construct the point

$$W = \Big(\big((X \vee Y) \wedge a\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big).$$

One can imagine this drawing to be a perspective view onto a plane in 3-dimensional space, where line a represents the horizon. Dragging the line a further and further away from X, Y, Z the perspective view looks more and more like a usual parallelogram construction.

The fourth vertex W is a function of X, Y, Z, therefore we introduce the notation $W = X +_Y Z$, and write $W = \{XYZ\}$. We write O instead of Y if it is fixed as origin, i.e., let $X + Z = X +_O Z$.

Since the operations $\vee$ and $\wedge$ are symmetric in both arguments the law $(X, Z) \mapsto X + Z$ is commutative. But the choice of the origin O is completely arbitrary, thus, the free change of the origin should be facilitated by a more general version of the associative law (called the *para-associative law*):

$$X +_O (U +_P V) = (X +_O U) +_P V$$

where O and P may be different points. Thus, to cope with the problem of collinear points we can use the para-associative law:

$$(X +_O P) +_P V = X +_O (P +_P V) = X +_O V.$$

It turns out that for any fixed origin O the plane $\mathbb{R}^2$ with $X + Z = X +_O Z$ is a commutative group with neutral element O.

Trapezoidal addition: in order to obtain a more general scheme we can introduce two special lines—as if they played the role of the ideal lines—and to define the point addition as:

$$W = \Big(\big((X \vee Y) \wedge a\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge b\big) \vee X\Big),$$

see Fig. 2.7. Note that when the lines a, b and the point Y are kept fixed, the law given by $(X, Z) \mapsto W$ depends nicely on the parameters Y, a, b.

If instead of "parallelograms" we use trapezoids, i.e., $b = i$, the constructions will depend on the choice of some line a in the plane and the underlying set of our constructions will be the set $\mathbb{G} = \mathbb{R}^2 \setminus a$ of all points of the plane $\mathbb{R}^2$ not on a.

Fixing a point Y not on a, and two other points X, Z such that the line $Y \wedge Z$ is not parallel to a we can construct the point

$$W = \Big(\big((X \vee Y) \wedge i\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big).$$

Observe that the map $W = \{XYZ\}$ is not symmetric in X and Z, therefore the law $(X, Z) \mapsto W = \{XYZ\}$ for fixed Y is not commutative. However, the operation $\{XYZ\}$ is associative, moreover, the following generalized associativity law holds:

$$\{XO\{UPV\}\} = \{\{XOU\}PV\}.$$

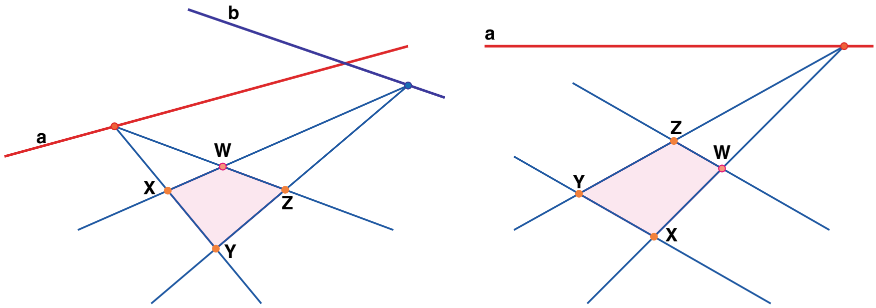


Fig. 2.7 While the quadrangle $XYZW$ is not a parallelogram, its construction has something in common with the one of a parallelogram: the picture illustrates the fundamental process of passing from a commutative, associative law—vector addition, corresponding to usual parallelograms—to a non-commutative law: $W = \Big(\big((X \vee Y) \wedge b\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big)$. Trapezoidal addition, i.e., $b = i$, the point W is provided as $W = \Big(\big((X \vee Y) \wedge i\big) \vee Z\Big) \wedge \Big(\big((Z \vee Y) \wedge a\big) \vee X\Big)$

If we fix some element $E \in \mathbb{G}$, then E is a unit element for the binary product $XZ = \{XEZ\}$. Thus for three points X, E, V on a line, we can define a fourth point $W = \{XEV\}$ on the same line.

As a conclusion: for any choice of origin $E \in \mathbb{G}$, the set $\mathbb{G} = \mathbb{R}^2 \setminus a$ is a group with product $XZ = \{XEZ\}$. By using the generalized associativity law follows that $U = (EXE)$ is the inverse of X. The converse is also true: the ternary law $\{XYZ\}$ can be recovered from the binary product in the group $(\mathbb{G}, e)$ with neutral element e as $\{xyz\} = xy^{-1}z$.

We can translate these geometrical facts into analytic formulas by using coordinates of the real vector space $\mathbb{R}^2$. Then, vectors are written as $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ while their sum is defined by $x + y = (x_1 + y_1, x_2 + y_2)^T$. Recall that for two distinct points the affine line spanned by x and y is

$$x \vee y = \big\{tx + (1-t)y \mid t \in \mathbb{R}\big\}.$$

Note that the quadrangle with vertices $xyzw$ is a parallelogram if and only if $w = x - y + z$. Thus, for a fixed element $y \in \mathbb{R}^2$, the law $x +_y z = x - y + z$ defines a commutative group with neutral element y. For $y = 0$, we get back the usual vector addition.

The linear algebra of trapezoid geometry can be obtained by fixing a line a given by $a = \{x \in \mathbb{R}^2 \mid \alpha(x) = 0\}$ for some non-zero linear form $\alpha : \mathbb{R}^2 \to \mathbb{R}$. Since lines $y \vee x$ and $z \vee w$ are parallel, we have that $z + t(x - y)$ for some $t \in \mathbb{R}$. The point $u = a \wedge (y \vee z)$ should be of the form $t_u y + (1 - t_u)z$—since u is on $y \vee z$.

Thus, from $\alpha(u) = 0$ follows that

$$t_u = \frac{1}{1 - \alpha(y)\alpha(z)^{-1}}.$$

Note that $|w - z|/|x - y| = |u - z|/|u - y|$, i.e., $|t| = |t_u|/|1 - t_u|$, from which follows that $t = \alpha(z)\alpha(y)^{-1}$.

Thus, on the set $\mathbb{G}$ defined by

$$\mathbb{G} = \{x \in \mathbb{R}^2 \mid \alpha(x) \neq 0\}$$

the point w is defined by

$$\{xyz\} = w = \alpha(z)\alpha(y)^{-1}(x - y) + z. \tag{2.1}$$

Observe that $\alpha(w) = \alpha(x)\alpha(y)^{-1}\alpha(z)$. For a fixed point $e \in \mathbb{G}$ such that $\alpha(e) = 1$ we have that $\mathbb{G}$ is a group with neutral element e and product

$$x \cdot z = (xez) = \alpha(z)(x - e) + z. \tag{2.2}$$

The corresponding group inverse of x is given by

$$x^{-1} = \alpha(x)^{-1}(e - x) + x. \tag{2.3}$$

The set $\mathbb{G}$ is open dense in $\mathbb{R}^2$. Moreover, the group law with the corresponding inversion map are smooth of class C^∞. It can be shown that $\mathbb{G}$ is isomorphic to $(\mathbb{R}, +) \times (\mathbb{R}^\times, \cdot)$, i.e., it is isomorphic to the affine group of the real line:

$$\mathrm{GA}(1, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

To bring this example closer to the feedback setting let us consider a special configuration: take the line a such that $\alpha = (-p,\ 1)$, i.e., the points of this line are $\lambda(1, p)^T$. Set $e = y = (0, 1)$ as the unit element and note that $\alpha(y) = 1$. Consider the set

$$\mathbb{G}_p = \{(k, 1)^T \in \mathbb{R}^2 \mid 1 - pk \neq 0\}$$

and the points $z = (k_z, 1)$ and $x = (k_x, 1)$ from $\mathbb{G}_p$.

Then, we have that

$$x \cdot_p z = \alpha(z)(x - e) + z = (1 - pk_z)\begin{pmatrix} k_x \\ 0 \end{pmatrix} + \begin{pmatrix} k_z \\ 1 \end{pmatrix} = \begin{pmatrix} k_x + k_z - pk_z k_x \\ 1 \end{pmatrix},$$

and

$$x^{-_p} = \alpha(x)^{-1}(e - x) + e = \begin{pmatrix} -(1 - pk_x)^{-1}k_x \\ 1 \end{pmatrix}.$$

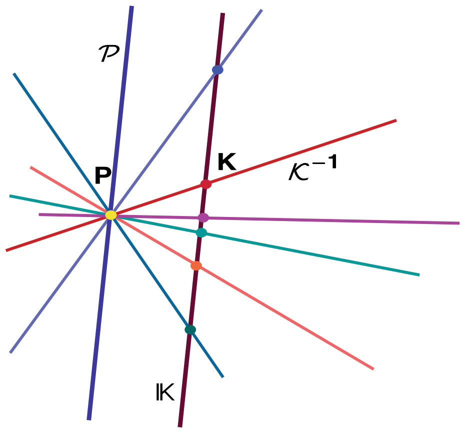


Fig. 2.8 Affine parametrization

In other words, if we fix p and consider all those k for which the matrix

$$F_{p,k} = \begin{pmatrix} 1 & k \\ p & 1 \end{pmatrix} \quad \text{or} \quad F_{-p,-k} = \begin{pmatrix} 1 & -k \\ -p & 1 \end{pmatrix} \quad \text{is nonsingular,}$$

then we obtain exactly the set $\mathbb{G}_p$. Moreover, on this set we have managed to define a group structure, $(\mathbb{G}_p, +_p)$ with unit element 0 defined by

$$k_1 +_p k_2 = k_1 + k_2 - pk_1k_2, \quad k^{-_p} = -(1 - pk)^{-1}k. \tag{2.4}$$

Observe that for $p = 0$ we obtain the usual addition on the real line.

The significance of the result for control is straightforward: take p as a plant and k as a controller. Then condition $1 - pk \neq 0$ selects exactly the controllers that renders the loop well-defined. By taking an arbitrary parallel line with p, its intersection with any other non-parallel line will work as a parametrization of these controllers.

Thus, we have the affine picture sketched on Fig. 2.8. In what follows we are going to provide further explanations in the context of the stable feedback loop.

2.3 The Standard Feedback Loop

A central concept of control theory is that of the feedback and the stability of the feedback loop. For practical reasons our basic objects, the systems, i.e., plants and controllers, are causal. Stability is actually a continuity property of a certain map, more precisely a property of boundedness and causality of the corresponding map. Boundedness here involves some topology. In what follows we consider linear systems, i.e., the signals are elements of some normed linear spaces and an operator means a linear map that acts between signals. Thus, boundedness of the systems is regarded as boundedness in the induced operator norm.

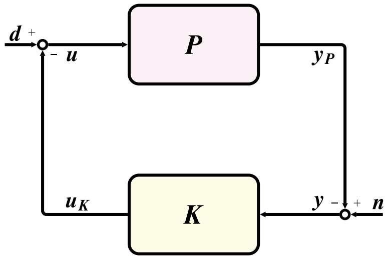


Fig. 2.9 Feedback connection

To fix the ideas let us consider the feedback-connection depicted on Fig. 2.9. It is convenient to consider the signals

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, \quad p = \begin{pmatrix} u \\ y_P \end{pmatrix}, \quad k = \begin{pmatrix} u_K \\ y \end{pmatrix}, \quad z = \begin{pmatrix} u \\ y \end{pmatrix} \in \mathscr{H},$$

where $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and we suppose that the signals are elements of the Hilbert space $\mathscr{H}_1, \mathscr{H}_2$ (e.g., $\mathscr{H}_i = \mathscr{L}^{n_i}[0, \infty)$) endowed by a resolution structure which determines the causality concept on these spaces. In this model the plant P and the controller K are linear causal maps. For more details on this general setting, see [12].

The feedback connection is called well-posed if for every $w \in \mathscr{H}$ there is a unique p and k such that $w = p + k$ (causal invertibility) and the pair (P, K) is called stable if the map $w \to z$ is a bounded causal map, i.e., the pair (P, K) is called well-posed if the inverse

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} S_u & S_c \\ S_p & S_y \end{pmatrix} = \begin{pmatrix} (I - KP)^{-1} & -K(I - PK)^{-1} \\ -P(I - KP)^{-1} & (I - PK)^{-1} \end{pmatrix} \tag{2.5}$$

exists (causal invertibility), and it is called stable if all the block elements are stable.

2.3.1 *Youla Parametrization*

A fundamental result concerning feedback stabilization is the description of the set of the stabilizing controllers. A standard assumption is that among the stable factorizations there exists a special one, called double coprime factorization, i.e., $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and there are causal bounded systems $U, V, \tilde{U}$ and $\tilde{V}$, with invertible V and $\tilde{V}$, such that

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \tilde{\Sigma}_P \Sigma_P = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2.6}$$

an assumption which is often made when setting the stabilization problem, [12, 38]. The existence of a double coprime factorization implies feedback stabilizability, actually $K_0 = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ is a stabilizing controller. In most of the usual model classes actually there is an equivalence.

For a fixed plant P let us denote by $\mathbb{W}_P$ the set of well-posed controllers, while $\mathbb{G}_P \subset \mathbb{W}_P$ denotes the set of stabilizing controllers.

Given a double coprime factorization the set of the stabilizing controllers is provided through the well-known Youla parametrization, [23, 41]:

$$\mathbb{G}_P = \{K = \mathfrak{M}_{\Sigma_P}(Q) \mid Q \in \mathbb{Q}, (V + NQ)^{-1} \text{ exists}\},$$

where $\mathbb{Q} = \{Q \mid Q \text{ stable }\}$ and

$$\mathfrak{M}_{\Sigma_P}(Q) = (U + MQ)(V + NQ)^{-1}. \tag{2.7}$$

For a recent work that covers most of the known control system methodologies using a unified approach based on the Youla parameterization, see [20].

Here $\mathfrak{M}_T(Z)$ is the Möbius transformation corresponding to the symbol T defined by

$$\mathfrak{M}_T(Z) = (B + AZ)(D + CZ)^{-1}, \text{ with } T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

on the domain $\mathrm{dom}_{\mathfrak{M}_T} = \{Z \mid (D + CZ)^{-1} \text{ exists}\}$. Note that

$$Q_K = \mathfrak{M}_{\tilde{\Sigma}_P}(K) = (\tilde{V}K - \tilde{U})(\tilde{M} - \tilde{N}K)^{-1}, \tag{2.8}$$

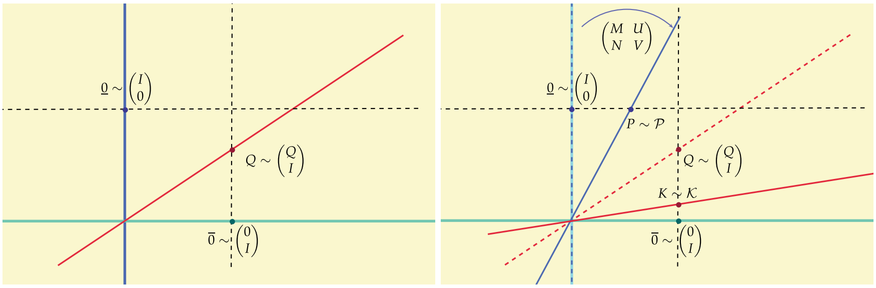
and thus $Q = 0_K$ corresponds to $K_0 = UV^{-1}$.

Since the dimensions of the controller and plant are different, it is convenient to distinguish the zero controller and zero plant by an index, i.e., 0_K and 0_P, respectively.

Observe that the domain of (2.8) is exactly $\mathbb{W}_P$; thus we can introduce the corresponding extended parameter set $\mathbb{Q}_P^{wp} = \{Q_K = \mathfrak{M}_{\tilde{\Sigma}_P}(K) \mid K \in \mathbb{W}_P\}$. Note, that Q_0, i.e., $\mathfrak{M}_{\tilde{\Sigma}_P}(0_K) = -\tilde{U}\tilde{M}^{-1} = -M^{-1}U$, is not in $\mathbb{Q}$, in general. The content of the Youla parametrization is that K is stabilizing exactly when $Q_K \in \mathbb{Q}$, see Fig. 2.10.

2.4 Group of Controllers

In order to design efficient algorithms that operate on the set of controllers that fulfil a given property, e.g., stability or a prescribed norm bound, it is important to have an operation that preserves that property, i.e., a suitable blending method. Available approaches use the Youla parameters in order to define this operation for stability in a trivial way. As these approaches ignore the well-posedness problem by assuming strictly proper plants, they do not provide a general answer to the problem.



(a) Trivial case (b) Transformed case

Fig. 2.10 Youla parametrization

In the particular case when $P = 0_P$ we have $\mathbb{G}_P = \mathbb{Q}$, i.e., mere addition preserves well-posedness and stability. Moreover, the set of these controllers forms the usual additive group $(\mathbb{Q}, +)$ with neutral element 0_K and inverse element $Q \to -Q$. In the general case, however, addition of controllers neither ensure well-posedness nor stability.

2.4.1 *Indirect Blending*

The most straightforward approach to obtain a stability preserving operation is to find a suitable parametrization of the stabilizing controllers, where the parameter space possesses a blending operation. As an example for this indirect (Youla based) blending is provided by the Youla parametrization. However, this mere addition on the Youla parameter level does not lead, in general, to a “simple” operation on the level of controllers:

$$K = \mathfrak{M}_{\Sigma_P}((\mathfrak{M}_{\tilde{\Sigma}_P}(K_1) + \mathfrak{M}_{\tilde{\Sigma}_P}(K_2))). \tag{2.9}$$

The unit element of this operation is the controller K_0 which defines Σ_P, see Fig. 2.11. Its implementation involves three nontrivial transformations.

Note that an obstruction might appear if the sum of the Youla parameters are not in the domain of $\mathfrak{M}_{\Sigma_P}$, e.g., for non strictly proper plants where some of the non strictly proper parameters are out-ruled.

We can formulate this process as a group homomorphism between the usual addition of parameters $\mathbb{Q}$ and the group of automorphisms $Q \mapsto \tau_Q$ associated to the space formed by simple translations, i.e.,

$$\tau_Q = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}, \quad \tau_{Q_1}\tau_{Q_2} = \tau_{Q_1+Q_2}.$$

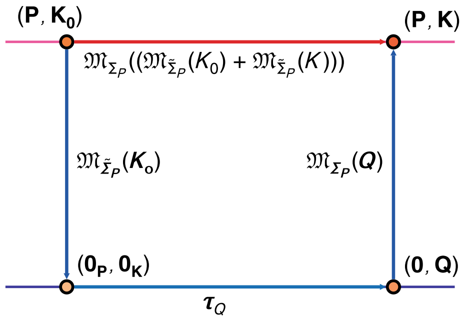


Fig. 2.11 Youla based blending

2.4.2 Direct Blending

The observation that

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & I - PK_1 \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - PK_2 \end{pmatrix} \tag{2.10}$$

leads to operation

$$K = K_1(I - PK_2) + K_2 = K_1 \boxdot_P K_2, \tag{2.11}$$

under which well-posed controllers form a group $(\mathbb{W}_P, \boxdot_P)$. The unit of this group is the zero controller $K = 0_K$ and the corresponding inverse elements are given by

$$K^{\boxminus_P} = -K(I - PK)^{-1}. \tag{2.12}$$

Note that

$$I - PK^{\boxminus_P} = (I - PK)^{-1}. \tag{2.13}$$

Clearly not all elements of $\mathbb{W}_P$ are stabilizing, e.g., 0_K is not stabilizing for an unstable plant.

Theorem 2.1 $(\mathbb{G}_P, \boxdot_P)$ *with the operation (blending) defined in* (2.11) *is a semi-group.*

Note, that

$$(I - PK)^{-1} = (I - PK_2)^{-1}(I - PK_1)^{-1}. \tag{2.14}$$

By using the notation

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I - PK \end{pmatrix} = R_P T_K^{(P)}$$

we have the group homomorphism $T^{(P)}_{K_1} T^{(P)}_{K_2} = T^{(P)}_{K_1 \boxdot_P K_2}$ and $K = \mathfrak{M}_{R_P T^{(P)}_K R_P^{-1}}(0_K)$.

On the level of Youla parameters the corresponding operation is more complex:

$$
\begin{aligned}
Q_{K_2} \odot_P Q_{K_1} &= \tilde{V} U + \tilde{V} M Q_{K_1} + Q_{K_2} \tilde{M} V + Q_{K_2} \tilde{N} M Q_{K_1} = \\
&= (Q_{K_2} - Q_0) \tilde{M} (V + N Q_{K_1}) + Q_{K_1} = \\
&= Q_{K_2} + (\tilde{V} + Q_{K_2} \tilde{N}) M (Q_{K_1} - Q_0), \qquad (2.15)\\
Q_{K^{\boxminus_P}} &= Q_0 - M^{-1} K \tilde{M}^{-1} = \\
&= Q_0 - (Q_K - Q_0)(I + V^{-1} N Q_K)^{-1} V^{-1} \tilde{M}^{-1}. \qquad (2.16)
\end{aligned}
$$

Note that $(\mathbb{G}_P, \boxdot_P)$ and $(\mathbb{Q}, \odot_P)$ are related by only a semigroup homomorphism, while $(\mathbb{W}_P, \boxdot_P)$ and $(\mathbb{Q}^{wp}_P, \odot_P)$ are related, however, through a group homomorphism.

2.4.3 *Strong Stability*

If a plant is stabilizable in general it is not obvious whether there exists a stable controller as a stabilizing one. If such a controller exists, then we call it a strongly stabilizing controller. While their synthesis is non-trivial, in practical applications strongly stabilizing controllers are preferred, see [13, 14].

The semigroup $(\mathbb{G}_P, \boxdot_P)$ does not have a unit, in general. However, if there is a stabilizing controller K_0 such that

$$K_0^{\boxminus_P} = -K_0 (I - P K_0)^{-1}$$

is also a stabilizing controller, i.e., K_0 is stable, then $(\mathbb{G}_P, \boxtimes_P)$ with

$$K_1 \boxtimes_P K_2 = K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2$$

is a semigroup with a unit (K_0). This may happen only if the plant is strongly stabilizable.

If we denote by $\mathbb{S}_P$ the set of strongly stabilising controllers, then if this set is not empty, then.

Theorem 2.2 $(\mathbb{S}_P, \boxtimes_P)$ *with the operation (blending) defined as*

$$
\begin{aligned}
K &= K_1 \boxtimes_P K_2 = K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2 = \\
&= K_2 + (K_1 - K_0)(I - P K_0)^{-1}(I - P K_2) \qquad (2.17)
\end{aligned}
$$

is the group of strongly stable controllers, where $K_0 \in \mathbb{S}_P$ is arbitrary. The corresponding inverse is given by

$$K^{\boxtimes_P^{-1}} = K_0 - (K - K_0)(I - PK)^{-1}(I - PK_0). \tag{2.18}$$

Opposed to the possible expectations, we not only have simple expressions for these operations in the Youla parameter space, but the formulae also resemble (2.11) and (2.12):

$$Q_K = Q_{K_2} \otimes Q_{K_1} = Q_{K_2} + Q_{K_1} + Q_{K_2} V^{-1} N Q_{K_1}, \tag{2.19}$$

$$Q_K^{\otimes^{-1}} = -Q(I + V^{-1}NQ)^{-1}. \tag{2.20}$$

It is important to note that while (2.19) keeps the strong stabilizability, as a property, invariant it does not guarantee that the property is fulfilled. This means that the formula also makes sense for parameters that does not correspond to stable controllers.

2.4.4 Example: State Feedback

In this section we provide some examples in order to illustrate the blending properties of these newly defined operators. To do this, let us consider first the state feedback case, i.e., fix the plant $P = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$ parametrized by its state space description and consider the stabilizing (state feedback) controllers given by:

$$K_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_1 \end{array}\right], \quad \text{and} \quad K_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_2 \end{array}\right],$$

respectively.

In order to applying operation (2.11) we need to compute the terms PK_2, K_1PK_2, $K_1 + K_2$, etc. To do this we apply the formulae (2.41), (2.42) and (2.43), respectively. Thus, we obtain first

$$PK_2 = \left[\begin{array}{cc|c} A & 0 & BF_2 \\ 0 & 0 & 0 \\ \hline I & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & BF_2 \\ \hline I & 0 \end{array}\right],$$

where the last equality is obtained by eliminating the uncontrollable and unobservable modes.

Analogously follows

$$K_1 P K_2 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & A & BF_2 \\ \hline 0 & F_1 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & BF_2 \\ \hline F_1 & 0 \end{array}\right],$$

and $K_1 + K_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_1 + F_2 \end{array}\right]$. Finally we obtain that the dynamic controller as

$$K = K_1 + K_2 - K_1 P K_2 = \left[\begin{array}{c|c} A & BF_2 \\ \hline -F_1 & F_1 + F_2 \end{array}\right].$$

In general this controller is not stable, however, it is stabilizing.

Indeed, one can assemble the closed loop system

$$\begin{aligned} \dot{x} &= Ax - BF_1 x_c + B(F_1 + F_2)x, \\ \dot{x}_c &= Ax_c + BF_2 x. \end{aligned}$$

Then, by applying the usual change of state variables $[x\ x - x_c]$, the corresponding closed loop matrix is

$$A_{cl} = \begin{pmatrix} A + BF_2 & BF_1 \\ 0 & A + BF_1 \end{pmatrix},$$

i.e., K is a stabilizing controller, as expected. Moreover, one can also observe the blending of the assigned spectrum.

Taking a stabilizing feedback $K_0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F_0 \end{array}\right]$ by (2.42) and (2.43) we have

$$(I - PK_0)^{-1} = \left[\begin{array}{c|c} A + BF_0 & BF_0 \\ \hline I & I \end{array}\right],$$

i.e., computing $K_0^{\boxminus_P}$ according to (2.12) gives

$$K_0^{\boxminus_P} = -K_0(I - PK_0)^{-1} = \left[\begin{array}{c|c} A + BF_0 & BF_0 \\ \hline -F_0 & -F_0 \end{array}\right].$$

This is a stable plant, as it was expected. However, it is not a stabilizing controller, in general: by taking the basis $[x\ x + x_c]$, the corresponding closed loop system can be expressed as $A_{cl} = \begin{pmatrix} A & BF_0 \\ 0 & A \end{pmatrix}$, which is not stable, in general.

It is obvious, that every static stabilizing state feedback controller is a strongly stabilizing one. Thus, by fixing a given stabilizing state feedback controller, say K_0, we are going to computing a blending controller according to (2.17). Proceeding as before, we have

$$(I - PK_0)^{-1}(I - PK_2) =$$
$$= \left[\begin{array}{cc|c} A + BF_0 & -BF_0 & BF_0 \\ 0 & A & BF_2 \\ \hline I & -I & I \end{array}\right] = \left[\begin{array}{c|c} A + BF_0 & B(F_0 - F_2) \\ \hline I & I \end{array}\right].$$

This leads to

$$K = K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2 = \left[\begin{array}{c|c} A + BF_0 & B(F_2 - F_0) \\ \hline -(F_1 - F_0) & F_1 + F_2 - F_0 \end{array}\right],$$

which is clearly stable. Note that the degree of the controller (n) is less than the expected one ($2n$). We can also verify, that this controller is stabilizing: the matrix of the closed loop system in the usual basis ($[x\ x - x_c]$) can be expressed as

$$A_{cl} = \begin{pmatrix} A + BF_2 & B(F_1 - F_0) \\ 0 & A + BF_1 \end{pmatrix},$$

i.e., K is a stabilizing controller, as expected. Again, we can also observe the blending of the assigned spectrum.

Taking a stabilizing feedback $K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right]$ and computing $K^{\boxtimes_P^{-1}}$ according to (2.18) leads to

$$K^{\boxtimes_P^{-1}} = \left[\begin{array}{c|c} A + BF & B(F - F_0) \\ \hline -(F - F_0) & -F + 2F_0 \end{array}\right],$$

which is clearly stable.

The corresponding closed loop matrix in the basis $[x\ x + x_c]$ can be written as

$$A_{cl} = \begin{pmatrix} A + BF_0 & -B(F - F_0) \\ 0 & A + BF_0 \end{pmatrix},$$

i.e., the closed loop system is stable, as we have already expected.

2.5 A Geometry Based Controller Parametrization

In what follows we fix a stabilizing controller, say K_0, and in the formulae we associate, according to (2.5), the corresponding sensitivities to this controller. Considering

$$\hat{\Sigma}_{P,K_0} = \begin{pmatrix} UV^{-1} & M - UV^{-1}N \\ V^{-1} & -V^{-1}N \end{pmatrix}$$

we obtain the lower LFT representation of the Youla parametrization, i.e.,

$$K = \mathfrak{M}_{\Sigma_{P,K_0}}(Q) = \mathfrak{F}_l(\hat{\Sigma}_{P,K_0}, Q), \tag{2.21}$$

see, e.g., [42]. Rearranging the terms one has

$$K = \mathfrak{F}_l(\Psi_{K_0,P}, R), \quad \text{with} \quad \Psi_{K_0,P} = \begin{pmatrix} K_0 & I \\ I & S_p \end{pmatrix} \tag{2.22}$$

and

$$R \in \mathbb{R}^{Y}_{K_0} = \{\, \tilde{V}^{-1} Q V^{-1} \mid Q \in \mathbb{Q} \,\}. \tag{2.23}$$

This fact was already observed for a while, e.g., [24] or [3], where it was used as a starting point for a Youla parametrization based gain scheduling scheme of rational LTI systems. We have recalled this result with the intention to demonstrate how our previous ideas on the geometry of stabilizing controllers can be applied in order to find significantly new information on an already known configuration.

2.5.1 A Coordinate Free Parametrization

In order to relate a Möbius transform to an LFT we prefer to use the formalism presented in [36]. Thus, recall that $\hat{\Sigma}_P$ is the Potapov–Ginsburg transform of Σ_P and formulae like (2.22) can be easily obtained by using the group property of the Möbius transform. Accordingly, we have that

$$K = \mathfrak{M}_{\Gamma_{P,K_0}}(R) = \mathfrak{F}_l(\Psi_{P,K_0}, R), \tag{2.24}$$

$$R = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}}(K) = \mathfrak{F}_l(\Phi_{P,K_0}, K), \tag{2.25}$$

where

$$\Gamma_{P,K_0} = \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix}, \quad \Psi_{P,K_0} = \hat{\Gamma}_{P,K_0}, \tag{2.26}$$

$$\Gamma^{-1}_{P,K_0} = \begin{pmatrix} I & -K_0 \\ S_p & S_y \end{pmatrix}, \quad \Phi_{P,K_0} = \begin{pmatrix} -K_0 S_y^{-1} & S_u^{-1} \\ S_y^{-1} & P \end{pmatrix}. \tag{2.27}$$

Observe that (2.25) is defined exactly on $\mathbb{W}_P$ and let the restriction on the stabilizing controllers be denoted by $\mathbb{R}_{K_0} = \{\mathfrak{F}_l(\Phi_{P,K_0}, K) \mid K \in \mathbb{G}_P\}$. Apparently, apart the structure of the set $\mathbb{R}^{Y}_{K_0}$ these formulae do not depend on any special factorization. Moreover, they can be also obtained directly, i.e., without any reference to some factorization of the plant or of the controller, starting from

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} (K - K_0) \begin{pmatrix} 0 & I \end{pmatrix}$$

and applying two times the matrix inversion lemma to obtain first

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix}^{-1} - \begin{pmatrix} S_u \\ S_p \end{pmatrix} R \begin{pmatrix} S_p & S_y \end{pmatrix},$$

with $R = (K - K_0)(I + S_p(K - K_0))^{-1}$ and then

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} R(I - S_pR)^{-1} \begin{pmatrix} 0 & I \end{pmatrix}. \tag{2.28}$$

Thus, it would be desirable to provide, if it exists, a coordinate free description of $\mathbb{R}_{K_0}$. Exactly this is the point where the geometric view and the coordinate free results of Sect. 2.4 can be applied.

As a starting point observe that

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix} = \tag{2.29}$$

$$= \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_pR)^{-1} \end{pmatrix}. \tag{2.30}$$

Analogous to (2.10) we have the factorization

$$\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} = \begin{pmatrix} S_u & 0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & S_u^{-1}K \\ 0 & I - PK \end{pmatrix}. \tag{2.31}$$

By using the notations

$$R_{(P,K_0)} = \begin{pmatrix} S_u & 0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix}$$

$$T_K^{(P,K_0)} = \begin{pmatrix} S_u & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & S_u^{-1}K \\ 0 & I - PK \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix}$$

we have

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = R_{(P,K_0)} T_K^{(P,K_0)}$$

and

$$T_{K_1}^{(P,K_0)} T_{K_2}^{(P,K_0)} = T_{K_1 \boxdot_P K_2}^{(P,K_0)},$$

moreover

$$K = \mathfrak{M}_{R_{(P,K_0)} T_K^{(P,K_0)} R_{(P,K_0)}^{-1}}(0_K) = \mathfrak{M}_{\Gamma_{P,K_0}}(R),$$

see (2.30) for the last equality. Thus, it is immediate that the operation (2.11) is a natural choice for this new configuration, too.

2.5.2 *Geometric Description of the Parameters*

Considering (2.15) and keeping in mind that $R = \tilde{V}^{-1} Q V^{-1}$ we have the blending rule on $\mathbb{R}_{K_0}^Y$:

$$R_2 \odot_{P,K_0} R_1 = K_0 + S_u R_1 + R_2 S_y - R_2 S_y S_p R_1. \tag{2.32}$$

For the stable controllers the parameter blending is more simple:

$$R_2 \otimes_{P,K_0} R_1 = R_2 + R_1 - R_2 S_p R_1, \tag{2.33}$$

$$R^{\otimes_{P,K_0}^{-1}} = -R(I - S_p R)^{-1}, \tag{2.34}$$

see (2.19) and (2.20).

Observe that $K_0 = \tilde{V}^{-1} \tilde{V} U V^{-1} \in \mathbb{R}_{K_0}^Y$ and that the corresponding controller is

$$K = K_0 \boxdot_P K_0 = [2K_0]_{\boxdot_P}.$$

Based on (2.32) it is easy to show that to the controller $K = [nK_0]_{\boxdot_P}$ corresponds the parameter $R = (I + \cdots + S_u^{n-1}) K_0 \in \mathbb{R}_{K_0}^Y$. Thus, if K_0 is stable, then all these parameters are stable. However, the corresponding controllers are not necessarily stable.

Theorem 2.3 *The algebraic structures defined by* (2.32) *and* (2.33) *holds also on* $\mathbb{R}_{K_0}$, *i.e., they can be introduced in a complete coordinate free way.*

Due to lack of space, we do not continue to deduce all the formulae, e.g., inverse, shifted blending, etc., for the parameters. Instead we show, in what follows, that the operation (2.32) can be obtained directly, without the Youla parametrization. To do so, observe that

$$I - PK = (I - PK_0)(I - S_p R_1)^{-1},$$

thus

$$\begin{pmatrix} I & S_u^{-1} K \\ 0 & I - PK \end{pmatrix} = \begin{pmatrix} I & S_u^{-1}(K_0 + S_u R) \\ 0 & S_o^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R)^{-1} \end{pmatrix}.$$

Then, according to (2.30) we have

$$\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} = \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & R_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R_1)^{-1} \end{pmatrix}$$
$$\begin{pmatrix} I & S_u^{-1}(K_0 + S_u R_2) \\ 0 & S_o^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R_2)^{-1} \end{pmatrix}. \tag{2.35}$$

Now, keeping in mind that

$$R = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}}(K) = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}\begin{pmatrix} I & K \\ P & I \end{pmatrix}}(0_K) = \mathfrak{M}_{\Gamma^{-1}_{P,K_0}\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix}}(0_K),$$

the assertion follows after evaluating (2.35).

We have already seen that $\{0, K_0\} \subset \mathbb{R}_{K_0}$. Moreover, we have seen that $\mathbb{Q} \subset \mathbb{R}^Y_{K_0}$ by an identification of $Q \to \tilde{V} Q V$, i.e., $R \to Q$. It turns out that this inclusion is also a coordinate free property, i.e., the inclusion holds regardless the existence of any coprime factorization.

Theorem 2.4 *The inclusion* $\mathbb{Q} \subset \mathbb{R}_{K_0}$ *holds.*

Indeed, by taking a controller $K \in \mathscr{K}_{K_0}$, where

$$\mathscr{K}_{K_0} = \{ K = \mathfrak{F}_l(\Psi_{K_0,P}, Q) = K_0 + Q(I - S_p Q)^{-1} \mid Q \in \mathbb{Q}\}, \tag{2.36}$$

after some standard computations, that are left out for brevity, we obtain

$$(I - PK)^{-1} = (I - S_p Q)(I - P K_0)^{-1} \tag{2.37}$$
$$(I - KP)^{-1} = (I - K_0 P)^{-1}(I - Q S_p) \tag{2.38}$$
$$(I - PK)P^{-1} = -(I - S_p Q) S_p \tag{2.39}$$
$$\begin{aligned} K(I - PK)^{-1} &= -S_c + (I - K_0 P)^{-1} Q (I - K_0 P)^{-1} - \\ &\quad - (I - K_0 P)^{-1} Q + Q. \end{aligned} \tag{2.40}$$

Thus $\mathscr{K}_{K_0} \subset \mathbb{G}_P$, as desired.

From a mathematical point of view, there is a small missing here. When a double coprime factorization exists, we should also prove that the set defined by (2.23), and the set defined by (2.28) are equal, i.e., $\mathbb{R}^Y_{K_0} = \mathbb{R}_{K_0}$. But this is equivalent to the fact that the Youla characterization of stabilizing controllers is exhaustive. This is a highly nontrivial issue and it is beyond the scope of this paper to address this topic in general. We should mention, however, that this property holds for discrete time systems, see, e.g., [12].

2.6 From Geometry to Control

As it was already pointed out in the introduction of this paper we have found very useful to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction. In the previous sections some examples were presented to illustrate this point. Now, it is time to demonstrate the way that starts from the abstract level and ends into a directly control relevant result.

The reader customised with system classes, like LTI, LPV (linear parameter varying), nonlinear, switching, etc. might find our presentation of the geometric ideas quite informal. We stress that this is a "feature" of the method. Recall that geometry—and also group theory—does not deal with the existence and the actual nature of the objects that are the primitives of the given geometry but rather captures the "rules" they obeys to. It gives the abstract structures that can be, for a given application, associated with actual objects, i.e., responds to the question "what can be done with these objects" rather than "how to synthesise the object having a given property (e.g., stability)".

We illustrate this fact by the example of the Youla parametrization. A basic knowledge is to place the topic in the context of finite rank LTI systems, i.e., those associated with rational transfer functions $\mathscr{R}$, and to interpret the result only in this context. However, we should not confine ourselves to this class: it is clear that an LTI plant can be also stabilized by more "complex" controllers, e.g., nonlinear ones, see, e.g., the IQC approach of [35]. This is also clear from the geometry: nothing prevents the Youla parameter to be any stable plant (not necessarily linear) in order to generate the stabilizing controller. Moreover, the nature of the parameter (e.g., nonlinear) is inherited by the controller through the Möbius transform.

We stress that the geometric picture behind the Youla parametrization has been applied under the hood even in the cases when the classes at hand do not have a sound input-output description, e.g., the class of switching systems or even the LPV systems. For the difficulties around these systems when we want to cast them exclusively into in input-output framework see, e.g., [7]. These difficulties does not prevent engineers to reduce the design of the switching controllers to switching between the corresponding values of the parameters, see, e.g., [3, 26, 37]. Moreover, the idea can be extended also for plants that are switching systems themselves, [7, 17], or LPV plants, [40].

Observe that in all these examples the authors spend a considerable amount of effort to solve the existential problem, i.e., how to obtain K_0. In all these cases this problem is cast in a state space framework and the taxonomy of the methods revolves around the type of the Lyapunov function (quadratic vs. polyhedral norm, constant Lyapunov matrix vs. parameter varying) involved that is used as stability certifier.

The motivation behind the increased complexity of the controller is that some additional performance demand is imposed either for the closed loop or for the controller, which cannot be fulfilled in the LTI setting. Concerning closed loop performances, the advantage of the Youla based approaches is that the performance transfer function is affine in the design parameter.

As an example consider the strong stabilizability problem. It is a standard knowledge that in $\mathscr{R}$ the problem does not always have a solution. However, it is less known that if one considers time variant (LTV) controllers, the problem is always solvable, see [21]. Moreover, for the discrete time case the problem is solvable in the disc algebra $\mathscr{A}$ or even in $\mathscr{H}_\infty$, see [28, 29].

To conclude this section we point out some additional properties of the parametrization presented in Sect. 2.5. As a consequence of (2.22) and (2.36), for every controller K_0 there is a stable perturbation ball Δ, contained in the image of the ball with radius $\frac{1}{\|S_p\|}$ under the map $x(1-x)^{-1}$, such that the pair $(P, K_0+\delta)$ is stable for all $\delta \in \Delta$. In particular, if the controller K_0 is strongly stabilizing, then all the controllers from $K_0 + \Delta$ are strongly stabilizing. This fact reveals the role of the sensitivity S_p in relation to the robustness of the stabilizing property of K_0. Due to the symmetry, analogous role is played by S_c for P.

This knowledge, together with (2.32) can be exploited to generate a hole branch of strongly stabilizing controllers starting from an initial one, K_0, with this property; e.g., one has to choose arbitrarily a stable R with a sufficiently small norm (less than $\frac{1}{\|S_p\|}$) and then apply (2.32) iteratively.

2.7 Conclusions

In this work we have shown that based on the direct blending operation the set $\mathbb{R}_{K_0}$ of stabilizing controllers can be defined and characterized in a completely coordinate free way, without any reference to a coprime factorization. For practical purposes it is also interesting to know that $K_0 \in \mathbb{R}_{K_0}$, moreover $\mathbb{Q} \subset \mathbb{R}_{K_0}$ holds as a coordinate free property, too. We emphasize, that a fairly large set of stabilizing controllers can be constructed (parametrized) just starting from the knowledge of a single stabilizing controller, without any additional knowledge (e.g., factorization). This underlines an important property of the geometric (and also input-output) view: describes the structure of the given set—in our case those of the stabilizing controllers – but does not provide a direct method to find any of the actual objects at hand. To do so, we need to ensure (e.g., by a construction algorithm) the existence at least of a single element with the given property.

Up to this point only projective geometric structures were considered. In order to qualify a given controller K as a stabilizing one (validation problem) metric aspects should be also considered, i.e., euclidean, hyperbolic, etc. geometries; e.g., concerning the Youla parametrization $Q_K \in \mathbb{Q}$, or in the geometric parametrization $R_K \in \mathbb{R}_{K_0}$, should be decided. It is subject of further research how these geometries find their way to control theory and vice versa.

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Appendix

A state space realization for the sum of systems is given by

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right], \tag{2.41}$$

while the product of the systems can be expressed as:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]. \tag{2.42}$$

Note that these realizations are not necessarily minimal. If D is invertible then a realization of the inverse system is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]. \tag{2.43}$$

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