

Domination of the rectangular queen's graph

Sándor Bozóki

Institute for Computer Science and
Control (SZTAKI), Budapest, Hungary;
Corvinus University of Budapest

bozoki.sandor@sztaki.hu

Péter Gál

Kofax-Recognita Zrt., Budapest, Hungary
galpetya@gmail.com

István Marosi

Kofax-Recognita Zrt., Budapest, Hungary
imarosi@gmail.com

William D. Weakley

Purdue University Fort Wayne, USA
weakley@pfw.edu

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Abstract

The queens graph $Q_{m \times n}$ has the squares of the $m \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column, or diagonal of the board. A set D of squares of $Q_{m \times n}$ is a *dominating set* for $Q_{m \times n}$ if every square of $Q_{m \times n}$ is either in D or adjacent to a square in D . The minimum size of a dominating set of $Q_{m \times n}$ is the *domination number*, denoted by $\gamma(Q_{m \times n})$.

Values of $\gamma(Q_{m \times n})$, $4 \leq m \leq n \leq 18$, are given here, in each case with a file of minimum dominating sets (often all of them, up to symmetry) in an online [appendix](#). In these ranges for m and n , monotonicity fails once: $\gamma(Q_{8 \times 11}) = 6 > 5 = \gamma(Q_{9 \times 11}) = \gamma(Q_{10 \times 11}) = \gamma(Q_{11 \times 11})$.

Let $g(m)$ [respectively $g^*(m)$] be the largest integer such that m queens suffice to dominate the $(m+1) \times g(m)$ board [respectively, to dominate the $(m+1) \times g^*(m)$ board with no two queens in a row]. Starting from the elementary bound $g(m) \leq 3m$, domination when the board is far from square is investigated. It is shown (Theorem 2) that $g(m) = 3m$ can only occur when $m \equiv 0, 1, 2, 3, \text{ or } 4 \pmod{9}$, with the online appendix showing that this does occur for $m \leq 40, m \neq 3$. Also (Theorem 4), if $m \equiv 5, 6, \text{ or } 7 \pmod{9}$ then $g^*(m) \leq 3m - 2$, and if $m \equiv 8 \pmod{9}$ then $g^*(m) \leq 3m - 4$. It is shown that equality holds in these bounds for $m \leq 40$.

Lower bounds on $\gamma(Q_{m \times n})$ are given. In particular, if $m \leq n$ then $\gamma(Q_{m \times n}) \geq \min\{m, \lceil (m+n-2)/4 \rceil\}$.

Two types of dominating sets (orthodox covers and centrally strong sets) are developed; each type is shown to give good upper bounds of $\gamma(Q_{m \times n})$ in several cases.

Three questions are posed: whether monotonicity of $\gamma(Q_{m \times n})$ holds (other than from $(m, n) = (8, 11)$ to $(9, 11)$), whether $\gamma(Q_{m \times n}) = (m+n-2)/4$ occurs with

$m \leq n < 3m + 2$ (other than for $(m, n) = (3, 3)$ and $(11, 11)$), and whether the lower bound given above can be improved.

A set of squares is *independent* if no two of its squares are adjacent. The minimum size of an independent dominating set of $Q_{m \times n}$ is the *independent domination number*, denoted by $i(Q_{m \times n})$. Values of $i(Q_{m \times n})$, $4 \leq m \leq n \leq 18$, are given here, in each case with some minimum dominating sets. In these ranges for m and n , monotonicity fails twice: $i(Q_{8 \times 11}) = 6 > 5 = i(Q_{9 \times 11}) = i(Q_{10 \times 11}) = i(Q_{11 \times 11})$, and $i(Q_{11 \times 18}) = 9 > 8 = i(Q_{12 \times 18})$.

Mathematics Subject Classifications: 05C69, 05C99

1 Introduction

Let m and n be positive integers. We will identify the $m \times n$ chessboard with a rectangle in the Cartesian plane, having sides parallel to the coordinate axes. We place the board so that the center of every square has integer coordinates, and refer to each square by the coordinates (x, y) of its center. Unless otherwise noted, squares have edge length one, and the board is placed so that the lower left corner has center $(1, 1)$; sometimes it is more convenient to use squares of edge length two or to place the board with its center at the origin of the coordinate system. By symmetry it suffices to consider the case $m \leq n$, which we will assume throughout: the board has at least as many columns as rows.

The square (x, y) is in *column* x and *row* y . Columns and rows will be referred to collectively as *orthogonals*. The *difference diagonal* (respectively *sum diagonal*) through square (x, y) is the set of all board squares with centers on the line of slope $+1$ (respectively -1) through the point (x, y) . The value of $y - x$ is the same for each square (x, y) on a difference diagonal, and we will refer to the diagonal by this value. Similarly, the value of $y + x$ is the same for each square on a sum diagonal, and we associate this value to the diagonal. Orthogonals and diagonals are collectively referred to as *lines* of the board.

The queens graph $Q_{m \times n}$ has the squares of the $m \times n$ chessboard as its vertices; two squares are adjacent if they are in some line of $Q_{m \times n}$. A set D of squares of $Q_{m \times n}$ is a *dominating set* for $Q_{m \times n}$ if every square of $Q_{m \times n}$ is either in D or adjacent to a square in D . The minimum size of a dominating set is the *domination number*, denoted by $\gamma(Q_{m \times n})$. A set of squares is *independent* if no two squares in the set are adjacent.

Almost all previous work on queen domination has concerned square boards. The problem of finding values of $\gamma(Q_{n \times n})$ has interested mathematicians for over 150 years. The first published work is that of De Jaenisch [13] in 1862, who gave minimum dominating sets and minimum independent dominating sets of $Q_{n \times n}$ for $n \leq 8$. His work was briefly summarized by Rouse Ball [19] in 1892, who considered several other questions about queen domination. In 1901, W. Ahrens [1, Chapter X] gave minimum dominating sets for $Q_{9 \times 9}$, and in 1902-3, K. von Szily [20, 21] gave minimum dominating sets of $Q_{n \times n}$ for $10 \leq n \leq 13$ and $n = 17$. Proof that these sets were minimum had to wait for later work, described below. De Jaenisch, Ahrens, and von Szily also worked extensively to find the number of different minimum dominating sets for each n , often giving lists with one set

from each symmetry class. Many of these results were collected by Ahrens in the 1910 edition [2] of his book, and can also be found in its later editions.

More detail and some examples from recent work on domination of $Q_{n \times n}$ can be found in [26].

The first published work on nonsquare boards of which we are aware is in Watkins [22]: the values $\gamma(Q_{5 \times 12}) = 4$ and $\gamma(Q_{6 \times 10}) = 4$ (see Problem 8.4 on p. 132 and Figure 8.19 on p. 137), found by D. C. Fisher.

Say that two minimum dominating sets of $\gamma(Q_{m \times n})$ are *equivalent* if there is an isometry of the $m \times n$ chessboard that carries one to the other.

We have computed $\gamma(Q_{m \times n})$ for rectangular chessboards with $4 \leq m \leq n \leq 18$. Results are given in Table 1; for most m and n we give a file of minimum dominating sets with one from every equivalence class, unless the number of equivalence classes is large. The online appendix at <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v26i4p45/HTML> includes the computational results. For each set, we describe its symmetry and say whether it can be obtained by one of the constructions in Section 3.

$n \setminus m$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
4	2														
5	2	3													
6	3	3	3												
7	3	3	4	4											
8	3	4	4	5	5										
9	4	4	4	5	5	5									
10	4	4	4	5	5	5	5								
11	4	4	5	5	6	5	5	5							
12	4	4	5	5	6	6	6	6	6						
13	4	5	5	6	6	6	7	7	7	7					
14	4	5	6	6	6	6	7	7	8	8	8				
15	4	5	6	6	6	7	7	7	8	8	8	9			
16	4	5	6	6	7	7	7	8	8	8	9	9	9		
17	4	5	6	7	7	7	8	8	8	9	9	9	9	9	
18	4	5	6	7	7	8	8	8	8	9	9	9	9	9	9

Table 1: Values of $\gamma(Q_{m \times n})$, $4 \leq m \leq n \leq 18$ (OEIS A274138)

The computation was done with a backtracking algorithm. The backtrack condition minimizes the number of queens placed. If a solution is found with k queens, then the remaining search space is limited to at most $k - 1$ queens. The algorithm places a single queen in a position covering the top left cell and does a recursive call to cover all remaining cells. Some heuristics are used also to find the first solution faster: the first queen is placed in the middle of the board (actually in the closest to middle position attacking the top

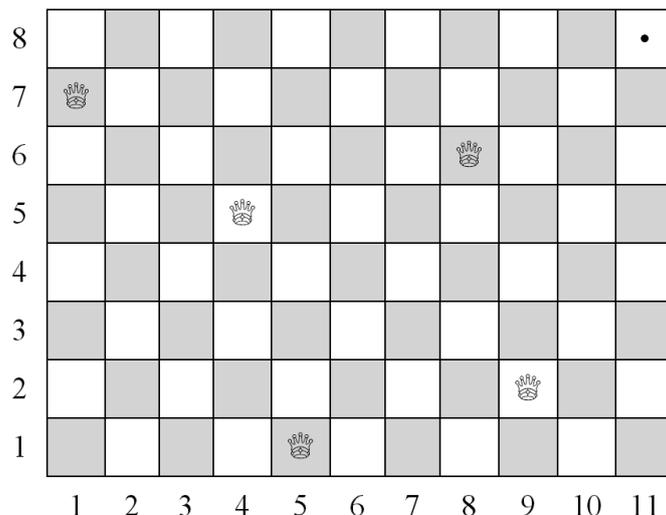


Figure 1: Five queens dominate $Q_{8 \times 11}$ except for one square (●)

left cell); other possible attacking positions are only tried later. Frequently this position is part of a minimal solution.

Once it is shown that there is no solution with $k - 1$ queens, a search for other solutions with k queens is made.

Cockayne [7, Problem 1] introduced monotonicity

$$\gamma(Q_{n \times n}) \stackrel{?}{\leq} \gamma(Q_{(n+1) \times (n+1)})$$

as an open problem (see also in Chartrand, Haynes, Henning and Zhang [6, Conjecture 1.2.1 on page 7]).

A remarkable observation about $Q_{8 \times 11}$: six queens (with bold typeface in Table 1) are necessary to dominate it, though five queens are sufficient (and necessary) to dominate each of $Q_{9 \times 11}$, $Q_{10 \times 11}$, $Q_{11 \times 11}$. A possible explanation for this is given later. We note that five queens can cover all but one square of $Q_{8 \times 11}$. One of the 8 placements is in Figure 1.

We extend Cockayne's question to the rectangular case.

Question 1. *Column-wise monotonicity: Does $\gamma(Q_{m \times n}) \leq \gamma(Q_{m \times (n+1)})$ hold for $m \leq n$? Row-wise monotonicity: Does $\gamma(Q_{m \times n}) \leq \gamma(Q_{(m+1) \times n})$ hold for $m \leq n$, $(m, n) \neq (8, 11)$?*

We discuss one type of internal symmetry of minimum dominating sets that frequently occurs. A *foursome* is a set of four squares $(x + a, y + b)$, $(x - a, y - b)$, $(x - b, y + a)$, $(x + b, y - a)$, where either each of x, y, a, b is an integer or each is half an odd integer, and a and b are unequal and nonzero. The *center* of the foursome is the point (x, y) , which need not be a square center. For examples, see Figure 1 above, the first minimum dominating sets given for $Q_{9 \times 9}$ and $Q_{11 \times 11}$, as well as the first four minimum dominating sets given for $Q_{11 \times 12}$.

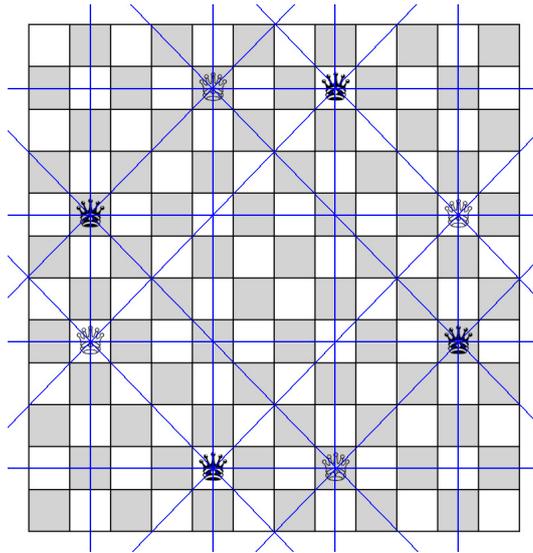


Figure 2: The four squares with white queens are a foursome; the lines through its squares are shown. Reflecting this foursome across any of the four lines through its center gives another foursome (the squares with black queens) that occupies the same lines.

If a foursome F is flipped across any of the four lines through its center, the result is another foursome F' that occupies the same lines as F ; this is illustrated in Figure 2.

Thus if a dominating set D of $Q_{m \times n}$ contains F , we may replace F in D with F' and obtain a dominating set D' of the same size as D . Usually D and D' are not equivalent.

As an example, we analyze the minimum dominating sets of $Q_{11 \times 17}$, which have size 8. Up to equivalence there are 131 solutions, shown in the [file](#). Of these, 85 have no foursomes, 41 have exactly one foursome, four ([#125](#), [69](#), [70](#), [62](#)) have exactly two foursomes, and one ([#76](#)) has 3 foursomes.

We may define a relation on the set $\mathcal{S}(11, 17)$ of minimum dominating sets of $Q_{11 \times 17}$ by saying that two sets are related if either they are equivalent, or flipping a foursome of the first set yields a set equivalent to the second set. This relation is reflexive and symmetric, and its transitive closure gives a partition $\mathcal{P}(11, 17)$ of $\mathcal{S}(11, 17)$, which may also be regarded as a partition of the set of (isometric) equivalence classes, as we will do.

For example, solution [#125](#) has two foursomes: one centered at $(12, 6)$ with $(a, b) = (4, 2)$, and one centered at $(9, 7)$ with $(a, b) = (3, -1)$. Flipping the first gives solution [#124](#). If instead we flip the second, we get the reflection of [#125](#) across its vertical line of symmetry. This implies that one part of the partition $\mathcal{P}(11, 17)$ contains just the equivalence classes of [125](#) and [124](#), and we denote this part by $\{125, 124\}$. It is then straightforward to see that $\mathcal{P}(11, 17)$ has 85 parts with one member and 20 parts with two members: $\{8, 9\}$, $\{20, 24\}$, $\{19, 23\}$, $\{21, 25\}$, $\{22, 26\}$, $\{15, 27\}$, $\{13, 6\}$, $\{12, 5\}$, $\{128, 129\}$, $\{73, 68\}$, $\{127, 130\}$, $\{126, 131\}$, $\{103, 97\}$, $\{125, 124\}$, $\{39, 44\}$, $\{96, 95\}$, $\{63, 62\}$, $\{72, 70\}$, $\{101, 77\}$, $\{100, 78\}$. There are also two parts with three members:

{80, 71, 69} and {79, 76, 75}.

It would be possible to reduce the size of the [appendix](#) by giving for each (m, n) one solution from each part of the partition $\mathcal{P}(m, n)$ rather than one solution from each isometry equivalence class. But when two solutions differ by the flip of a foursome, it is not clear which is most useful to see, so we have not done this.

2 Lower bounds on queen domination numbers

We begin by looking at what happens when the board is far from square.

Proposition 1. *If $n \geq 3m - 2$, then $\gamma(Q_{m \times n}) = m$.*

Proof. Each queen attacks all squares in her own row, but at most three squares in any other row. Thus $m - 1$ queens occupy at most $m - 1$ rows and cover at most $3(m - 1)$ squares in any row that does not contain a queen. On the other hand, m queens are certainly sufficient. \square

Note that $\gamma(Q_{3 \times 6}) = 2$ (see the set given immediately after Theorem 2) and $\gamma(Q_{5 \times 12}) = 4$ (see the database), but as shown by our computations, for $m = 4, 6, 7$, $\gamma(Q_{m \times n})$ reaches m before n reaches $3m - 2$.

We change viewpoint slightly, focusing on the size of the dominating set rather than the dimensions of the board. For each positive integer m , let $g(m)$ be the largest integer such that m queens can cover the $(m + 1) \times g(m)$ board. (Proposition 1 asserts $g(m) \leq 3m$.) Let $g^*(m)$ be the largest integer such that m queens, no two in a row, can cover the $(m + 1) \times g^*(m)$ board.

Theorem 2. *If $g(m) = 3m$ then $m \equiv 0, 1, 2, 3$, or $4 \pmod{9}$.*

Take the board to be a rectangle in the Cartesian plane with sides parallel to the axes, board center at the origin, and squares of edge length two.

Assume that D is a dominating set of size m for the $(m + 1) \times 3m$ board. One of two cases occurs:

- 1. there is only one empty row, which without loss of generality has index h , $0 \leq h \leq m$, and each other row contains exactly one square of D , or*
- 2. there are exactly two empty rows; in this case m is even, row 0 contains two squares of D , the empty rows are indexed $\pm h$ for some h , $0 < h \leq m$, and each other row contains exactly one square of D .*

In either case:

- if $m \equiv 0$ or $4 \pmod{9}$, then $h \equiv 0 \pmod{3}$;*
- if $m \equiv 1$ or $3 \pmod{9}$, then $h \not\equiv 0 \pmod{3}$.*

Proof. The $3m$ column indices are $x = 1 - 3m, 3 - 3m, \dots, 3m - 3, 3m - 1$ and the $m + 1$ row indices are $y = -m, 2 - m, \dots, m - 2, m$.

Let $D = \{(x_i, y_i) : 1 \leq i \leq m\}$ be a dominating set of the $(m + 1) \times 3m$ -board. Since $|D| = m$ and there are $m + 1$ rows, at least one row contains no square of D . Let h be the index of such a row. Since $|D| = m$, and a queen covers at most three squares of any line she does not occupy, the $3m$ squares of row h are each covered exactly once by D . For each $(x, y) \in D$, the squares of row h covered by (x, y) are $(x - (y - h), h), (x, h), (x + (y - h), h)$. We can add up the squares of the column indices of the squares in row h in two ways, giving the equation

$$(1 - 3m)^2 + (3 - 3m)^2 + \dots + (3m - 1)^2 = \sum_{i=1}^m [(x_i - (y_i - h))^2 + x_i^2 + (x_i + (y_i - h))^2].$$

This reduces to

$$2 \binom{3m+1}{3} = 3 \sum_{i=1}^m x_i^2 + 2 \sum_{i=1}^m y_i^2 - 4h \sum_{i=1}^m y_i + 2mh^2. \quad (1)$$

For a particular dominating set D we may regard this as a quadratic equation in h , so there are at most two empty rows.

If there are two empty rows h_1, h_2 , then (1) implies $2mh_1^2 - 4h_1 \sum_{i=1}^m y_i = 2mh_2^2 - 4h_2 \sum_{i=1}^m y_i$ and then

$$\sum_{i=1}^m y_i = \frac{m(h_1 + h_2)}{2}. \quad (2)$$

As $|D| = m$, there is exactly one row, say l , with two queens, and all rows except h_1, h_2, l have just one queen. Thus $\sum_{i=1}^m y_i = -h_1 - h_2 + l$. With (2) this implies $-h_1 - h_2 + l = m(h_1 + h_2)/2$ and then $l = (m + 2)(h_1 + h_2)/2$. From $-m \leq l \leq m$ we have $-2m \leq (m + 2)(h_1 + h_2) \leq 2m$ so $-2 < h_1 + h_2 < 2$. But $h_1 + h_2$ is even since all row indices have the same parity. Thus $h_1 + h_2 = 0$, so $l = 0$ and all row indices are even, which implies m is even. So there is h , $0 < h \leq m$, such that the empty rows are $\pm h$. Here $\sum_{i=1}^m y_i = 0$ and $\sum_{i=1}^m y_i^2 = 2 \binom{m+2}{3} - 2h^2$, and (1) becomes

$$2 \binom{3m+1}{3} - 4 \binom{m+2}{3} = 3 \sum_{i=1}^m x_i^2 + 2(m-2)h^2. \quad (3)$$

If instead there is only one empty row h , then we may assume $0 \leq h \leq m$ by flipping across the x -axis if necessary. Then $\sum_{i=1}^m y_i = -h$ and $\sum_{i=1}^m y_i^2 = 2 \binom{m+2}{3} - h^2$, and (1) gives

$$2 \binom{3m+1}{3} - 4 \binom{m+2}{3} = 3 \sum_{i=1}^m x_i^2 + 2(m+1)h^2. \quad (4)$$

The left sides of (3) and (4) reduce to $m(25m^2 - 6m - 7)/3$. Multiplying either of (3) and (4) by 3 and reducing modulo 9 gives the congruence

$$m(25m^2 - 6m - 7) \equiv -3(m-2)h^2 \pmod{9}. \quad (5)$$

For $m \equiv 5, 6, 7$ or $8 \pmod{9}$, (5) leads to $h^2 \equiv -1 \pmod{3}$ or another impossibility. For $m \equiv 0$ or $4 \pmod{9}$, (5) implies $h^2 \equiv 0 \pmod{3}$ and thus $h \equiv 0 \pmod{3}$. For $m \equiv 1$ or $3 \pmod{9}$, (5) gives $h^2 \equiv 1 \pmod{3}$, so $h \equiv \pm 1 \pmod{3}$. (For $m \equiv 2 \pmod{9}$, (5) is satisfied for any h .) \square

Only one example of the second case of Theorem 2 is known: for $m = 2$, the set $D = \{(-3, 0), (3, 0)\}$ covers the 3×6 board. A computer search shows that there is no other example with $m \leq 40$. It seems likely that no other example exists; if for some m there is such a set $D = \{(x_i, y_i) : 1 \leq i \leq m\}$, we have been able to show that $0 = \sum x_i = \sum x_i y_i = \sum x_i^2 y_i = \sum (x_i^3 y_i + 2x_i y_i^2)$.

By computer search (see the [file](#)) we have shown that for all m such that $m \leq 40$, $m \neq 3$, and $m \equiv 0, 1, 2, 3$, or $4 \pmod{9}$, there exist sets of m queen squares dominating $Q_{(m+1) \times 3m}$, with h taking all values not ruled out by Theorem 2. We believe $g(m) = g^*(m) = 3m$ holds for all $m \neq 3$ with $m \equiv 0, 1, 2, 3$, or $4 \pmod{9}$.

The following is immediate from Theorem 2.

Corollary 3. *If $m \equiv 5, 6, 7$, or $8 \pmod{9}$ then $\gamma(Q_{(m+1) \times 3m}) = m + 1$.*

We now examine the cases $m \equiv 5, 6, 7, 8 \pmod{9}$.

Theorem 4. *If $m \equiv 5, 6$, or $7 \pmod{9}$ then $g^*(m) \leq 3m - 2$. If $m \equiv 8 \pmod{9}$ then $g^*(m) \leq 3m - 4$.*

Take the board to be a rectangle in the Cartesian plane with sides parallel to the axes, board center at the origin, and squares of edge length two.

For $m \equiv 6$ or $7 \pmod{9}$, assume that D is a set of m squares that dominates the $(m + 1) \times (3m - 2)$ board, occupying all but row h . If $m \equiv 6 \pmod{9}$ then $h \not\equiv 0 \pmod{3}$. If $m \equiv 7 \pmod{9}$ then $h \equiv 0 \pmod{3}$.

Proof. Let m be a positive integer and j an integer with $0 \leq j \leq 4$. Let D be a dominating set of m queens for the $(m + 1) \times (3m - j)$ -board, with only row h empty.

The $3m - j$ column indices form the set $S_{\text{col}} = \{j + 1 - 3m, j + 3 - 3m, \dots, 3m - j - 3, 3m - j - 1\}$ and the $m + 1$ row indices form the set $S_{\text{row}} = \{-m, 2 - m, \dots, m - 2, m\}$.

It will be useful to consider row h extended beyond the board, and to look at the congruence classes modulo 3 of the column indices. To this end, for each integer i , let $C'_i = \{x \in \mathbb{Z} : x \equiv i \pmod{3}\}$. The restriction of C'_i to column indices of the $(m + 1) \times (3m - j)$ -board is $C_i = C'_i \cap S_{\text{col}}$. We write c_i for the size of C_i . By symmetry, $C_{-1} = \{-x : x \in C_1\} = -C_1$ so $c_{-1} = c_1$.

For each integer i let $R_i = \{y \in S_{\text{row}} : y \equiv i \pmod{3}\}$. Write r_i for the size of R_i . By symmetry, $R_{-1} = -R_1$ so $r_{-1} = r_1$.

Lemma 5. *Choose $s, b \in \{-1, 0, 1\}$ such that $s \equiv j$ and $b \equiv k + 1 \pmod{3}$. Then $c_{-1} = c_1 = c_0 + s$ and $r_{-1} = r_1 = r_0 - b$.*

Proof. The facts that $c_{-1} = c_1$, $c_{-1} + c_0 + c_1 = 3m - j$ and the c_i 's differ by at most one imply the first equation, and the second is similar. \square

Let p be the number of squares (x, y) of D with $y \not\equiv h \pmod{3}$. For each $i \in \{-1, 0, 1\}$, let t_i be the number of squares (x, y) of D such that $x \equiv i \pmod{3}$ and $y \equiv h \pmod{3}$. Let $t = t_{-1} + t_0 + t_1$. Thus $|D| = p + t_{-1} + t_0 + t_1 = p + t$.

Let (x, y) be a square of D . The squares in the extension of row h covered by (x, y) are $(x - (y - h), h), (x, h), (x + (y - h), h)$, some of which may be off the $(m + 1) \times (3m - j)$ -board. Their x -coordinates $x - (y - h), x, x + (y - h)$ are an arithmetic progression with difference $y - h$. If $y \equiv h \pmod{3}$ then all of $x - (y - h), x, x + (y - h)$ are in C'_x . If $y \not\equiv h \pmod{3}$ then the three values $x - (y - h), x, x + (y - h)$ are different modulo 3, so they contribute one member to each of C'_{-1}, C'_0, C'_1 . Thus for each $i \in \{-1, 0, 1\}$, $p + 3t_i$ is the number of covers (with multiplicity) of squares (x, h) with $x \in C'_i$.

For each $i \in \{-1, 0, 1\}$ let a_i be the number of "wasted covers" by D of squares (x, h) with $x \in C'_i$. That is, a_i counts every cover of any square (x, h) that is off the board ($|x| > 3m - j - 1$) and all but one cover of each multiply covered square (x, h) on the board. Thus each a_i is nonnegative, and since each of the m squares of D covers 3 squares of the extended row h , $a_{-1} + a_0 + a_1 = j$.

Since each square in row h of the board is covered by D , we get a system of equations:

$$p + 3t_{-1} = c_{-1} + a_{-1} \tag{6}$$

$$p + 3t_0 = c_0 + a_0 \tag{7}$$

$$p + 3t_1 = c_1 + a_1 \tag{8}$$

$$a_{-1} + a_0 + a_1 = j. \tag{9}$$

As $c_{-1} = c_1$, subtracting (6) from (8) shows

$$a_{-1} \equiv a_1 \pmod{3}. \tag{10}$$

For a dominating set D as hypothesized to exist, it is necessary that the total number t of squares (x, y) in D with $y \equiv h \pmod{3}$ is one less than the number r_h of rows in R_h . When $t = r_h - 1$, we will say h is *eligible* for t .

Theorem 2 covers the case $j = 0$, so we pass to less wide boards, only considering $m \equiv 5, 6, 7, 8 \pmod{9}$.

Let $j = 1$. From (10) we have $a_{-1} = a_1 = 0$ and then $a_0 = 1$. Here the s of Lemma 5 is 1 so $c_{-1} = c_1 = m, c_0 = m - 1$. Then equations (6-8) imply $t_{-1} = t_0 = t_1$, so $t \equiv 0 \pmod{3}$, and the following analysis shows that no h is eligible for any t for $m \equiv 5, 6, 7, 8 \pmod{9}$.

If $m \equiv 5 \pmod{9}$ then $r_{-1} = r_1 = r_0 = (m + 1)/3 \equiv -1 \pmod{3}$.

If $m \equiv 6 \pmod{9}$ then $r_{-1} = r_1 = m/3 \equiv -1 \pmod{3}, r_0 = (m/3) + 1 \equiv 0 \pmod{3}$.

If $m \equiv 7 \pmod{9}$ then $r_{-1} = r_1 = (m + 2)/3 \equiv 0 \pmod{3}, r_0 = (m - 1)/3 \equiv -1 \pmod{3}$.

If $m \equiv 8 \pmod{9}$ then $r_{-1} = r_1 = r_0 = (m + 1)/3 \equiv 0 \pmod{3}$.

Let $j = 2$. Since (9) here implies all $a_i \leq 2$, (10) gives $a_{-1} = a_1$, and then (6-8) imply $t_{-1} = t_1$. The s of Lemma 5 is -1 so $c_{-1} = c_1 = m, c_0 = m + 1$. There are two possibilities:

$(a_{-1}, a_0, a_1) = (0, 2, 0)$, when equations (6-8) imply $t_0 = t_1 + 1$, so $t \equiv 1 \pmod{3}$, or

$(a_{-1}, a_0, a_1) = (1, 0, 1)$, when equations (6-8) imply $t_0 = t_1$, so $t \equiv 0 \pmod{3}$.

Then examining the values of the r_i 's found above for $m \equiv 5, 6, 7, 8 \pmod{9}$, we see:
 For $m \equiv 5 \pmod{9}$, all h are eligible for $t \equiv 1 \pmod{3}$ and none for $t \equiv 0 \pmod{3}$;
 For $m \equiv 6 \pmod{9}$, $h \not\equiv 0 \pmod{3}$ are eligible for $t \equiv 1 \pmod{3}$ and none for $t \equiv 0 \pmod{3}$;
 For $m \equiv 7 \pmod{9}$, $h \equiv 0 \pmod{3}$ is eligible for $t \equiv 1 \pmod{3}$ and none for $t \equiv 0 \pmod{3}$;
 For $m \equiv 8 \pmod{9}$, no h is eligible for either t .

We continue with $m \equiv 8 \pmod{9}$.

Let $j = 3$. Here $c_{-1} = c_0 = c_1 = m - 1$ so from equations (6-8) we see all a_i 's are congruent modulo 3. Either $(a_{-1}, a_0, a_1) = (1, 1, 1)$, and then $t_{-1} = t_0 = t_1$, so $t \equiv 0 \pmod{3}$, or one of the a_i 's is 3 and the other two are zero, which gives $t \equiv 1 \pmod{3}$. For $m \equiv 8 \pmod{9}$, neither of these gives an eligible h , as before.

Let $j = 4$. Here there are more possibilities for (a_{-1}, a_0, a_1) , but the only helpful one for $m \equiv 8 \pmod{9}$ is $(2, 0, 2)$, which gives $t \equiv -1 \pmod{3}$, with all h eligible. \square

Computer search reveals (see the [file](#)) that for $m \leq 40$ and $m \equiv 5, 6, 7 \pmod{9}$, all minimum dominating sets of $Q_{m+1 \times 3m-2}$ have just one empty row, and all eligible values of h actually occur. For $m \equiv 8 \pmod{9}$, Theorem 4 does not say any h are ineligible; indeed, our search has found solutions with one empty row for all the h values. The only board size in the $3 \leq m \leq 40$ range where the minimum dominating sets found have two empty rows is $Q_{9 \times 20}$. Those dominating sets demonstrate numerous patterns of pairs of empty rows, as shown in this [file](#).

We next show that by “pasting together” dominating sets of a certain type, we can extend the range of values for which the bounds of Theorem 4 are known to be exact.

Say that a *topless* dominating set for the $(m + 1) \times n$ board is a dominating set of size m having one square in each row except the top row, which is empty.

Proposition 6.

- (A) *Suppose that for $i = 1, 2$ there is a topless dominating set of m_i queens for the $(m_i + 1) \times n_i$ board. Then there is a dominating set of $m_1 + m_2$ queens for the $(m_1 + m_2 + 1) \times (n_1 + n_2)$ board.*
- (B) *Let k be a positive integer. Suppose that for each l , $1 \leq l \leq k$, and for $i = 0, 1, 2$, there exist topless dominating sets of size $9l + i$ for the $(9l + i + 1) \times (27l + 3i)$ board. Then for each $m \neq 3$, $1 \leq m \leq 9k + 8$, there is a dominating set of size m for the $(m + 1) \times (3m - j)$ board, where $j = 0$ if $m \equiv 0, 1, 2, 3$, or $4 \pmod{9}$, $j = 2$ if $m \equiv 5, 6$, or $7 \pmod{9}$, and $j = 4$ if $m \equiv 8 \pmod{9}$.*

Proof. We return to our usual scheme of indexing columns and rows from the bottom left board corner. For (A): For $i = 1, 2$, let S_i be a topless dominating set of m_i queens for the $(m_i + 1) \times n_i$ board. On the $(m_1 + m_2 + 1) \times (n_1 + n_2)$ board, the squares in which the columns indexed 1 to n_1 and the rows indexed 1 to $m_1 + 1$ meet form a copy of the $(m_1 + 1) \times n_1$ board. Place a copy S'_1 of S_1 on that. The squares in which the columns

indexed $n_1 + 1$ to $n_1 + n_2$ and the rows indexed $m_1 + 1$ to $m_1 + m_2 + 1$ meet form a copy of the $(m_2 + 1) \times n_2$ board. Place a copy S'_2 of S_2 , rotated by a half turn, on that. The union of the two copies is a set $S_1 \oplus S_2$ of $m_1 + m_2$ squares that leaves only row $m_2 + 1$ empty on the $(m_1 + m_2 + 1) \times (n_1 + n_2)$ board. As S'_1 covers the left n_1 squares of that row and S'_2 covers the remainder, $S_1 \oplus S_2$ dominates the $(m_1 + m_2 + 1) \times (n_1 + n_2)$ board.

For (B): We will use topless dominating sets of the $(m + 1) \times (3m - j)$ board for $(m, j) = (1, 0), (5, 2), (8, 4)$, and $(11, 0)$. Assume the hypotheses; we then need to prove the existence of (minimum) dominating sets of size m for the $(m + 1) \times (3m - j)$ boards in the ranges claimed. For $m < 18$ the database (see the files [here](#) and [here](#)) contain the claimed dominating sets. For other board sizes of $m = 9l, 9l + 1, 9l + 2, 2 \leq l \leq k$, we are assuming there are such sets (in fact, topless). For $m = 9l + 3$, we use part (A) to “paste together” topless sets for $m = 9l + 2$ and $m = 1$. For $m = 9l + 4$, we paste together topless sets for $m = 9(l - 1) + 2$ and $m = 11$. Pasting together a topless set for each of $m = 9l, 9l + 1, 9l + 2$ with a topless set for $m = 5$ gives the result for $m = 9l + 5, 9l + 6, 9l + 7$. Finally, pasting together topless sets for $m = 9l$ and $m = 8$ gives the result for $m = 9l + 8$. \square

As mentioned after Theorem 2, computer search gave topless dominating sets as required in Proposition 6(B) for $k = 4$. It follows that $g^*(m)$ equals the bound of Theorem 4 for $m \leq 44, m \neq 3$.

We next develop a lower bound for $\gamma(Q_{m \times n})$ for more general m, n .

Raghavan and Venkatesan [18] and Spencer [7, 23] independently proved that

$$\gamma(Q_{n \times n}) \geq \left\lceil \frac{n - 1}{2} \right\rceil. \quad (11)$$

It has been shown [10] that $\gamma(Q_{n \times n}) = (n - 1)/2$ only for $n = 3, 11$. Both of these values are significant for our work here, as we now discuss.

A central queen on $Q_{3 \times 3}$ shows $\gamma(Q_{3 \times 3}) = 1$. This simple fact has a useful generalization: if C is a central sub-board of $Q_{m \times n}$ such that every square of $Q_{m \times n}$ has a line meeting C , then a subset of C that occupies all those lines is a dominating set of $Q_{m \times n}$. More than a hundred years ago, Szily [20, 21] gave dominating sets of this type for $Q_{13 \times 13}$ and $Q_{17 \times 17}$, which were later shown to be minimum. We found that $Q_{13 \times 16}$ has a minimum dominating set (solution #23 in the database) of this *centrally strong* form and have also used this idea to produce good upper bounds of $Q_{m \times n}$ for some m, n , as shown below.

It follows from [24] that there are exactly two minimum dominating sets for $Q_{11 \times 11}$. Placing the origin of our coordinate system at board center, these sets are $\mathcal{D} = \{(0, 0), \pm(2, 4), \pm(4, -2)\}$ (see Figure 3) and the reflection of \mathcal{D} across the column $x = 0$. So up to equivalence \mathcal{D} is the unique minimum dominating set of $Q_{11 \times 11}$, consisting of a foursome and a queen at its center. This amazing set has an influence on many other values of $\gamma(Q_{m \times n})$.

First, since \mathcal{D} fits on $Q_{9 \times 9}$, by omitting edge rows and columns of $Q_{11 \times 11}$ we get dominating sets of $Q_{m \times n}$ for $(m, n) = (10, 11), (10, 10), (9, 11), (9, 10), (9, 9)$, and these

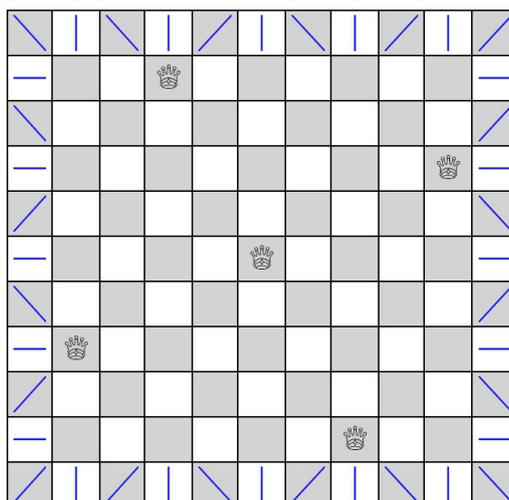


Figure 3: The minimum dominating set \mathcal{D} of $Q_{11 \times 11}$ is shown, with the unique cover of each edge square (see Corollary 8) indicated.

turn out to be minimum dominating sets. In a sense, the observed failure of monotonicity, $\gamma(Q_{8 \times 11}) = 6 > 5 = \gamma(Q_{9 \times 11})$, occurs simply because \mathcal{D} does not fit on $Q_{8 \times 11}$.

Also, by adding edge rows or columns to $Q_{11 \times 11}$ and adding corner squares to \mathcal{D} , we obtain minimum dominating sets for the values $\gamma(Q_{11 \times 12}) = 6$, $\gamma(Q_{11 \times 13}) = 7$, $\gamma(Q_{12 \times 12}) = 6$, $\gamma(Q_{12 \times 13}) = 7$, $\gamma(Q_{12 \times 14}) = 8$, $\gamma(Q_{13 \times 13}) = 7$, $\gamma(Q_{13 \times 14}) = 8$, $\gamma(Q_{14 \times 14}) = 8$, and $\gamma(Q_{15 \times 15}) = 9$. Finally, it is shown in [25] that \mathcal{D} gives a set implying $\gamma(Q_{53 \times 53}) = 27$.

It was observed by Eisenstein et al. [8] that if a dominating set D of $Q_{n \times n}$ contains no edge squares, the facts that there are $4(n - 1)$ edge squares and every queen covers eight edge squares imply $|D| \geq \lceil (n - 1)/2 \rceil$. This suggests the bound (11).

A similar approach leads one to guess the bound of our next theorem, but some care is needed to handle the general case.

Theorem 7. *Let m, n be positive integers with $m \leq n$. Then*

$$\gamma(Q_{m \times n}) \geq \min \left\{ m, \left\lceil \frac{m + n - 2}{4} \right\rceil \right\}. \quad (12)$$

Proof. It suffices to show that if $\gamma(Q_{m \times n}) \leq m - 1$ then $\gamma(Q_{m \times n}) \geq (m + n - 2)/4$. So we assume that $\gamma(Q_{m \times n}) \leq m - 1$.

First, suppose $\gamma(Q_{m \times n}) = m - 1$. Then by Proposition 1 we have $n < 3m - 2$, which implies $m - 1 > (m + n - 2)/4$ as needed.

Thus we may take $\gamma(Q_{m \times n}) \leq m - 2$ and let D be a minimum dominating set of $Q_{m \times n}$. Since $m \leq n$, there are at least two rows and at least two columns that do not contain squares of D . Let a be the index of the leftmost empty column, b the index of the rightmost empty column, c the index of the lowest empty row, d the index of

the highest empty row. The board has a rectangular sub-board U with corner squares (a, c) , (a, d) , (b, c) , and (b, d) . Let E be the set of edge squares of this sub-board. We say that U is the *box* of D and E is the *box border* of D ; these sets are defined for any square set D with $|D| \leq m - 2$. Here $|E| = 2(d - c) + 2(b - a)$.

Removing columns a and b and rows c and d divides the board into nine regions (some possibly empty). Let C be the set of squares of D inside U ; that is, $C = \{(x, y) \in D : a < x < b \text{ and } c < y < d\}$. Let T_{nw} be the set of squares of D in the “northwest” region of the $m \times n$ board; that is, $T_{nw} = \{(x, y) \in D : x < a \text{ and } y > d\}$. Similarly we label seven more subsets of D by their “geographic direction” from the central region: $T_n, T_{ne}, T_e, T_{se}, T_s, T_{sw}$, and T_w . Let $R = T_{nw} \cup T_{ne} \cup T_{sw} \cup T_{se}$, the set of those squares of D whose orthogonals do not meet U . Let $S = T_n \cup T_e \cup T_s \cup T_w$, the set of those squares of D having exactly one orthogonal that meets U . Then D is the disjoint union of R , S , and C .

Since each column to the left of column a contains at least one square of D ,

$$|T_{sw}| + |T_w| + |T_{nw}| \geq a - 1. \tag{13}$$

Similarly,

$$|T_{se}| + |T_e| + |T_{ne}| \geq n - b, \tag{14}$$

$$|T_{sw}| + |T_s| + |T_{se}| \geq c - 1, \tag{15}$$

$$|T_{nw}| + |T_n| + |T_{ne}| \geq m - d. \tag{16}$$

Adding inequalities (13)-(16) and using the definitions of R and S gives

$$2|R| + |S| \geq m + n - 2 - (d - c) - (b - a). \tag{17}$$

Each square in R covers at most two squares of E , as the square’s orthogonals and one of its diagonals miss E . Each square in S covers at most six squares of E , as one of the square’s orthogonals misses E . Each square in C covers eight squares of E . Since D is a dominating set, D covers all squares of E , so

$$2|R| + 6|S| + 8|C| \geq 2(d - c) + 2(b - a). \tag{18}$$

Adding two times (17) to (18) gives

$$6|R| + 8|S| + 8|C| \geq 2(m + n - 2). \tag{19}$$

Since $|D| = |R| + |S| + |C|$, adding $2|R|$ to both sides of (19) gives

$$8|D| \geq 2(m + n - 2 + |R|).$$

Thus

$$\gamma(Q_{m \times n}) = |D| \geq (m + n - 2 + |R|)/4, \tag{20}$$

which implies the desired conclusion. \square

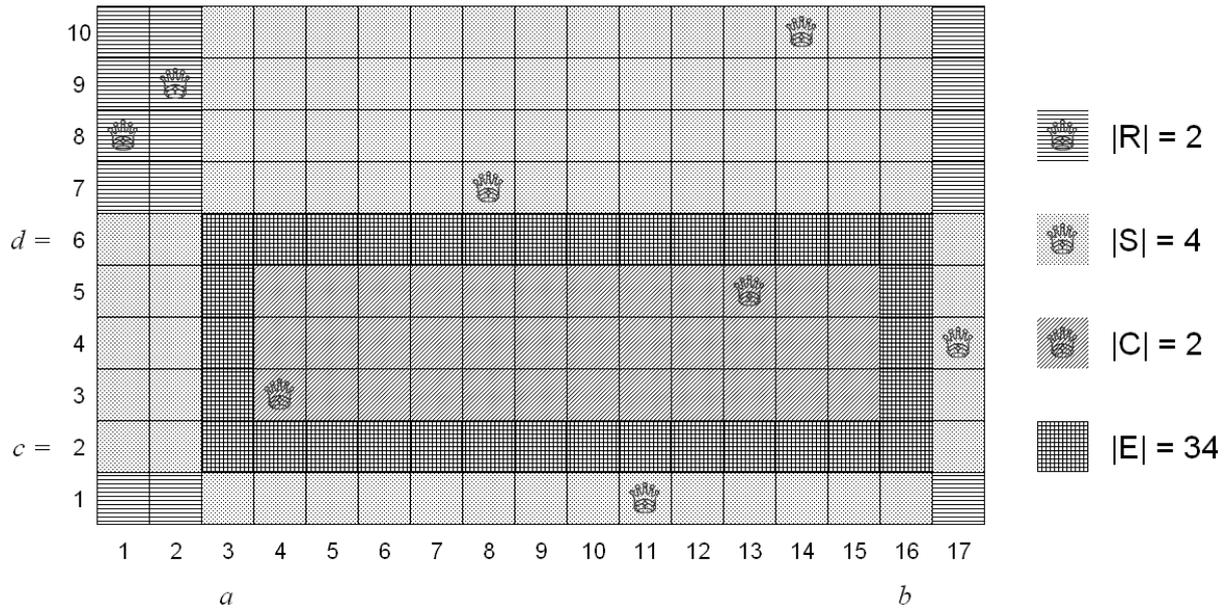


Figure 4: Illustration of the proof of Theorem 7 with $Q_{10 \times 17}$: R, S, C are the sets of queen squares having respectively 0, 1, or 2 orthogonals meeting the box of D , and E is the box border of D .

A diagram illustrating the proof for $Q_{10 \times 17}$ is given in Figure 4.

There are 120 pairs (m, n) satisfying $4 \leq m \leq n \leq 18$. Of these, the bound (12) is achieved for 40 pairs (28 with $m \leq 6$), for 76 pairs the bound is exceeded by one and for the pairs (12, 14), (13, 17), (14, 16), and (15, 15) the bound is exceeded by two.

We next explore when $\gamma(Q_{m \times n}) = (m + n - 2)/4$. From Proposition 1 it follows that for any positive integer m , if $n = 3m + 2$ then $\gamma(Q_{m \times n}) = m$, and here $m = (m + n - 2)/4$. So we restrict to $n < 3m + 2$.

Corollary 8. *Suppose $m \leq n < 3m + 2$ and $\gamma(Q_{m \times n}) = (m + n - 2)/4$. Let D be a minimum dominating set of $Q_{m \times n}$. Then $|D| \leq m - 2$, each box border square is covered exactly once by D , and D is independent.*

Proof. From $n < 3m + 2$ we have $|D| = (m + n - 2)/4 \leq m - 2$, so the box of D is defined. Since $|D| = (m + n - 2)/4$, in this setting we have equality in inequalities (13)–(20), so each square of the box border E is covered exactly once by D . Thus any line meeting E contains at most one square of D . Every square of D must be diagonally adjacent to four squares of E , so if any line containing a square of D does not meet E , it is an orthogonal. From (20) we see that here the set R of “corner squares” in D is empty, so every square of D has at least one orthogonal meeting E ; then since (13)–(16) are equations here, each orthogonal that misses E contains exactly one square of D . Thus D is independent. \square

Rarely does a minimum dominating set cover each of its box border squares uniquely; see Figure 3 for an example. We also note that the minimum dominating sets #1-4

for $Q_{11 \times 12}$ have this property. Each of these sets consists of a foursome centered at $(13/2, 13/2)$ plus the corner squares $(1, 1)$ and $(12, 1)$, so is not independent.

As mentioned earlier, $\gamma(Q_{n \times n}) = (n - 1)/2$ is achieved only for $n = 3, 11$. Considering Corollary 8, we suspect that the answer to the following question is no.

Question 2. Does $\gamma(Q_{m \times n}) = (m + n - 2)/4$ with $m \leq n < 3m + 2$ occur, other than for $(m, n) = (3, 3)$ and $(11, 11)$?

We next extend the method of proof used in [7, 18, 23] for the lower bound (11) to show that the dimensions of the box of D give a lower bound for $|D|$.

Proposition 9. Let $m \leq n$ and let D be a dominating set of $Q_{m \times n}$ of size at most $m - 2$. Let m' be the number of rows and n' the number of columns of the box of D . Then:

If $m' > n'$ then $|D| \geq \lceil \frac{n}{2} \rceil$;

If $m' \leq n'$ then $|D| \geq \lceil \frac{n-1-(n'-m')}{2} \rceil$.

Proof. Let a, b, c, d be defined as in the proof of Theorem 7. (Then $m' = d - c + 1$ and $n' = b - a + 1$.) Since $m' \leq m \leq n$, we may choose an integer e such that e through $e + m' - 2$ are indices of columns of the board. Set $S = \{(x, c), (x, d) : e \leq x \leq e + m' - 2\}$ and $P = \{(x, y) \in D : x < e \text{ or } x > e + m' - 2\}$. Then no square is diagonally adjacent to more than two squares of S and no square of P is orthogonally adjacent to any square of S . As the $2(m' - 1)$ squares of S are covered by D , $2(m' - 1) \leq 2|P| + 4(|D| - |P|)$, which implies

$$|D| \geq \left\lceil \frac{m' - 1 + |P|}{2} \right\rceil. \quad (21)$$

If $m' > n'$ then we can choose e so that all columns that do not meet S are occupied, so $|P| \geq n - (m' - 1)$. If $m' \leq n'$ we can choose e so that S is contained in the top and bottom edges of U , and then $|P| \geq n - n'$. In both cases, (21) implies the conclusion. \square

As $\gamma(Q_{n \times n}) = (n - 1)/2$ only for $n = 3, 11$, we have $\gamma(Q_{n \times n}) \geq \lceil n/2 \rceil$ for all other positive integers n . There is much evidence that this lower bound is quite good. Work from [4, 5, 11, 14, 17, 23, 24] reported in [17] shows that for n from 1 to 120, excluding 3 and 11, we have $\lceil n/2 \rceil \leq \gamma(Q_{n \times n}) \leq \lceil n/2 \rceil + 1$. In this range, $\gamma(Q_{n \times n}) = \lceil n/2 \rceil$ is known for 46 values of n and $\gamma(Q_{n \times n}) = \lceil n/2 \rceil + 1$ is known for $n = 8, 14, 15, 16$. Also, $\gamma(Q_{(4k+1) \times (4k+1)}) = 2k + 1$ is known for $1 \leq k \leq 32$.

For $m < n$, we have little evidence that the bound (12) is good. We were not able to use the methods of the proofs of Theorem 7 and Proposition 9 to improve on this bound. Also, a computer search using a greedy algorithm for some larger m, n did not supply evidence about lower bounds for $\gamma(Q_{m \times n})$.

The statement of Proposition 9 leads one to consider the quantity $n/2$. We have checked that when $4 \leq m \leq n \leq 18$, $\gamma(Q_{m \times n}) \geq \min\{m - 1, \lfloor n/2 \rfloor - 1\}$. This bound and the bound (12) are close only when m and n are close. So we ask the following.

Question 3. For m, n with $m \leq n$, what is a good general lower bound for $\gamma(Q_{m \times n})$? In particular, is it true that $\gamma(Q_{m \times n}) \geq \min\{m - 1, \lfloor n/2 \rfloor - 1\}$?

3 Construction of dominating sets

Given dimensions m and n , we would like a general approach that would allow us to construct minimum dominating sets of $Q_{m \times n}$, or at least reasonably small dominating sets. We have two difficulties to consider.

The first difficulty was just discussed: in general we know the value of $\gamma(Q_{m \times n})$ only approximately.

The second difficulty is that construction of a dominating set of $Q_{m \times n}$ generally means specifying most or all of the lines that the set is to occupy. There are some restrictions that the indices of the lines must satisfy, as we now describe.

Let $D = \{(x_i, y_i) : 1 \leq i \leq l\}$ be a set of l squares of $Q_{m \times n}$ that occupies difference diagonals $(d_i)_{i=1}^l$ and sum diagonals $(s_i)_{i=1}^l$. Since the square (x, y) is on the difference diagonal with index $y - x$ and the sum diagonal with index $y + x$, summing over D gives

$$\sum_{i=1}^l d_i = \sum_{i=1}^l y_i - \sum_{i=1}^l x_i \text{ and } \sum_{i=1}^l s_i = \sum_{i=1}^l y_i + \sum_{i=1}^l x_i. \quad (22)$$

The Parallelogram Law $2x^2 + 2y^2 = (y - x)^2 + (y + x)^2$ gives a quadratic constraint

$$2 \sum_{i=1}^l x_i^2 + 2 \sum_{i=1}^l y_i^2 = \sum_{i=1}^l d_i^2 + \sum_{i=1}^l s_i^2 \quad (23)$$

on the line indices.

In each of the two constructions given below, we will refer to lines that must be occupied for domination as *required* lines and other lines as *auxiliary* lines.

Both constructions produce a number of minimum dominating sets, but neither can produce a dominating set of $Q_{m \times n}$ of size less than $\lfloor n/2 \rfloor$. This is a little evidence for the possible bound mentioned in Question 3.

3.1 Domination by orthodox covers

This idea generalizes [25, Section 2]. Let D be a set of squares of $Q_{m \times n}$. If it is possible to place the origin of the coordinate system so that every even column and every even row contains a square of D , we will say D is an *orthodox* set. That is, an orthodox set is one that occupies at least every other column and every other row of $Q_{m \times n}$.

Say that square (x, y) of $Q_{m \times n}$ is *even* if $x + y$ is even, *odd* if $x + y$ is odd. We divide the even squares of $Q_{m \times n}$ into two classes: (x, y) is *even-even* if both x and y are even, *odd-odd* if both are odd. If D is an orthodox set and each odd-odd square of $Q_{m \times n}$ is covered by some square of D , we say D is an *orthodox cover*. For example, solution #10 for $Q_{7 \times 11}$ given in Table 1 is an orthodox cover; take the origin at $(6, 3)$ to see this.

It is clear from the definition that an orthodox cover dominates every even square of $Q_{m \times n}$, and since every odd square of $Q_{m \times n}$ is on one even-indexed orthogonal, all odd squares are also dominated: an orthodox cover is a dominating set of $Q_{m \times n}$. Many orthodox covers appear in the [appendix](#), and are labeled there as such.

Since $Q_{m \times n}$ has at least $\lfloor n/2 \rfloor$ even-indexed columns, an orthodox set on $Q_{m \times n}$ has at least $\lfloor n/2 \rfloor$ members. Generally, we expect that most of the squares of D will be even-even, to help dominate the odd-odd squares diagonally. When n is considerably larger than m , there are more possibilities of placing queens on odd squares that occupy even columns. Also, it is possible sometimes to achieve a dominating set of size less than $\lfloor n/2 \rfloor$ by a minor modification, as is shown in solution #1 for $Q_{7 \times 12}$ in Table 1. If the center of the square there labeled (6, 3) is taken to be the origin of the coordinate system, the dominating set shown misses being an orthodox set only by not occupying the rightmost column. Thus the three odd-odd squares in that column are not covered along their column, as they would be by an orthodox set. But the queen on a dark square covers those three squares, and the odd squares of its column, and thus completes a dominating set of size 5.

Minimum dominating set #147 for $Q_{12 \times 16}$ is an orthodox cover with a single queen on a dark square, at (6, 10). The squares covered only by this queen are the dark squares in its column and (1, 10). Replacing (6, 10) with the white square (6, 5) covers those squares and, adding a row 0 to the board, also the square (1, 0). In fact, the full set now covers all of row 0 and is thus a minimum dominating set of $Q_{13 \times 16}$; it is solution #15 for $Q_{13 \times 16}$, rotated by a half-turn.

There are many ways to create orthodox covers, and we will only give one example. An approach is to regard $Q_{m \times n}$ as the union of overlapping copies of $Q_{m \times m}$; for odd m , this allows us to use [25, Theorem 1], which gives sufficient conditions for an orthodox set on $Q_{m \times m}$ to be an orthodox cover.

Example 1. *An orthodox cover implying $\gamma(Q_{13 \times 19}) \leq 10$.*

We take the origin of the coordinate system to be the center of $Q_{13 \times 19}$, and regard $Q_{13 \times 19}$ as the union of two copies of $Q_{13 \times 13}$, centered at $(\pm 3, 0)$. From [25, Theorem 1], if we regard the center of $Q_{13 \times 13}$ as the origin, an orthodox set on $Q_{13 \times 13}$ dominates if the set occupies the sum and difference diagonals with indices in $\{-6, -2, 0, 2, 6\}$. Asking this on both copies of $Q_{13 \times 13}$, we wish to have our orthodox set occupy the sum and difference diagonals which (on $Q_{13 \times 19}$) have indices in $\{-6, -2, 0, 2, 6\} \pm 3$, which is $\{\pm 1, \pm 3, \pm 5, \pm 9\}$, so there will be two auxiliary difference diagonal indices d_1, d_2 and two auxiliary sum diagonal indices s_1, s_2 . The required column indices are $\pm 1, \pm 3, \pm 5, \pm 7, \pm 9$, so there will be no auxiliary column indices. The required row indices are $0, \pm 2, \pm 4, \pm 6$, so there will be three auxiliary row indices r_1, r_2, r_3 .

From (22) we have $d_1 + d_2 = r_1 + r_2 + r_3 = s_1 + s_2$ and (23) gives $d_1^2 + d_2^2 + s_1^2 + s_2^2 = 420 + 2(r_1^2 + r_2^2 + r_3^2)$. We attempt to find a solution with symmetry by a half-turn about the board center: this means $r_1 = 0, r_2 = -r_3, d_1 = -d_2$, and $s_1 = -s_2$. Then the quadratic constraint simplifies to $d_1^2 + s_1^2 = 210 + 2r_2^2$, of which one solution is $d_1 = 13, s_1 = 7, r_2 = 2$. Now all lines are specified, and it is not difficult to find the solution $D = \{\pm(9, 0), \pm(7, -6), \pm(5, 2), \pm(3, 2), \pm(1, -4)\}$.

3.2 Domination by centrally strong sets

We begin by considering a board C which is to be a central sub-board of a larger board B . Say that C has m_1 rows and n_1 columns, with $m_1 \geq n_1$ and m_1, n_1 not both even. It is convenient here to have the board squares of side length two, and place C with its center at the origin. Thus if, for example, m_1 is odd and n_1 is even, then each square has center (x, y) with x an odd integer and y an even one.

We then wish to choose a nonnegative integer k and a set D of squares of B (actually, all or almost all in C) such that D contains at least one square from the extension of each orthogonal of C to B , and D contains exactly one square from the extension of each difference diagonal of C , except none from the highest k and the lowest k extended difference diagonals; similarly for sum diagonals. Let

$$m = m_1 + 2n_1 - 2k, \quad n = 2m_1 + n_1 - 2k, \quad g = m_1 + n_1 - 2k - 1. \quad (24)$$

Then $m \leq n$, and it is straightforward to verify that if C is taken to be the central $m_1 \times n_1$ sub-board of the $m \times n$ board B , then D is a dominating set of $Q_{m \times n}$ and $|D| = g$. Such a D will be called a *centrally strong* set, as it generalizes the idea discussed for square boards in [25, page 234]. We note that our definition requires each square of D to have both diagonals among the required ones, and thus both have indices of absolute value at most $m_1 + n_1 - 2k - 2$, but this does not imply $D \subseteq C$. If in fact $D \subseteq C$, we say that D is a *strict* centrally strong set.

A number of strict centrally strong sets occur in the [appendix](#), and are labeled there as such. We note that these sets can only occur when $m \leq n < 2m$; this follows from (24) and the fact that since there will be $n_1 - 2k - 1$ auxiliary row indices, this quantity is nonnegative.

One merit of this construction is that a single centrally strong D gives an upper bound for $\gamma(Q_{m \times n})$ for several pairs (m, n) since D is confined to a small central region of the $m \times n$ board, especially if D is strict. For example, there is a strict centrally strong set $D = \{\pm(-5, 0), \pm(-3, 4), \pm(-1, 6), \pm(1, 2), \pm(3, 6)\}$ with $m_1 = 9, n_1 = 6$, and $k = 2$, and $|D| = 10$, which shows that $\gamma(Q_{m \times n}) \leq 10$ when $9 \leq m \leq 17$ and $6 \leq n \leq 20$. For some of these pairs (m, n) , this bound is poor, but for the six pairs with $m + n \geq 35$, combining with the bound (12) gives $9 \leq \gamma(Q_{m \times n}) \leq 10$, and 10 is a useful upper bound for some of the smaller boards also.

The simplest centrally strong sets occur with $m_1 \geq 1, n_1 = 1$ and $k = 0$, where we get m_1 queens occupying all squares of the $m_1 \times 1$ board C , and the following bound (which we have stated in terms of $m = m_1 + 2$). For $3 \leq m \leq 10$ at least, this bound gives the exact value of $\gamma(Q_{m \times (2m-3)})$.

Proposition 10. *For $m \geq 3$, $\gamma(Q_{m \times (2m-3)}) \leq m - 2$.*

We next consider the effect of (22) and (23) on the search for centrally strong sets. Symmetry and the requirement that each difference diagonal contains exactly one square of D imply that the sum of the difference diagonal indices of D is zero. Similarly the sum of the sum diagonal indices of D is zero, and then (22) implies that $\sum_{(x,y) \in D} x = 0$ and

$\sum_{(x,y) \in D} y = 0$. As we require a centrally strong set to occupy all (extended) columns of the sub-board, we regard the n_1 indices of these columns as required column indices; by symmetry their sum is zero. As C has n_1 columns and g occupied squares, there will be $g - n_1 = m_1 - 2k - 1$ auxiliary column indices, each having parity opposite to that of n_1 . Since $\sum_{(x,y) \in D} x = 0$ and all required column indices sum to zero, so do the auxiliary column indices. Similarly there will be m_1 required row indices and $g - m_1 = n_1 - 2k - 1$ auxiliary row indices, with sum zero, each having parity opposite to that of m_1 . If D is strict, then all indices of occupied columns have absolute value at most $n_1 - 1$ and all indices of occupied rows have absolute value at most $m_1 - 1$. (We have required that m_1, n_1 not both be even because if they were, there would be an odd number of auxiliary row indices, each odd, so their sum could not be even, thus not zero.)

Using the identities $\sum_{i=1}^j (2i - 1)^2 = \binom{2j+1}{3}$ and $\sum_{i=1}^j (2i)^2 = \binom{2j+2}{3}$, we see that the sum of the squares of the indices of all occupied diagonals of C is $4\binom{g+1}{3}$, the sum of the squares of the required column indices is $2\binom{n_1+1}{3}$ and the sum of the squares of the required row indices is $2\binom{m_1+1}{3}$. Letting \sum_{orth} denote the sum of the squares of the auxiliary column indices and auxiliary row indices, the quadratic constraint (23) gives

$$\sum_{orth} = 2 \left[\binom{g+1}{3} - \binom{m_1+1}{3} - \binom{n_1+1}{3} \right]. \quad (25)$$

Combined with Proposition 10, part (a) of the following proposition shows how small a centrally strong set can be. In parts (b) and (c), we limit the values of m_1, n_1, k that need be considered when constructing centrally strong sets.

We say that a value of k for which there exists a centrally strong set on $Q_{m_1 \times n_1}$ is *feasible* for (m_1, n_1) .

Proposition 11.

- (a) For any centrally strong set D with $n_1 > 1$, $|D| \geq n/2$.
- (b) For any (m_1, n_1) , it is only necessary to use the largest feasible k to determine all upper bounds for $\gamma(Q_{m \times n})$ implied by centrally strong sets from (m_1, n_1) .
- (c) If k is feasible for (m_1, n_1) and $k + 1$ is feasible for $(m_1, n_1 + 2)$, the latter gives the more useful result.

Proof. (a): As the number $n_1 - 2k - 1$ of auxiliary row indices is nonnegative, $n_1 \geq 2k + 1$. If $n_1 = 2k + 1$ then $g = m_1$ by (24), and then the fact that the right side of (25) is nonnegative implies $n_1 = 1$ and $k = 0$, the situation of Proposition 10. Thus for $k \geq 1$ we have $n_1 \geq 2k + 2$, which by (24) is equivalent to $|D| \geq n/2$.

(b): Suppose for some integer $h > 0$ that both k and $k - h$ are feasible for (m_1, n_1) . Then the triple m_1, n_1, k gives a dominating set D of size g on $Q_{m \times n}$, where m, n, g are determined by (24), and similarly the triple $m_1, n_1, k - h$ gives a dominating set D' of size $g + 2h$ on $Q_{(m+2h) \times (n+2h)}$. However, by repeating $2h$ times the process of adding an edge

row and edge column to the board and the new corner square to the dominating set, we can construct from D a dominating set of $Q_{(m+2h) \times (n+2h)}$ of the same size as D' .

(c): Using (24), if m_1, n_1, k gives a dominating set of size g for $Q_{m \times n}$, then $m_1, n_1 + 2, k + 1$ gives a dominating set of size g for $Q_{(m+2) \times n}$. \square

Example 2. A centrally strong set implying $\gamma(Q_{13 \times 16}) \leq 8$.

Let $m_1 = 7$ and $n_1 = 4$, and $k = 1$. Then a strict centrally strong set D is to have one auxiliary row index, which from $\sum_{(x,y) \in D} y = 0$ must be zero, and two auxiliary column indices, say c_1, c_2 , each in $\{-3, -1, 1, 3\}$. From $\sum_{(x,y) \in D} x = 0$ we see $c_2 = -c_1$ and from (25) we have $c_1^2 + c_2^2 = 18$, so we can take $c_1 = 3$ and $c_2 = -3$. We then easily obtain $D = \{\pm(1, -6), \pm(3, 4), \pm(3, 0), \pm(3, -2)\}$; see solution #23 for $Q_{13 \times 16}$. (Recall that board squares have edge length two, column indices are even integers, and row indices are odd integers here.) Using (24) this gives $\gamma(Q_{13 \times 16}) \leq 8$ (and equality holds by our computer search).

We give two infinite families of strict centrally strong sets, each including a minimum dominating set found by von Szily [20, 21].

Example 3. Strict centrally strong sets for $n_1 = 5, k = 1$ and odd $m_1 \geq 5$, and for $n_1 = 7, k = 2$ and odd $m_1 \geq 7$.

In our approach described above, all orthogonal indices would be even here; we have divided by two, thus returning to a board with squares of edge length one.

For $n_1 = 5, k = 1$, and $m_1 \equiv 1 \pmod{4}$, D consists of $\pm(-1, \frac{m_1-1}{2})$, $(0, 0)$, and $\pm(0, 2i)$ and $\pm(2, \frac{m_1+5}{2} - 4i)$ for $1 \leq i \leq \frac{m_1-1}{4}$. With $m_1 = 5$, this gives a minimum dominating set of $Q_{13 \times 13}$ found by von Szily [20]; see also solution #41.

For $n_1 = 5, k = 1$, and $m_1 \equiv -1 \pmod{4}$, D consists of $\pm(\pm 1, \frac{m_1-1}{2})$, $\pm(-1, \frac{m_1-3}{2})$, $(0, 0)$, $\pm(0, 2i)$ for $1 \leq i \leq \frac{m_1-7}{4}$, and $\pm(2, \frac{m_1+3}{2} - 4i)$ for $1 \leq i \leq \frac{m_1-3}{4}$.

These sets show that for $i \geq 3$, if $2i - 1 \leq m \leq 2i + 7$ and $5 \leq n \leq 4i + 1$, then $\gamma(Q_{m \times n}) \leq 2i + 1$.

Now let $n_1 = 7$.

For $m_1 = 7$, let $D = \{i(1, 2) + j(2, -1) : -1 \leq i, j \leq 1\}$. This gives a minimum dominating set of $Q_{17 \times 17}$ found by von Szily [21]; see also solution #21.

For $m_1 = 9$, let $D = \{(0, 0), \pm(1, 4), \pm(2, -3), \pm(1, 2) + j(2, -1) : -1 \leq j \leq 1\}$. This gives $\gamma(Q_{19 \times 21}) \leq 11$, which is the best we know.

The following complicated description of a placement is the result of unifying four cases depending on the residue of m_1 modulo 8. Any odd $m_1 \geq 11$ has a unique expression $m_1 = 11 + 2(l_1 + l_2)$ with l_1 an integer and either $l_2 = l_1$ or $l_2 = l_1 + 1$. (Here $l_1 = \lfloor (m_1 - 11)/4 \rfloor$ and $l_2 = \lceil (m_1 - 11)/4 \rceil$.)

Start with $(0, 0), \pm(1, 2), \pm(2, -3), \pm(3, -1), \pm((-1)^{l_1}, -2l_1 - 5), \pm((-1)^{l_2+1}, -2l_2 - 4)$. Add $\pm(2, 4j)$ and $\pm(2, 4j+1)$ for $1 \leq j \leq \lceil l_2/2 \rceil$, and add $\pm(2, -4j-2)$ and $\pm(2, -4j-3)$ for $1 \leq j \leq \lfloor l_2/2 \rfloor$. If $l_2 = l_1$ then add $\pm(2, (-1)^{l_1}(2l_1 + 4))$.

These sets show that for $i \geq 4$, if $2i - 1 \leq m \leq 2i + 9$ and $7 \leq n \leq 4i + 1$, then $\gamma(Q_{m \times n}) \leq 2i + 1$.

Above we have described two approaches to the construction of dominating sets. In both, once a set of lines to be occupied by the dominating set is specified, it is necessary to see whether one can find such a dominating set. A fast backtrack search idea of Hitotumatu and Noshita [12], explained and amplified by Knuth [15], was used by Östergård and Weakley [17] to find values and bounds of $\gamma(Q_{n \times n})$ up to $n = 120$. This approach and also the algorithm of Neuhaus [16] can be applied to rectangular boards as well. But as mentioned, neither of our constructions can produce a dominating set of size less than $\lfloor n/2 \rfloor$ for $Q_{m \times n}$ (with $m \leq n$). Thus a resolution of Question 3 would be needed to determine whether extensive search based on these constructions is useful.

The complexity of computing minimum dominating set of queens is another open question [9, Section 5]. Backtracking algorithms, dynamic programming, and treewidth technique are analyzed extensively by Fernau [9, Sections 2-4].

Applications of backtracking algorithms to a variety of domination problems are studied in the doctoral dissertation of Bird [3]; in particular, he examines how recursive backtracking search can be split among multiple processes by partitioning the search tree. We give some of his results on queens at the end of the next section.

4 Independent domination

We have calculated the independent domination number $i(Q_{m \times n})$, $4 \leq m \leq n \leq 18$, as shown in the table. Each table entry is linked to some minimum independent dominating sets. In these ranges for m and n , monotonicity fails twice: $i(Q_{8 \times 11}) = 6 > 5 = i(Q_{9 \times 11}) = i(Q_{10 \times 11}) = i(Q_{11 \times 11})$, and $i(Q_{11 \times 18}) = 9 > 8 = i(Q_{12 \times 18})$. The first instance is essentially the same failure as for $\gamma(Q_{m \times n})$. The second is similar in that the only (up to symmetry) independent dominating set of size 8 for $Q_{12 \times 18}$ does not fit on $Q_{11 \times 18}$.

From the definitions it is clear that $\gamma(Q_{m \times n}) \leq i(Q_{m \times n})$, and this appears to be an excellent lower bound for $i(Q_{m \times n})$. In the table, we have highlighted the entries where these two numbers are unequal. We know of no case where $\gamma(Q_{m \times n}) + 1 < i(Q_{m \times n})$.

Bird [3, Chapter 5] reports the new values $i(Q_{n \times n}) = \gamma(Q_{n \times n}) = (n/2) + 1$ for $n = 20, 22, 24$, $i(Q_{19 \times 19}) = i(Q_{21 \times 21}) = 11$, and $i(Q_{23 \times 23}) = 13$. He also gives the number of minimum dominating sets and the number of minimum independent dominating sets, up to equivalence, for $Q_{n \times n}$ up to $n = 18$.

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$n \setminus m$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
4	3														
5	3	3													
6	3	3	4												
7	3	4	4	4											
8	4	4	4	5	5										
9	4	4	4	5	5	5									
10	4	4	4	5	5	5	5								
11	4	4	5	5	6	5	5	5							
12	4	4	5	5	6	6	6	6	7						
13	4	5	5	6	6	6	7	7	7	7					
14	4	5	6	6	6	6	7	7	8	8	8				
15	4	5	6	6	7	7	7	7	8	8	9	9			
16	4	5	6	6	7	7	7	8	8	8	9	9	9		
17	4	5	6	7	7	7	8	8	8	9	9	9	9	9	
18	4	5	6	7	7	8	8	9	8	9	9	9	10	10	10

Table 2: Values of independent domination number $i(Q_{m \times n})$, $4 \leq m \leq n \leq 18$ (OEIS A299029). Highlighted cells indicate where $\gamma \neq i$.

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