The $\mathcal{H}_\infty$ Control Performance Group

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Abstract: Conventional robust control design algorithms generate only one solution that fulfils the suboptimal $\mathcal{H}_\infty$ norm criterion and thus, leaves no room for further controller tuning. Often, the designed controller is not suitable, because it is either unstable or some structural properties needs to be also satisfied. Then, the designer has to modify the original control problem and to perform the entire synthesis again. This paper proposes a method for improving the $\mathcal{H}_\infty$ control synthesis, by introducing extra flexibility into the design process. Based on the formulation of all controllers belonging to a given performance level and Lyapunov function candidate, the paper reveals the group structure, corresponding to performance problem. Based on this group structure, efficient systematic algorithms can be developed for $\mathcal{H}_\infty$ controller tuning.

Keywords: performance blending; robust control; geometry

1 Introduction

The most typical robust performance problem, can be cast as a suboptimal normalized $\mathcal{H}_\infty$ design, where for a fix (given) generalized plant description P we seek all controllers K that internally stabilize the loop and achievesthe performance guarantee $\| \mathfrak{F}_l(P,K) \| < 1$. Through a practical design problem often it would be desirable to perform a search on a set of controllers that guarantee a given performance level in order to select a suitable one for a specific implementation goal. A typical example is to find a stable controller, or a stable controller that achieves a closed loop performance that was included in the $\mathcal{H}_\infty$ design specification. This problem leads to an iterative design process. In order to implement such an iterative algorithm, a controller blending method is needed which keeps invariant the stability of the loop and the prescribed $\mathcal{H}_\infty$ performance level.

It is a fact, that by applying the Youla parametrization, the closed-loop will be an affine expression $\mathfrak{F}_l(\bar{P},Q)$, defined by the stable parameter Q and the stable matrix $\bar{P} = \begin{pmatrix} n_{zw} & n_{zu} \\ \tilde{n}_{yw} & 0 \end{pmatrix}$. Recall that the Youla parametrization, provided as $\mathcal{K}_{stab} = \{K = \mathfrak{M}_{\Sigma_P}(Q) \mid Q \in \mathbb{Q}, (V+NQ)^{-1} \text{exists}\}$, where $\mathbb{Q} = \{Q \mid Q \text{ stable}\}$

and $\mathfrak{M}_{\Sigma_P}(Q) = (U + MQ)(V + NQ)^{-1}$, is induced by a double coprime factorization of the plant, i.e., we have stable matrices such that

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix}\begin{pmatrix} M & U \\ N & V \end{pmatrix} = \tilde{\Sigma}_P \Sigma_P = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{1}$$

with $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and a stabilizing controller $K_0 = UV^{-1} = \tilde{V}^{-1}\tilde{U}$. For a recent work that covers most of the known control system methodologies using a unified approach based on the Youla parameterization, see [7].

With a further simplification, i.e., an inner(co-inner)-outer factorization we can consider a parametrization where n_{zu} and n_{yw} are isometries. Then we have the invariance relation $\| \mathfrak{F}_l(\bar{P}, Q_1) - \mathfrak{F}_l(\bar{P}, Q_2) \| = \| Q_1 - Q_2 \|$ of the Euclidean distance. However, this is not the invariance we are interested in.

The starting point of this paper is the fact that solutions of the suboptimal $\mathcal{H}_\infty$ design are parametrized by the elements of the unit ball. One of the most well-known approach to arrive to this conclusion assumes either left or right invertibility of P and uses the scattering framework by augmenting the plant, if necessary, to obtain a well defined Potapov-Ginsburg transform $\hat{P}$, see [1, 9] for details. Then, a J-inner outer factorization $\hat{P} = \hat{\Theta}_a \hat{R}$, with a block tridiagonal structure of the outer factor that corresponds to the structure of the augmentation, solves the problem. The controllers are given by $\mathfrak{M}_{\hat{R}^{-1}}(H_a)$ with $H_a = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}$, $\| H \| < 1$, while the closed loop is given by $\mathfrak{M}_{\hat{\Theta}_a}(H_a)$. Recall that Θ_a is an inner function, thus

$$\| \mathfrak{F}_l(P, K) \| = \| \mathfrak{M}_{\hat{\Theta}_a}(H_a) \| = \| \mathfrak{F}_l(\Theta_a, H_a) \| < 1 \tag{2}$$

For the details on J-inner and J-lossless functions see [2] and [9].

These facts motivate our interest in the unit corresponding ball: if we would like to blend controllers and guarantee a prescribed performance level, we should blend elements of the unit ball. One possible approach is to consider the action of the J-unitary operators on this ball – they obviously form a group considering the composition of operators– and to express the desired operation as a group homomorphism. This is the same idea (the indirect approach) that we follow with the addition of the Youla parameters to blend stable controllers:

$$K = \mathfrak{M}_{\Sigma_P}((\mathfrak{M}_{\tilde{\Sigma}_P}(K_1) + \mathfrak{M}_{\tilde{\Sigma}_P}(K_2))) \tag{3}$$

We can formulate this process in more technical terms as follows: considering the parameter space $\mathbb{Q}$, the group of automorphisms associated to this space is formed by simple translations $Q \mapsto \tau_Q$, with $\tau_Q = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}$, $\tau_{Q_1}\tau_{Q_2} = \tau_{Q_1+Q_2}$.

In this particular case, the group homomorphism between the composition of translations and the addition of parameters is trivially combined with the Möbius transform that defines the Youla parametrization. The only obstruction might appear for non-strictly proper plants, where some of the non-strictly proper parameters, are out-ruled. While this approach does not provide an exhaustive characterization of the topic, one can define a blending, that preserves stability and it is defined directly in terms of the plant and controller, without the necessity to use any factorization, see [17, 18].

The group actions that correspond to the addition of stable plants seen for the Youla parametrization are the hyperbolic motions of the unit ball, determined by the J-unitary operators. Therefore, to fulfil our program for the $\mathcal{H}_\infty$ problem, a suitable parametrization is needed that relates the J-unitary operators to the elements of the unit ball. Moreover, due to the increase in the plant order, we might encounter serious difficulties. While most of the results presented in this paper remain valid in a more general, operator valued, setting, here we restrict our attention to the state space solutions and blending of full order $\mathcal{H}_\infty$ controllers.

We cannot define directly, an operation on the unit ball, in a trivial way, that bears a nice algebraic structure. The map

$$\varphi_a(z) = a + \sqrt{1-|a^*|^2}z(1+a^*z)^{-1}\sqrt{1-|a|^2}$$

is called a translation in the unit disc $\mathbb{D}$. It can be shown that φ_a is an analytic automorphism of $\overline{\mathbb{D}}$. Moreover, $\varphi_a^{-1} = \varphi_{-a}$. In the general case, the analytic automorphisms $\varphi_a(z)$ are called Möbius-Potapov-Harris transformations, see [11, 3, 5, 6]. The elementary Blaschke transformation defines the hyperbolic translations, the Möbius addition $a \oplus z = \varphi_a(z)$, like the translation group on the Euclidean plane. However, elementary translations of the hyperbolic plane do not form a group. Moreover, Möbius addition in the disc is neither commutative nor associative.

One can introduce the concept of "gyrator" $\mathrm{gyr}: \mathbb{D} \times \mathbb{D} \to \mathrm{Aut}(\mathbb{D},\oplus)$, that measures the extent to which Möbius addition, deviates from associativity and commutativity:

$$\mathrm{gyr}[a,b]z = \ominus (a \oplus b) \oplus \{a \oplus (b \oplus z)\},$$

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a), \qquad (\mathrm{gyro-commutative\ \ law}),$$

i.e., gyrations represent rotations of the disc $\mathbb{D}$ about its center. Thus, in terms of elementary translations and rotations the group structure of the hyperbolic transformations can be characterized by using the concept of the gyrogroups, that was introduced and applied mainly in the context of Einstein's special relativity, see, e.g., [20, 19] and the references cited therein. The group operation, the Blaschke group, can be expressed as $(a,\alpha) \odot (b,\beta) = (a \oplus \alpha b, \mathrm{gyr}[a,\alpha b]\alpha\beta)$. Elements of the set $\mathbb{D} \times \mathrm{Aut}(\mathbb{D},\oplus)$ are called motions of the gyrogroup in the sense that each element $(a,\phi) \in \mathbb{D} \times \mathrm{Aut}(\mathbb{D},\oplus)$ gives rise to the motion $(l_a,\phi)z \mapsto z \oplus \phi a$. Moreover, every biholomorphic mapping h is of the form $h = \varphi_{h(0)}(uxv) = u\varphi_{h^{-1}(0)}(x)v$, where u and v are unitary operators. The metric defined as

$$\rho(a,b) = \ln\frac{1+\|\varphi_a(b)\|}{1-\|\varphi_a(b)\|} = \operatorname{arctanh}(\| \varphi_a(b) \|)$$

is invariant with respect to biholomorphic automorphisms and provides an extension of the Poincaré disk model of the hyperbolic geometry to the operator ball [8].

It turns out that when we consider the solution of different quadratic performance problems by using a state space description and LMI techniques, the solution sets are parametrized by elements of a matrix unit ball, see [14, 15, 16]. This paper presents in details an explicit parametrization of these suboptimal $\mathcal{H}_\infty$ controllers and the corresponding induced operation on the parameter space. In contrast to the operator valued case, in this context one can implement the necessary operations easily.

Concerning the structure of the presentation: for the sake of completeness in Section 2 we summarize the basic results related to the LMI-based suboptimal $\mathcal{H}_\infty$ controller synthesis problem, while Section 3 presents the result that provides all the solutions of the problem that correspond to a fixed Lyapunov matrix. Additional standard facts and notations are summarized in the Appendix. As a counterpart of the indirect approach for the controller blending based on the Youla parameters for stability, Section 4 presents the main result of the paper for performance problems by providing a parametrization of the J-unitary matrices and the group operation of this parameter space that corresponds to the hyperbolic motions defined by these J-unitary matrices.

2 LMI-based $\mathcal{H}_\infty$ Synthesis for LTI Systems

In this section we recall the main steps of LMI-based robust control synthesis. The synthesis starts from the state-space model of the augmented plant comprising the nominal plant model and all necessary weighting functions:

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \begin{pmatrix} A & B_p & B \\ C_p & D_p & E_p \\ C & F_p & 0 \end{pmatrix} \begin{pmatrix} x \\ w \\ u \end{pmatrix} \tag{4}$$

Here u is the control input, y is the measured output, z is the performance output and w collects the external (performance) inputs, such as noises, disturbances, reference signals, etc. The controller is a finite dimensional, linear time invariant system described as

$$\begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix} \tag{5}$$

With this controller, the closed loop system admits the following description:

$$\begin{pmatrix}\dot{\xi}\\ z\end{pmatrix}=\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{pmatrix}\begin{pmatrix}\xi\\ w\end{pmatrix} \qquad \text{where}$$

$$\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{pmatrix}=\begin{pmatrix}A+BD_cC & BC_c & B_p+BD_cF_p\\ B_cC & A_c & B_cF_p\\ C_p+E_pD_cC & E_pC_c & D_p+E_pD_cF_p\end{pmatrix}$$

$$=\begin{pmatrix}A & 0 & B_p\\ 0 & 0 & 0\\ C_p & 0 & D_p\end{pmatrix}+\begin{pmatrix}0 & B\\ I & 0\\ 0 & E_p\end{pmatrix}\begin{pmatrix}A_c & B_c\\ C_c & D_c\end{pmatrix}\begin{pmatrix}0 & I & 0\\ C & 0 & F_p\end{pmatrix} \qquad (6)$$

The aim of the control design is to minimize the induced $\mathcal{L}_2$ norm between w and z of $T_{zw}=\mathcal{D}+\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}$ of, i.e., to find a stable controller (5) so that the closed loop (6) satisfies the performance relation

$$\int_0^\infty \begin{pmatrix}w(t)\\ z(t)\end{pmatrix}^T\begin{pmatrix}-\gamma^2 I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}w(t)\\ z(t)\end{pmatrix}dt\le -\varepsilon\int_0^\infty w(t)^T w(t)dt, \varepsilon>0 \qquad (7)$$

where the performance bound $\gamma>0$ is minimized to be as small as possible. If $\mathcal{X}$ defines a quadratic storage function $V(x)=x^T\mathcal{X}x$ the dissipativity relation

$\frac{dV(x)}{dt}+\begin{pmatrix}w\\ z\end{pmatrix}^T\begin{pmatrix}-\gamma^2 I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}w\\ z\end{pmatrix}<0$ leads to the matrix inequality $\quad \mathcal{X}>0,$

$$\begin{pmatrix}I & 0\\ 0 & I\\ \mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{pmatrix}^T\begin{pmatrix}0 & 0 & \mathcal{X} & 0\\ 0 & -\gamma^2 I & 0 & 0\\ \mathcal{X} & 0 & 0 & 0\\ 0 & 0 & 0 & I\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & I\\ \mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{pmatrix}<0 \qquad (8)$$

which is nonlinear (quadratic) in the unknown variables. To render it linear, $\mathcal{X}$ is partitioned as $\mathcal{X}=\begin{pmatrix}X & U\\ U^T & *\end{pmatrix}$ *and* $\mathcal{X}^{-1}=\begin{pmatrix}Y & V\\ V^T & *\end{pmatrix}$, where $\dim X=\dim A$ and $\dim *=\dim A_c$.

If we consider $\ker\begin{pmatrix}0 & I & 0\\ B^T & 0 & E_p^T\end{pmatrix}=\begin{pmatrix}\Phi^1\\ 0\\ \Phi^2\end{pmatrix}$ and $\ker\begin{pmatrix}I & 0 & 0\\ 0 & C & F_p\end{pmatrix}=\begin{pmatrix}0\\ \Psi^1\\ \Psi^2\end{pmatrix}$, then, by an application of the elimination lemma, (8) is equivalent to the following set of LMIs:

$$\begin{pmatrix}Y & I\\ I & X\end{pmatrix}>0 \qquad (9)$$

$$(*)^T\begin{pmatrix}0 & X & 0 & 0\\ X & 0 & 0 & 0\\ 0 & 0 & -\gamma^2 I & 0\\ 0 & 0 & 0 & I\end{pmatrix}\begin{pmatrix}I & 0\\ A & B_p\\ 0 & I\\ C_p & D_p\end{pmatrix}\Psi<0 \qquad (10)$$

$$(*)^T \begin{pmatrix} 0 & Y_\gamma & 0 & 0 \\ Y_\gamma & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & \gamma^2 I \end{pmatrix} \begin{pmatrix} -K^T & -C_p^T \\ I & 0 \\ -B_p^T & -D_p^T \\ 0 & I \end{pmatrix} \Phi > 0 \tag{11}$$

where $\Phi = \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix} = \ker\begin{pmatrix} B^T & E_p^T \end{pmatrix}$ and $\Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} = \ker\begin{pmatrix} C & F_p \end{pmatrix}$ and $Y_\gamma = \gamma^2 Y$.

Once we have determined X, Y and the minimal performance level γ_*, the corresponding Lyapunov matrix $\mathcal{X}_*$ can be computed as follows: compute full rank U, V such that $UV^T = I - XY$ by using an SVD decomposition and set $\mathcal{X}_* = \begin{pmatrix} Y & V \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ X & U \end{pmatrix}$ to obtain the desired closed-loop Lyapunov matrix.

The last step of the synthesis procedure is the construction of a stable controller for the previously determined Lyapunov matrix and performance bound. By substituting $\mathcal{X}_*$ and γ_* in (8) one can easily recognize that (8) – due to the special structure (6) of the closed loop system – has exactly the same structure as the LMI in the Elimination Lemma. As a consequence, one possible controller candidate can be determined by using the basiclmi procedure.

3 Parameterization of the Controllers

In what follows we present an approach for characterizing all solutions of the design equations based on the following results:

Lemma 3 () *Let $P \in \mathbb{R}^{(m+n)\times(m+n)}$ be a given symmetric (Hermitian) matrix with inertia $in(P) = (m, 0, n)$. Let the matrix M be defined such that $P = M^* J M$, where, $J = diag(-I_m, I_n)$. Then all solutions $Z \in \mathbb{R}^{n\times m}$ of inequality*

$$\begin{pmatrix} I \\ Z \end{pmatrix}^* P \begin{pmatrix} I \\ Z \end{pmatrix} < 0 \tag{12}$$

can be expressed as $Z = T_{M^{-1}}(H)$, where H is an arbitrary contraction: $H^T H < I$.

Theorem 1 *Consider the quadratic matrix inequality*

$$\begin{pmatrix} I \\ AKB + C \end{pmatrix}^T P \begin{pmatrix} I \\ AKB + C \end{pmatrix} < 0 \tag{13}$$

in the unstructured unknown K. Assume C is of dimension $n \times m$, P has inertia $(m, 0, n)$ and assume that A has full column- and C has full row rank, respectively. If the solvability conditions are satisfied then all solutions of (13) can be characterized as follows:

$$K = V_a \Sigma_a^{-1} Z \Sigma_b^{-1} U_b^T, \quad Z = T_N(H) \tag{14}$$

where V_a, Σ_a, Σ_b, U_b and N are constant matrices determined by A, B, C, P and H is an arbitrary contraction.

Remark 1 *The rank conditions on* A *and* B *have been introduced to ease the discussion. By slightly modifying the proof and the final formula* (14) *they can be relaxed.*

Proof. Suppose (13) has a solution, i.e., the solvability conditions hold. Compute first the SVD-decomposition of A and B:

$$A = U_a \begin{pmatrix} \Sigma_a \\ 0 \end{pmatrix} V_a^T, \qquad B = U_b (\Sigma_b \quad 0) V_b^T$$

Σ_a, Σ_b are diagonal matrices collecting the nonzero singular values of A and B. Then we have

$$\begin{aligned} AXB &= U_a \begin{pmatrix} \Sigma_a \\ 0 \end{pmatrix} V_a^T K U_b (\Sigma_b \quad 0) V_b^T \\ &= U_a \begin{pmatrix} \Sigma_a & 0 \\ 0 & 0 \end{pmatrix} \widetilde{K} \begin{pmatrix} \Sigma_b & 0 \\ 0 & 0 \end{pmatrix} V_b^T = U_a \begin{pmatrix} \Sigma_a \widetilde{K} \Sigma_b & 0 \\ 0 & 0 \end{pmatrix} V_b^T \end{aligned}$$

Introducing $Z = \Sigma_a \widetilde{K} \Sigma_b$ (13) reads as

$$(*)^T P \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I \\ U_a \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} V_b^T \end{pmatrix} < 0$$

Multiplying it from left and right by V_b^T and V_b we get

$$(*)^T P \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} V_b \\ U_a \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} < 0$$

which is the same as

$$(*)^T P \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} V_b & 0 \\ 0 & U_a \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ Z & 0 \\ 0 & 0 \end{pmatrix} < 0$$

The next step is reordering the rows of the rightmost matrix. For this, a permutation matrix Π is introduced:

$$\Pi = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \qquad \Pi \begin{pmatrix} I & 0 \\ 0 & I \\ Z & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}$$

Then (13) amounts to

$$(*)^T P \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} V_b & 0 \\ 0 & U_a \end{pmatrix} \Pi^T \begin{pmatrix} I & 0 \\ Z & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} < 0$$

Denoting the inner matrix product by $\tilde{P}$ and partitioning it according to the blocks of the outer terms we arrive at the following inequality:

$$\begin{pmatrix} I & 0 \\ Z & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ * & \tilde{P}_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ Z & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} < 0$$

or, equivalently

$$\begin{pmatrix} \begin{pmatrix} I \\ Z \end{pmatrix}^T \tilde{P}_{11} \begin{pmatrix} I \\ Z \end{pmatrix} & \begin{pmatrix} I \\ Z \end{pmatrix} \tilde{P}_{12} \begin{pmatrix} I \\ 0 \end{pmatrix} \\ * & \begin{pmatrix} I \\ 0 \end{pmatrix}^T \tilde{P}_{22} \begin{pmatrix} I \\ 0 \end{pmatrix} \end{pmatrix} < 0$$

If the analysis equation has a solution (which is assumed), then the bottom-right block is negative definite, i.e., $\bar{P}_{22} = \begin{pmatrix} I \\ 0 \end{pmatrix}^T \tilde{P}_{22} \begin{pmatrix} I \\ 0 \end{pmatrix} < 0$. Schur complement theorem can be applied now to transform the LMI to the form of (12):

$$\begin{pmatrix} I \\ Z \end{pmatrix}^T \left[\tilde{P}_{11} - \tilde{P}_{12} \begin{pmatrix} I \\ 0 \end{pmatrix} P_{22}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}^T \tilde{P}_{12}^T \right] \begin{pmatrix} I \\ Z \end{pmatrix} < 0 \tag{15}$$

Using this form Lemma 3 can be applied to generate all solutions of (15): denoting by $\bar{P} = M^* J M$ the inner matrix if one picks a particular solution given by the matrix $Z = T_{M^{-1}}(H)$, then the original unknown controller variable K can be computed as $K = V_a \Sigma_a^{-1} Z \Sigma_b^{-1} U_b^T$.

If we apply Theorem 1 to the synthesis inequality (8) evaluated at the previously constructed Lyapunov matrix $\mathcal{X}$ and performance level $\gamma = \gamma_*$ values then we can see that the controllers that guarantee the given performance level can be parameterized as follows:

$$K = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} = V_a \Sigma_a^{-1} Z \Sigma_b^{-1} U_b^T \tag{16}$$

with $Z = T_N(H)$ and H a contractive matrix. Throughout this paper it is assumed that the domain of the Möbius transform T_N is the entire contractive ball.

Remark 2 *An analogous result can be obtained along the classical two Riccati based approach, where the set of the controllers is described by a linear fractional transform defined on the set of the contractive transfer functions, for the details see, e.g., [21]. Then, by restricting the set of parameters on the set of contractive matrices, we obtain an analogous starting point as for the LMI case.*

4 The Matrix Blaschke Group

As we have already shown, for performance problems the parametrization of the solutions provides an immediate blending possibility by following the indirect approach. In contrast to the stabilization problem [17], the identification of the elements of this approach is not trivial. In what follows we present the group structure and a parametrization of the automorphism group of the unit ball.

Setting $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ we consider the associated group of J-unitary matrices Φ, i.e., those matrices for which $\Phi^* J \Phi = J$. There is a correspondence between the contractive ball and the J-unitary matrices: for every contraction H the matrix

$$\Phi_H = \begin{pmatrix} N_H & 0 \\ 0 & N_{H^*} \end{pmatrix} \begin{pmatrix} I & -H^* \\ -H & I \end{pmatrix}$$

is J-unitary. It is convenient to introduce the following notations: $D_H = (I - H^*H)$ and $N_H = D_H^{-1}$. Observe that we have the following properties:

$$N_H = N_H^*, \quad N_{(-H)} = N_H, \quad HN_H = N_{H^*}H,$$

$$N_{UH} = N_H, \quad N_{HU}U^* = U^*N_H,$$

for any unitary U. It is immediate that $\Phi_H = \Phi_H^*$ and that $\Phi_H^{-1} = \Phi_{-H}$.

Concerning the geometric content, recall that J-unitary matrices define the movements, i.e., hyperbolic translations, on the matrix unit ball that preserve the hyperbolic distance. Their Möbius transform defines the multidimensional generalization of the elementary Blaschke products:

$$B_H(Z) = \mathfrak{M}_\Phi(\underline{Z}) = N_{H^*}(Z - H)(I - H^*Z)^{-1}D_H =$$

$$-H + D_{H^*}Z(I - H^*Z)^{-1}D_H = \mathfrak{F}_l(\Psi, Z)$$

with $\Psi = \begin{pmatrix} -H & D_{H^*} \\ D_H & H^* \end{pmatrix}$. The elementary Blaschke products $B_H(Z)$ are biholomorphic automorphisms of the unit ball $\mathcal{B}$ and $\| B_H(Z) \| \leq B_{\|H\|}(\| Z \|)$. Moreover, every biholomorphic mapping h is of the form $h = B_{h(0)}(UZV) = UB_{h^{-1}(0)}(Z)V$, where U and V are unitary operators. The metric defined as

$$\rho(A, B) = \ln \frac{1 + \| B_A(B) \|}{1 - \| B_A(B) \|} = arctanh(\| B_A(B) \|)$$

is invariant with respect to biholomorphic automorphisms and provides an extension of the Poincaré disk model of the hyperbolic geometry to the operator ball. For details see, e.g., [4, 8, 10].

Note that

$$B_H(0) = -H, \quad B_H(H) = 0, \quad B_{-H}(0) = H \tag{17}$$

$$B_H \circ B_{-H} = B_{-H} \circ B_H = I \tag{18}$$

In contrast to the Euclidean geometry, where elementary translations form a group, in the hyperbolic world we do not have this property. This fundamental difference makes things more complicated: we cannot define a group structure merely on the contractive ball. However, based on the observation that every J-unitary matrix can be expressed as an elementary translation and a block diagonal unitary action, there is a remedy.

Theorem 2 *Every J-unitary matrix can be expressed as $\Phi = W_{U,V}\Phi_H$, where H is a suitable contraction and U and V are unitary matrices, with $W_{U,V} = diag\{U, V\}$.*

For the result in the general, operator valued context, see, e.g., [2]. Its proof relies on the existence and uniqueness properties of the polar decomposition. The following commutation formula is the basic observation for our purposes.

$$\Phi_H W_{U,V} = W_{U,V}\Phi_{V^*HU} \tag{19}$$

Its importance relies in the derivation of the formula that relates the action of the J-unitary group in terms of the three parameters (U, V, H). Observe that

$$\Phi_1\Phi_2 = W_{U_1,V_1}\Phi_{H_1}W_{U_2,V_2}\Phi_{H_2} = W_{U_1,V_1}W_{U_2,V_2}\Phi_{V_2^*H_1U_2}\Phi_{H_2} = W_{U,V}\Phi_H,$$

i.e., $\Phi_{(U_1,V_1,H_1)}\Phi_{(U_2,V_2,H_2)} = \Phi_{(U,V,H)}$

The operation $(U, V, K) = (U_1, V_1, H_1) \circ (U_2, V_2, H_2)$ defined by this homomorphism is obviously a group, called the Blaschke group. If we would like to provide an explicit expression of this homomorphism, we need to provide a formula for the product $\Phi_{H_1}\Phi_{H_2}$ of the elementary Blaschke factors, i.e., for $(U, V, H) = (I, I, H_1) \circ (I, I, H_2)$.

As a first step, observe that by definition we have

$$(U, V, H) = (U, V, 0) \circ (I, I, H)$$

$$(U_1U_2, V_1V_2, 0) = (U_1, V_1, 0) \circ (U_2, V_2, 0)$$

and we have already shown that

$$(U_1, V_1, H_1) \circ (U_2, V_2, H_2) = (U_1U_2, V_1V_2, 0) \circ (I, I, V_2^*H_1U_2) \circ (I, I, H_2) \tag{20}$$

Before arriving to the final formula, we need some relations that are interesting in their own right. First observe that by using the J-unitary property of Φ_H and the definition of B_H we have $\begin{pmatrix} I \\ B_H(Z) \end{pmatrix}^* J \begin{pmatrix} I \\ B_H(Z) \end{pmatrix} = (\star) J \begin{pmatrix} I \\ Z \end{pmatrix} (I - H^*Z)D_H$, i.e., the defect can be expressed as $D^2_{B_H(Z)} = I - B_H^*(Z)B_H(Z) = Q_H^*(Z)Q_H(Z)$, with the factor $Q_H(Z) = D_Z(I - H^*Z)^{-1}D_H$.

Thus, for the unitary matrix $E_H(Z)$ we get the expression $D_{B_H} = E_H^*(Z)Q_H(Z)$, i.e.,

$E_H(Z) = Q_H(Z)N_{B_H(Z)}$, and, by a direct verification results that

$$B_H(Z) = -B_Z(H)E_H(Z) \tag{21}$$

Now we can formulate one of the main results of this section:

Theorem 3 *The product of elementary J-unitary matrices is the J-unitary matrix given by*

$$\Phi_{H_1}\Phi_{H_2} = W_{U,V}\Phi_H,$$

where the contractive term and the unitary factor can be computed as

$$H = B_{-H_2}(H_1), \quad U = E_{-H_2}(H_1), \quad V = E_{-H_2^*}(H_1^*)$$

Proof: Indeed, from the identity $B_{H_1}(B_{H_2}(Z)) = VB_H(Z)U^*$ and applying (17) we get $0 = VB_H(B_{-H_2}(H_1))U^*$, *i.e.*, $H = B_{-H_2}(H_1)$. It also follows that $B_{H_1}(-H_2) = -VHU^*$. Thus

$$D^2_{B_{H_1}(-H_2)} = UD_H^2U^* = UD^2_{B_{-H_2}(H_1)}U^*$$

$$D^2_{B^*_{H_1}(-H_2)} = VD^2_{H^*}V^* = VD^2_{B^*_{-H_2}(H_1)}V^*$$

Putting together all these results we can obtain the expressions of the unitary factors as $U = E_{-H_2}(H_1)$, $V = E_{-H_2^*}(H_1^*)$, as it was claimed. Finally, we have that

$$(E_{-H_2}(H_1), E_{-H_2^*}(H_1^*), B_{-H_2}(H_1)) = (I, I, H_1) \circ (I, I, H_2)$$

Combining Theorem 3 with (20) we have obtained the explicit formula for the desired blending operation that defines the group homomorphism

$$\Phi_{(U_1,V_1,H_1)}\Phi_{(U_2,V_2,H_2)} = \Phi_{(U_1,V_1,H_1)\circ(U_2,V_2,H_2)} = \Phi_{(U,V,H)}$$

as follows:

Theorem 4 *Corresponding to our notations, the operation given by*

$$\begin{gathered}(U,V,H) = (U_1,V_1,H_1) \circ (U_2,V_2,H_2) = \\ (U_1U_2E_{-H_2}(V_2^*H_1U_2), V_1V_2E_{-H_2^*}(U_2^*H_1^*V_2), B_{-H_2}(H_1))\end{gathered} \tag{22}$$

defines a group structure.

Remark 3 *In the performance problem considered in this paper we are interested only in the contraction part, see* (16). *One might think that the map $(H_1, H_2) \to B_{-H_2}(H_1)$ is sufficient to define the blending, and that the unitary part does not play any role. Thus, it seems that in the matrix case, for practical purposes one needs only the elementary Blaschke maps according to $T_{\Phi_H}(0) = -H$.*

Remember, however, that $\Phi_{H_1}\Phi_{H_2} = W_{U,V}\Phi_H$, in general. Thus, the elementary Blaschke maps are not enough to define an automorphism group structure and we should use the formula $T_{\Phi_{H_1}}T_{\Phi_{H_2}}(0) = T_{\Phi_H}(0) = -VHU^*$, where the parameters are given by Theorem 4. At this point recall, that the controller is given by (16), where $Z = T_N(H) = (C + DH)(A + BH)^{-1}$. Thus, in an iterative process, the additional unitary factors may be used to maintain some structural constraints through the iteration. As an example, taking a generalized SVD of the pair (A, B), one can simplify the computation of the inverse during the iteration.

Conclusions

This paper proposes a method for improving the $\mathcal{H}_\infty$ control synthesis, which provides a starting point for developing algorithms that uses some sort of iteration. The paper is based on the observation that solutions of the quadratic performance problems, e.g., a suboptimal $\mathcal{H}_\infty$ design, are parametrized by the elements of the unit ball. Based on the formulation for all controllers belonging to a given performance level and Lyapunov function candidate, the paper reveals the group structure corresponding to the control performance problem.

The paper presents, in detail, an explicit parametrization of the hyperbolic motions of the matrix unit ball and the corresponding induced operation, on this parameter space. The obtained formula, leads to an indirect blending algorithm, for controllers, that guarantees a given performance level. In contrast to the operator valued case, in this context, one can implement the necessary operations easily. Based on this group structure, efficient systematic algorithms can be developed for $\mathcal{H}_\infty$ controller tuning.

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Appendix

Notations and basic results

The notations used in the paper are fairly standard. The kernel of a matrix M is denoted by $M_\perp$ and is interpreted as $MM_\perp = 0$. The inertia of a matrix M is denoted by $\text{in}(m,k,n)$ where m,k,n are the number of positive, zero and negative eigenvalues of M. The Möbius transformation of matrix K with respect to the matrix N is denoted by $T_N(K)$ and is defined by $T_N(K) = (C+DK)(A+BK)^{-1}$,

where $N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Lemma 2 (Projection lemma) For arbitrary A, B and a symmetric P, the LMI

$$K^T XB + B^T X^T A + P < 0 \tag{30}$$

in the unstructured X has a solution if and only if

$$A_\perp^T P A_\perp < 0 \qquad and \quad B_\perp^T P B_\perp < 0, \tag{31}$$

where $A_\perp = \ker(A)$ and $B_\perp = \ker(B)$.

If (31) is satisfied then one particular solution X of (30) can be determined by the numerical algorithm implemented in basiclmi MATLAB routine.

Lemma 3 (Elimination lemma) Consider the quadratic matrix inequality

$$\begin{pmatrix} I \\ AXB+C \end{pmatrix}^T P \begin{pmatrix} I \\ AXB+C \end{pmatrix} < 0 \tag{32}$$

in the unstructured unknown X. Assume C is of dimension $n \times m$ and P has inertia $(m,0,n)$. Then (32) has a solution if and only if

$$B_\perp^T \begin{pmatrix} I \\ C \end{pmatrix}^T P \begin{pmatrix} I \\ C \end{pmatrix} B_\perp < 0, \quad and \qquad A_\perp^T \begin{pmatrix} -C^T \\ I \end{pmatrix}^T P^{-1} \begin{pmatrix} -C^T \\ I \end{pmatrix} A_\perp > 0, \tag{33}$$

where $A_\perp = \ker(A)$ and $B_\perp = \ker(B)$.

Note, that solution of the H_∞ problem uses the Projection lemma, which is a special case of the Elimination lemma when $P = \begin{pmatrix} Q & S \\ S^* & 0 \end{pmatrix}$.

Möbius transformation and basic properties

Definition 1 *Let* $M \in \mathbb{F}^{(m+n)\times(m+n)}$ *($F = \mathbb{R}$ or $\mathbb{C}$) be partitioned as* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. *The Möbius transformation* T_M *is defined by the equation*

$T_M(X) = (C+DX)(A+BX)^{-1}$ *for* $X \in \mathbb{F}^{n\times m}$ *where* $(A+BX)^{-1}$ *exists. Denote by* $\text{dom}(T_M) = \{X \in \mathbb{F}^{n\times m} : \exists (A+BX)^{-1}\}$ *the domain of* T_M.

The dual Möbius transformation is defined by $T_M^d(Z) = (ZB+D)^{-1}(ZA+C)$, and $\mathrm{dom}(T_M^d) = \{Z \in \mathbb{F}^{n\times m} : \exists (ZB + D)^{-1}\}$.

Theorem 5 *Let* $M \in \mathbb{F}^{(m+n)\times(m+n)}$. *Then*

$$X \in dom(T_M^d) \quad \Leftrightarrow \quad X^* \in dom(T_{L^*M^*L}).$$

Moreover $T_M^d(X) = T_{L^*M^*L}^*(X^*)$, *where* $L = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}$. *If* $M \in \mathbb{F}^{(m+n)\times(m+n)}$ *is a nonsingular matrix, then* $T_M(X) = -T_{M^{-1}}^d(-X)$.

Corollary 1 $\quad -T_M^*(X) = T_{L^*M^{-*}L}(-X^*)$.

Let us consider the composition of two Möbius transformations.

Definition 2 *Let* M *and* N *be partitioned as* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $N = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$.

Composition of the transformations T_M *and* T_N *is* $(T_N \circ T_M)(X) = T_N(T_M(X))$.

Lemma 4 $(T_N \circ T_M)(X) = T_N\big(T_M(X)\big) = T_{NM}(X)$, *with* $X \in dom(T_M)$ *and* $T_M(X) \in \mathrm{dom}(T_N)$. *If* M *is nonsingular,* $Z \in \mathrm{dom}(T_M)$ *and* $T_M(X) = K$ *then* $K \in \mathrm{dom}(T_{M^{-1}})$ *and* $T_{M^{-1}}(K) = X$, *i.e.*, $\mathrm{dom}(T_M) = \mathrm{Range}(T_{M^{-1}})$.