Detection filter design for homogeneous multi-agent networks ⋆

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Abstract: A class of large-scale systems with decentralized information structures such as multi-agent systems can be represented by a linear system with a generalized frequency variable. In these models agents are modelled through a strictly proper SISO state space model while the supervisory structure, representing the information exchange among the agents, is represented via a linear state-space model. In this paper the fundamental problem of residual generation, as a basic detector filter design task, is solved in the context of homogeneous multi-agent networks. In order to reduce the dependence of the filter on the particular agent and to exploit the inherent informational structure of the network system the same local-global structure is imposed to the filter as of the original system. It is shown that a stable detection filter can be designed if the same structural conditions are fulfilled as in the unstructured LTI case.

Keywords: FDI filter design; fundamental problem of residual generation; homogeneous multi-agent networks;

1. INTRODUCTION AND MOTIVATION

Modern engineering systems in the areas of manufacturing, transportation, and telecommunications can be effectively represented as a network of agents that mutually interact and exchange information. Dynamical interactions among agents, and the intrinsic complexity of the physical networks make the analysis and control of multi-agent network systems quite a challenging task.

In order to make the analysis computationally tractable, the simplifying assumption that the agents can be described by the same transfer function is often introduced. Then, the overall dynamics can be represented as the interconnection of a scalar transfer matrix and of a feedback control block, that represents the communication exchange among the agents. Under these assumptions, Hara and co-authors have been able to describe the homogeneous multi-agent system dynamics as a linear system with generalized frequency variable, Hara et al. [2009]. A series of powerful results were derived regarding controllability, stability and stabilizability , H_2 and H_∞-norm computation of the overall system, see Harat et al. [2007], Hara et al. [2010, 2014]. This class of system descriptions has a potential to provide a theoretical foundation for analyzing and designing large-scale dynamical systems in a variety of areas.

Safety is of great importance in modern control, and one of the main requirements of this problem is its task of fault detection and isolation. There are various approaches to residual generation, see, e.g., the detection filter approach initiated by Massoumnia [1986] for LTI systems and used also by Edelmayer et al. [1997], Bokor and Balas [2004] for LTV and LPV systems, the dedicated observers and the parity space approaches Gertler [1998], just to mention a few.

In the so called "geometrical approach" to some fundamental problems of LTI control theory, such as disturbance decoupling, unknown input observer design, fault detection, a central role is played by the (A, B)-invariant and (C, A)-invariant subspaces and certain controllability and unobservability subspaces, Wonham [1985], Massoumnia et al. [1989]. The nonlinear version of this geometrical approach deals with certain distributions and codistributions, Hermann and Krener [1977], Isidori [1989].

In this paper we investigate the problem of the geometry based fault detection and isolation in the context of homogeneous multi-agent networks.

1.1 Problem statement

Let us consider the following model for hierarchical multiagent dynamical systems: the system consists of N identical SISO agents whose state space realization is expressed as

$$\begin{aligned} \dot{x}_i &= A_h x_i + b_h u_i \\ y_i &= c_h x_i \end{aligned} \tag{1}$$

and the transfer function is given by

$$h(s) = c_h(sI_{n_h} - A_h)^{-1} b_h, \tag{2}$$

where $c_h^T, b_h \in \mathbb{R}^{n_h}$ and $A_h \in \mathbb{R}^{n_h \times n_h}$.

In what follows we make the standing assumption that $h(s)$ is stable.

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The agents are connected to each other through the input and output according to the following rule:

$$\begin{aligned} \alpha &= A\beta + Bu \\ y &= C\beta + Du, \\ \beta &= (I_N \otimes h(s))\alpha, \end{aligned} \tag{3}$$

where $u \in \mathbb{R}^m, y \in \mathbb{R}^p, \alpha, \beta \in \mathbb{R}^N$ and A, B, C are real matrices of corresponding dimensions. If the connection is well-defined, the overall system will be given by the upper linear fractional transformation (LFT), see Figure 1:

$$\mathcal{G}(s) = \mathfrak{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, h(s)). \tag{4}$$

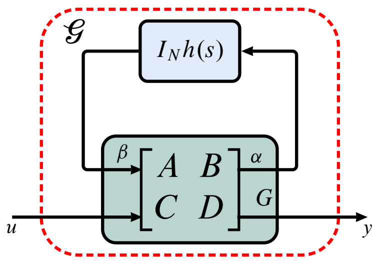

Fig. 1. Homogeneous multi-agent network

A short computation reveals that the state-space realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of $\mathcal{G}(s)$ can be expressed as

$$\mathcal{A} = I_N \otimes A_h + A \otimes (b_h c_h), \tag{5}$$

$$\mathcal{B} = B \otimes b_h, \quad \mathcal{C} = C \otimes c_h, \quad \mathcal{D} = D, \tag{6}$$

or

$$\mathcal{A} = A_h \otimes I_N + (b_h c_h) \otimes A, \tag{7}$$

$$\mathcal{B} = b_h \otimes B, \quad \mathcal{C} = c_h \otimes C, \quad \mathcal{D} = D. \tag{8}$$

Remark 1. Given (1) one can immediately check that for the aggregate state $\xi = \left[x_1^T, \cdots, x_N^T\right]^T$ we have

$$\dot{\xi} = (I_N \otimes A_h)\xi + (I_N \otimes b_h)\alpha, \tag{9}$$

$$\beta = (I_N \otimes c_h)\xi. \tag{10}$$

Then, from (3) we have

$$\begin{aligned} \dot{\xi} &= (I_N \otimes A_h + A \otimes (b_h c_h))\xi + (B \otimes b_h)u, \\ y &= (C \otimes c_h)\xi. \end{aligned}$$

Observe that this realization is free of interblock mixing. This fact motivates to call these realizations as *compatible* (to the given hierarchical structure).

Interblock mixing means that the states of the realization are obtained by a blending of the components of ξ defined by (9) that corresponds to the different blocks. Note that not all realizations of $\mathcal{G}$ are compatible, e.g., (7) is not a compatible realization. Thus, compatibilty of the realization reflects not only the fact that in the global state matrices the component local state matrices appear in a certain structure but also the corresponding state should be obtained as a stack of the local (agent) states. Observe that the dynamics is given entirely by the agents, the informational level (3) is only the glue that connect them in a certain structure.

Example 2. Let us consider a very simple example of formation control: there are N identical agents moving between the walls placed at l_1 and l_2 in the one-dimensional space. The position of the ith agent is represented as y_i.

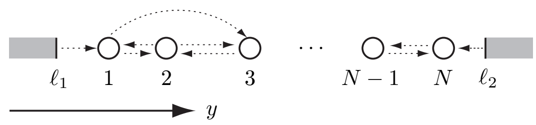

Fig. 2. Formation control example

We assume that the agents are collision-free and all the agents share the common dynamics $h(s)$.

The control objective for the ith agent is to control its position by collecting its relative position with respect to the other agents and the walls. In particular, the target (reference) position r_i of the ith agent is given by

$$r_i = F_i y + b_i, \tag{11}$$

where F_i characterizes the weighted relative position with respect to the other agents and b_i takes a nonzero value when the information of the wall positions is known and zero when no information is available for the agent i. Note that $F_{ij} \neq 0$ if j is an agent that the ith agent can sense, and zero otherwise.

In this paper we suppose that the fault occurs on the informational structure level described by (3), i.e., the faulty system is modeled as

$$\begin{aligned} \alpha &= A\beta + Bu + L\nu \\ y &= C\beta + Du, \\ \beta &= (I_N \otimes h(s))\alpha. \end{aligned} \tag{12}$$

With this model structure possible faulty communication channels, intruders or the presence of misbehaved agents can be described.

We would like to design a filter for fault detection and isolation (FDI) that has the same structure as the original system:

$$\mathcal{F}(s) = \mathfrak{F}_u(\begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}, h(s)), \tag{13}$$

i.e., our goal is to obtain an FDI filter with a compatible realization.

The main motivation behind this problem setting is to confine the design task on the level of the information structure and make it independent on the choice of the agents' dynamics $h(s)$. By using this strategy one should not redesign the FDI filter if the dynamics of the agents changes.

In this paper we propose a geometry based solution to this problem based on the detection filter design approach that solves the so called fundamental problem of residual generation (FPRG), see Massoumnia [1986], Massoumnia et al. [1989]. The nontrivial part of the extension of this method to homogeneous multi-agent networks is to guarantee the stability of the designed filter.

Section 2 recall the FPRG design problem for LTI systems, provides the solvability conditions and the filter equations. In order to apply the method to our problem, we need to stabilize the filter. Section 2.1 provides the theoretical background of the stabilization problem and shows that the classical solvability condition is also valid for the design of the filters having the imposed compatibility structure. Section 3 provides a short simulation example, while the paper is concluded by some conclusions.

2. THE FUNDAMENTAL PROBLEM OF RESIDUAL GENERATION

Let us consider first the following LTI system, that has two failure events:

$$\dot{x}(t) = Ax(t) + Bu(t) + L_1 v_1(t) + L_2 v_2(t), \quad (14)$$
$$y(t) = Cx(t) + Du(t). \quad (15)$$

The task of designing a residual generator that is sensitive to the fault associated with the L_1 direction and insensitive to the fault associated with the L_2 direction is called the fundamental problem of residual generation. More precisely, one has to design a residual generator.

If we denote by r_1 the output of the residual generator, then we should ensure that when a fault is present, i.e., $v_1 \neq 0$, then $r_1 \neq 0$ while if $v_1 = 0$ $\lim_{t\to\infty} \|r_1(t)\| = 0$, i.e., to guarantee a stability condition requirement of the residual generator.

In the solution of this problem a central role is played by the (C, A)-invariant subspaces and certain unobservability subspaces, see Massoumnia [1986], Massoumnia et al. [1989], or, in the nonlinear version of this problem, observability codistributions, see De Persis and Isidori [2000, 2001].

Let us recall some of the basic notions used to construct the detection filter: it is known that for LTI systems, a subspace $\mathcal{W}$ is (C,A)-invariant if $A(\mathcal{W} \cap \mathrm{Ker}C) \subset \mathcal{W}$. This is equivalent to the existence of a matrix G such that $(A+GC)\mathcal{W} \subset \mathcal{W}$. A (C,A)-unobservability subspace $\mathcal{S}$ is a subspace such that there exist matrices G and H with the property that $(A + GC)\mathcal{S} \subset \mathcal{S}$, i.e., $\mathcal{S}$ is (C,A)-invariant, and $\mathcal{S} \subset \mathrm{Ker}HC$. The family of (C,A)-unobservability subspaces containing a given set $\mathcal{L}$ has a minimal element $\mathcal{S}^*$. It is important to stress that efficient algorithms exist to compute these invariant subspaces, e.g., see Wonham [1985], Basile and Marro [2002].

Let us denote by $\mathcal{L}_i = \mathrm{Im}L_i$, $i = 1, 2$ and denote by $\mathcal{S}^*$ the smallest unobservability subspace containing $\mathcal{L}_2$. Then one has the following result:

Theorem 3. (FPRG). The fundamental problem of residual generation has a solution if and only if $\mathcal{S}^* \cap \mathcal{L}_1 = 0$, moreover, if the problem has a solution, the dynamics of the residual generator can be assigned arbitrary.

The conditions of Theorem 3 ensure that the fault to be detected is not hidden in the unobservability subspace of the detection filter. In fact, the fault direction will be decoupled from the rest of the fault directions since they are contained in the unobservability subspace of the residual generator.

The residual generator associated with fault direction $\mathcal{L}_1$ can be described by the following observer form:

$$\dot{w}(t) = Nw(t) - Gy(t) + (F + GD)u(t), \quad (16)$$
$$r_1(t) = Mw(t) - Hy(t) + HDu(t),$$

where u, y are the known input and measured output signals of the original LTI system, the components of w are the states of the residual generator and r_1 is the residual.

In order to construct the detection filter, denote by P the projection operator $P : \mathcal{X} \to \mathcal{X}/\mathcal{S}^*$ and then the state matrices can be determined as follows: H is a solution of the equation $\mathrm{Ker}HC = \mathrm{Ker}C + \mathcal{S}^*$, and M is the matrix associated to the unique solution of $MP = HC$.

Let us consider a gain matrix G_0 such that condition $(A+G_0C)\mathcal{S}^* \subset \mathcal{S}^*$ holds and let $A_0 = A + G_0C|_{\mathcal{X}/\mathcal{S}^*}$ denote its restriction to the factor space. It is a standard result, see, e.g., Wonham [1985], that on this factor space one can assign the eigenvalues arbitrarily, i.e., there is a gain matrix G_1 such that $N = A_0 + G_1M$ has the prescribed eigenvalues. Then set $G = PG_0 + G_1H$ and $F = PB$ to complete the design.

Noe that while D is present in the filters dynamic equations (16) it does not affect the computation of the invariant subspaces.

In order to assure the decoupling property it is sufficient that $\mathcal{S}^* \cap \mathcal{L}_1 = 0$ holds for any unobservability subspace $\mathcal{S}^*$ containing $\mathcal{L}_2$. Besides the fact that the minimal unobservability subspace can be determined by a well defined algorithm, minimality guarantees the necessity of the condition and the observability property of the constructed filter. In the LTI case this latter property makes possible the construction of a stable filter. While the geometrical ideas can be extended quite straightforward for the more general time varying situation the question of stability will be quite involved since the pole allocation property of observable pairs has no counterpart in the general theory.

This result can be extended to the case with multiple events, called the extension of the fundamental problem of residual generation (EFPRG). The EFPRG has a solution if and only if $\mathcal{S}_i^* \cap \mathcal{L}_i = 0$, where $\mathcal{S}_i^*$ is the smallest unobservability subspace containing $\overline{\mathcal{L}}_i := \sum_{j\neq i} \mathcal{L}_j$.

It is clear that one can apply this algorithm in the context of the problem set in this paper, too. Al the computations should be performed for the fictitious system defined by the state matrices $(A, [B\, L_1\, L_2], C, D)$. Observe, however, that the solvability of the FPRG problem on the global level, i.e, with a detection filter having a not necessarily compatible realization, does not imply the solvability of the FPRG problem set in this paper.

If the design condition of Theorem 3 is fulfilled, the only nontrivial problem is the stabilization condition needed to ensure the stability of the overall filter defined by

$$\mathcal{F}(s) = \mathfrak{F}_u(\begin{bmatrix} N & [-G,\ F+GD] \\ M & [-H,\ HD] \end{bmatrix}, h(s)). \quad (17)$$

This is actually a compatible stabilizing output injection gain computation, i.e., a compatible detectability problem in the context of the framework set by homogeneous multi-agent networks.

2.1 Compatible stabilizability and detectability

In the case of LTI system if the realization is minimal then it is obvious that we have stabilizability (detectability). This section shows that this is also true for the compatible case., i.e., when we impose the condition that the feedback (output injection) fit the structure imposed by the network. The nontriviality of the assertion is given by the fact that the informational level does not have access to the entire state, only to the outputs of the agents.

Let us consider the following setting:

$$\begin{aligned} \alpha &= A\beta + BF\beta \\ \beta &= (I_N \otimes h(s))\alpha, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \alpha &= A\beta + GC\beta \\ \beta &= (I_N \otimes h(s))\alpha, \end{aligned} \tag{19}$$

respectively. We call F a compatible stabilizing feedback gain and G a compatible stabilizing output injection gain, respectively, if the corresponding closed loop systems are stable. Compatible stabilizability (detectability) means that there exists a stabilizing compatible feedback (compatible output injection).

By applying (5) one has

$$\mathcal{A} = I_N \otimes A_h + A \otimes (b_h c_h) + BF \otimes (b_h c_h)$$

and

$$\mathcal{A} = I_N \otimes A_h + A \otimes (b_h c_h) + GC \otimes (b_h c_h),$$

respectively. This shows that in this case the stabilization problem reduces to a problem which is similar to a classical (LTI) static output feedback problem. It is known that the static output feedback problem does not always has a solution.

As a motivation background consider the following observer design problem associated to system (4):

$$\begin{aligned} \hat{\alpha} &= A\hat{\beta} - Gy + (B + LD)u \\ \hat{\beta} &= (I_N \otimes h(s))\hat{\alpha}, \end{aligned}$$

which leads to the error equation

$$\zeta = (A + GC)\eta \tag{20}$$

$$\eta = (I_N \otimes h(s))\zeta \tag{21}$$

with $\eta = \beta - \hat{\beta}$ and $\zeta = \alpha - \hat{\alpha}$. On the global level the error equation is

$$\dot{e} = (I_N \otimes A_h + A \otimes (b_h c_h) + GC \otimes (b_h c_h))e$$

which can be written as

$$\dot{e} = (I_N \otimes A_h + (I_N \otimes b_h)(A + GC)(I_N \otimes c_h))e,$$

with $e = \xi - \hat{\xi}$. As it is already clear from (20) this is a static output feedback connection for $I_N \otimes h(s)$.

The state feedback case in analogous and it is omitted in what follows, for brevity.

Concerning controllability, observability and minimality of the realizations we have the following results:

Theorem 4. (Hara et al. [2009]). If rank$B = N$ (rank$C = N$), then $(\mathcal{A}, \mathcal{B})$ is controllable ($(\mathcal{A}, \mathcal{C})$ is observable) if and only if (A_h, b_h) is controllable ((A_h, c_h) is observable).

If rank$B \neq N$ (rank$C \neq N$), then $(\mathcal{A}, \mathcal{B})$ is controllable ($(\mathcal{A}, \mathcal{C})$ is observable) if and only if (A_h, b_h, c_h) is a minimal representation and (A, B) is controllable ((A, C) is observable).

Theorem 5. (Hara et al. [2009]). Assume that $h(s)$ is strictly proper. Then the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of $\mathcal{G}(s)$ is minimal if and only if the realization (A_h, b_h, c_h) of $h(s)$ and the realization (A, B, C, D) of $G(s)$ is a minimal.

Thus, if (A, C) is observable and (A_h, b_h, c_h) is a minimal representation then one can always design a stable observer on the global level. Which is not clear for the first glance that this is also possible by using a structured variant, i.e., an observer having a compatible representation, using a compatible output injection gain, provided that A_h is stable.

To prove the assertion we use the following result, Hara and Tanaka [2010]:

Theorem 6. $\mathcal{A}$ in (5) is Hurwitz stable if and only if for all $\lambda \in \sigma(A)$ all the eigenvalues of $A_h + \lambda b_h c_h$ belong to the open left-half complex plane.

It is immediately obvious that the condition of the theorem is fulfilled if $\sigma(A) = \{0\}$. Since (A, C) is observable, there is a feedback gain G that places all the eigenvalues of $A + GC$ to the origin. Thus, we have the desired result:

Proposition 7. If A_h is stable and (A, C) is observable then there is a compatible stabilizing output injection gain.

Remark 8. The assumption on stability of A_h is essential, in general: e.g., with the minimal realization defined by $A_h = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b_h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c_h = [1\ 0]$ we have that $A_h + \lambda b_h c_h = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}$, which has as eigenvalues $\pm\sqrt{\lambda}$, i.e., it cannot be stable for any λ.

Compatible realizations impose a certain structure which leads to a reduced ability in stabilizing or in achieving certain performances (pole allocation) compared to the unconstrained versions. This might be a serious limitation of this type of solutions in practical applications.

We conclude this section with the main result of this paper:

Theorem 9. The fundamental problem of residual generation has a compatible solution if and only if $\mathcal{S}^* \cap \mathcal{L}_1 = 0$.

3. SIMULATION EXAMPLE

In this section we demonstrate the proposed method through a simulation example. At the informational level we set

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \; L_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \; L_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \; D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

These matrices defines the detection problem according to (14). The agents are supposed to obey the dynamics defined by $h(s) = \frac{s-1}{(s+1)(s+2)}$.

Following the steps of the proposed algorithm the filter matrices that corresponds to (16) are

$$N_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \; G_1 = \begin{bmatrix} -1.0 & -1.0 \\ -1.0 & 0 \\ 0 & 0 \end{bmatrix}, \; F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_1 = [0\ 0\ -1], \; H_1 = [1\ 0],$$

and

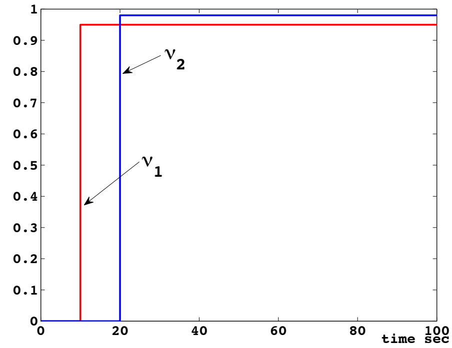

Fig. 3. Fault signals

$$N_2 = \begin{bmatrix} 0.2886 & 1.0774 & 0.2113 \\ 0.0773 & 0.2887 & 0.7887 \\ 0 & 0 & 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0.5774 & 0.7888 \\ 0.5773 & 0.2113 \\ 0 & 1 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.5774 & 0 \\ 0.5774 & 0 \\ 0 & 1 \end{bmatrix}, \; M_2 = \begin{bmatrix} 0.7071 & 0.7071 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.7071 & 0.7071 \end{bmatrix}$$

respectively.

The fault signals are depicted on Figure 3 while the detected residuals are on Figure 4.

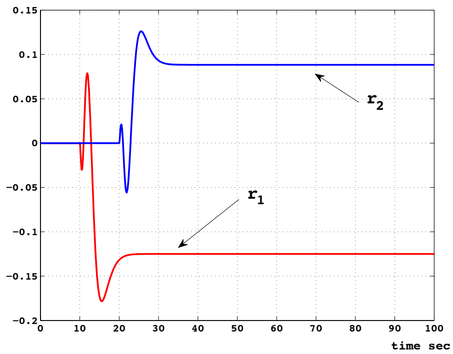

Fig. 4. Residuals ($\lambda = 0$)

We tried to influence the detection performance by tuning the pole placement in the second step of the algorithm in a way that still maintains the stability of the overall filter by using Theorem 6.

The re-designed filter dynamics are defined by

$$N_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1.0001 \\ 1 & 1 & 2.0001 \end{bmatrix}, \; G_1 = \begin{bmatrix} 1 & 1 \\ 2.0001 & 0 \\ 2.0001 & 0 \end{bmatrix}$$

and

$$N_2 = \begin{bmatrix} 1.0774 & 0.2887 & 0.2113 \\ 0.2888 & 0.0773 & 0.7887 \\ 1.0001 & 1.0001 & 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0.2113 & 0 \\ 0.7888 & 0.0001 \\ 1.0001 & 0.0001 \end{bmatrix}$$

respectively.

The obtained residuals are depicted on Figure 5.

The simulation examples show that the restriction imposed to the pole allocation by the stability condition of Theorem 6 seriously deteriorates the achievable detection performance (detection speed) compared to the unstructured filters.

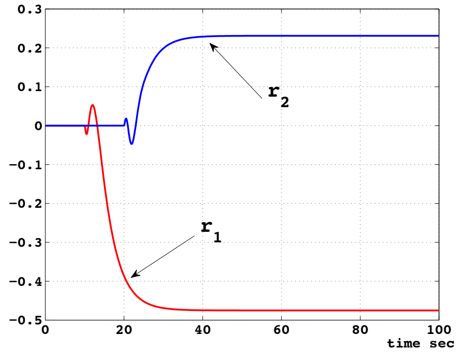

Fig. 5. Residuals (tuned)

4. CONCLUSION

The fundamental problem of residual generation, as a basic detector filter design task, has been set in the context of homogeneous multi-agent networks: in order to reduce the dependence of the filter on the particular agent and to exploit the inherent informational structure of the network system a compatibility structure is imposed to the filter. As a result the FDI filter is designed based only on the data determined by the informational structural (global) layer. It is shown that the design can be performed if and only if the associated classical LTI FPRG problem is solvable.

Compatible realizations impose, however, a certain structure to the filter to be designed. The resulting constraint leads to a reduced ability in stabilizing or in achieving certain performances (pole allocation) compared to the unconstrained versions. This might be a serious limitation of this type of solutions in practical applications.

APPENDIX

4.1 Notations and basic facts

For the matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ their Kronecker product $A \otimes B$ is the block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}. \qquad (22)$$

The Kronecker product has the following properties:

$$A \otimes (B + C) = A \otimes B + A \otimes C, \qquad (23)$$
$$(A + B) \otimes C = A \otimes C + B \otimes C, \qquad (24)$$
$$(kA) \otimes B = A \otimes (kB) = k(A \otimes B), \qquad (25)$$
$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \qquad (26)$$

If the matrices are nonsingular, then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \qquad (27)$$

while theu have compatible dimensions, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \qquad (28)$$

In general, the Kronecker product is not commutative. However, there exist permutation matrices P and Q such that

$$A \otimes B = P\,(B \otimes A)\,Q. \qquad (29)$$

If A and B are square matrices, then we can take $P = Q^T$.

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