

Hyperbolic geometrical approach to model reduction

Alexandros Soumelidis* József Bokor** Ferenc Schipp***
Zoltán Szabó****

* *Systems and Control Laboratory, Institute for Computer Science and Control, Budapest, Hungary (e-mail: soumelidis@sztaki.mta.hu)*

** *Institute for Computer Science and Control, Hungarian Academy of Sciences, Hungary, MTA-BME Control Engineering Research Group. (e-mail: bokor.jozsef@sztaki.mta.hu)*

*** *Department of Numerical Analysis, Eötvös Loránd University, Budapest, Hungary, (e-mail: schipp@numanal.inf.elte.hu)*

**** *Systems and Control Laboratory, Institute for Computer Science and Control, Budapest, Hungary (e-mail: szabo.zoltan@sztaki.mta.hu)*

Abstract: Model reduction of large scale systems is an actively researched area of modelling and control. The problem is more involved if uncertainties are also present and a computational tractable nominal model is needed for the design. Based on results of the Kolmogorov n -width theory the paper provides useful bounds for the worst case approximation error – both \mathcal{H}_2 and \mathcal{H}_∞ – in terms of the hyperbolic distance related to the sets of uncertain poles. A related model reduction strategy that uses only this a priori pole information is also proposed. The method is illustrated through numerical examples.

Keywords: Identification for control; Frequency domain identification; Nonparametric methods.

1. INTRODUCTION AND MOTIVATION

Simulation and control design of complex and large scale systems usually requires the derivation of a reduced order model of the real system. If the model is obtained from measurements, this process is called approximate model identification, while starting from a complex high order model obtained, e.g., from first principle analysis, the goal is to get an approximation that is close to the original model in some sort of system norm.

Approximate system identification was discussed in e Silva (1996); Heuberger et al. (2005) in association with the selection of an optimal orthogonal basis in an appropriate \mathcal{H}_2 -space. The optimality criterion was the Kolmogorov n -width and associated hyperbolic distance of system poles. Tóth (2008) proposed a fuzzy clustering procedure to find the appropriate model order for identification.

Similar to the identification task is the model reduction problem, which has gained a lot of interest in the context of modelling of large scale systems with moderate complexity, especially in the airspace industry. Standard approaches, e.g., balanced model truncation, are formulated in terms of the observability and controllability Grammians. For large systems, however, these methods encounter serious numerical difficulties in the solution of Lyapunov equations.

The recently developed iterative rational Krylov algorithm proposes a systematic method for selecting interpolation points for multipoint rational Krylov approximations based on an \mathcal{H}_2 -norm optimality criterion. This method has been applied to reduction of large-scale linear time

invariant (LTI) systems, although its extension to parameter dependent LTI systems remains an open question, Antoulas et al. (2001), Gugercin et al. (2008). A model-constrained adaptive sampling methodology was proposed in Bui-Thanh et al. (2007) for steady problems that are linear in state, but have nonlinear dependence on a set of parameters that describe geometry and PDE coefficients.

When uncertainty is present a method for model reduction for parametric uncertainties was shown in Dolgin and Zeheb (2005) while an approach for polytopic and affine uncertainty sets was proposed in Goncalves et al. (2009). This paper consider the case when the parametric uncertainty information can be formulated in the term of the poles, a case often encountered for aerospace applications.

Consider the set of stable linear time invariant (LTI) systems given by their rational transfer function analytic on the closed unit disc $\mathbb{D} := \mathbb{D} \cup \mathbb{T}$ where the disc is denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denotes its boundary on the complex plain \mathbb{C} . Assume that all the poles have multiplicity 1. Let us call the reflection of the original system poles p' as inverse poles, i.e., $p = p'/|p'|^2, p \in \mathbb{D}$. On \mathbb{D} let us introduce the first order rational functions $R_p(z) = 1/(1 - \bar{p}z)$ which are parameterized by $|p| < 1$.

For a given set $P \subset \mathbb{D}$ of inverse poles of the original system let us consider the set $\mathcal{R}_P = \text{span}\{R_p : p \in P\}$ and introduce the normalized ball related to this set as

$$\mathcal{B}_P^q = \{f \in \mathcal{R}_P \mid \|f\|_q \leq 1\},$$

where $\|\cdot\|_q$ is the norm of the Hardy-space \mathcal{H}_q . In this paper we consider $q = 2$ and $q = \infty$.

For a fixed $a \in \mathbb{D}$ let us associate n -dimensional linear subspace $X_n^a = \text{span}\{R_a^k : 0 \leq k < n\}$ of the Hardy-space \mathcal{H}_q . Concerning \mathcal{H}_2 , it is known that the first n elements of discrete Laguerre system form an orthogonal basis in X_n^a . In this specific context model reduction in \mathcal{H}_q norm means to approximate the transfer function $f \in \mathcal{B}_P^q$ from X_n^a such that $n < |P|$, where $|P|$ denotes the number of elements in P .

It is known that approximation of a plant in a system space, like \mathcal{H}_∞ , has to be done in this space norm. Approximation in space \mathcal{H}_∞ is, however, far more complicated than in Hilbert spaces. Therefore we discuss first the approximation in \mathcal{H}_2 and we show that the results can be used in \mathcal{H}_∞ -norm, too.

Examine first the approximation of $f \in \mathcal{B}_P^2$ from X_n^a . It is known that if $X \in \mathcal{H}_2$ is a closed subspace, then for all $f \in \mathcal{H}_2$ there exists a unique $g_0 \in X$ such that

$$\text{dist}(f, X) := \inf_{g \in X} \|f - g\| = \|f - g_0\|,$$

where $g_0 = P_X f$, i.e., the orthogonal projection of f onto the subspace X . The best approximation will be characterised by the quantity that is considered as a Kolmogorov n -width:

$$w_n(\mathcal{B}_P^2) = \inf_{a \in \mathbb{D}} \sup_{f \in \mathcal{B}_P^2} \text{dist}(f, X_n^a). \quad (1)$$

It will be shown that one can obtain the following bound on w_n :

$$\begin{aligned} w_n(\mathcal{B}_P^2) &\leq C_2 r^n(a_P), \\ r(a_P) &= \min_{a \in \mathbb{D}} \max_{p \in P} \rho(a, p) \quad (a \in \mathbb{D}), \end{aligned} \quad (2)$$

where ρ is the so-called hyperbolic (pseudo-)distance on the unit disc \mathbb{D} :

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|}.$$

This bound can be interpreted geometrically: a_P is the centre of the hyperbolic circle covering the set of poles P and $r_P := r(a_P)$ is its radius. The consequence is that the approximation of the system with lower order dynamics can be reduced to the determination of finding hyperbolic circles that cover the pole-sets or clusters with minimum radius. Then the approximate model will be defined in a basis like Laguerre, Kautz, Malmquist-Takenaka, GOBF parameterized by the circle centers.

This bound will also be given for using the \mathcal{H}_∞ norm. For the quantity

$$w_n(\mathcal{B}_P^\infty) = \inf_{a \in \mathbb{D}} \sup_{f \in \mathcal{B}_P^\infty} \inf_{g \in X_n^a} \|f - g\|_\infty$$

the following bound will be obtained:

$$w_n(\mathcal{B}_P^\infty) \leq C_\infty r_P^n.$$

To obtain good approximation to a high order system, it is recommended to use of Malmquist-Takenaka basis $\{\phi_k^a\}$ parameterized by a set of poles $\mathbf{a} = (a_n \in \mathbb{D}, n \in \mathbb{N})$. Setting $\mathbf{a}_k = (a_0, \dots, a_{k-1}) \in \mathbb{D}^k$ let us introduce the family of subspaces $X_n^{\mathbf{a}_k} = \text{span}\{\phi_k^a : 0 \leq k < n\}$ ($n \in \mathbb{N}^*$). It will be shown that for the n -width criterion

$$w_n(\mathcal{B}_P^\infty) = \inf_{\mathbf{a}_k \in \mathbb{D}^k} \sup_{f \in \mathcal{B}_P} \text{dist}(f, X_n^{\mathbf{a}_k}) \quad (3)$$

the following bound can be derived

$$\begin{aligned} w_n(\mathcal{B}_P^\infty) &\leq C_\infty \min_{\mathbf{a}_k \in \mathbb{D}^k} r(\mathbf{a}_k, P), \\ r(\mathbf{a}_k, P) &= \max_{p \in P} \prod_{j=0}^{k-1} |B_{a_j}(p)|, \end{aligned}$$

where B_{a_j} denotes the Blaschke function with parameter a_j .

Starting from these observations, this paper proposes a model reduction algorithm based on a Kolmogorov n -width criterion for uncertain plants, where the uncertainty measure is formulated by using hyperbolic distances. The necessary theoretical background is sketched in Section 2, where the key result is formulated in Theorem 2. Section 3 formulates the model reduction problem and provides the proposed model reduction algorithm based on Theorem 6. The method is illustrated by some numerical examples in Section 4.

2. N-WIDTHS AND THE HYPERBOLIC RADIUS

Denote by \mathcal{L}_2 the classical $\mathcal{L}_2(\mathbb{T})$ Hilbert space endowed with the inner-product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \bar{g}(e^{it}) dt \quad (4)$$

and with norm

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^2 dt \right)^{\frac{1}{2}}.$$

Let \mathcal{L}_∞ be the Banach space with the norm

$$\|f\|_\infty = \text{ess. sup}_{t \in \mathbb{T}} |f(e^{it})|.$$

Accordingly \mathcal{H}_2 will be the Hardy space of square integrable functions on \mathbb{T} with analytic continuation on the unit disc. Analogously we consider the space \mathcal{H}_∞ . For a classical introduction in \mathcal{H}_q theory see Duren (1970) and Garnett (1981).

The Blaschke functions defined on $\bar{\mathbb{D}}$ as

$$B_{\mathbf{b}}(z) = \epsilon B_b(z), \quad B_b(z) = \frac{z - b}{1 - \bar{b}z}, \quad (5)$$

with $\mathbf{b} = (b, \epsilon) \in \mathbb{B} = \mathbb{D} \times \mathbb{T}$ play an important role in the sequel. b is called the parameter while $b' = 1/\bar{b}$ is the pole of of the Blaschke function B_b . Some features of the Blaschke function are mentioned as follows:

- $B_{\mathbf{b}} : \mathbb{D} \mapsto \mathbb{D}$ and $B_{\mathbf{b}} : \mathbb{T} \mapsto \mathbb{T}$ are bijections.
- $B_{\mathbf{b}}(z)$ is an inner function in the space \mathcal{H}_2 , i.e., $|B_{\mathbf{b}}(e^{it})| = 1$ ($t \in [-\pi, \pi]$).
- The Blaschke functions $B_{\mathbf{b}}$ are isometries with respect to the metric

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|} = |B_{z_1}(z_2)|, \quad (B_{z_1} := B_{(z_1, 1)}), \quad (6)$$

which is called – following Poincaré – a pseudo-hyperbolic metric (see, e.g., Ahlfors (1973) for details), moreover,

$$\rho(B_{\mathbf{b}}(z_1), B_{\mathbf{b}}(z_2)) = \rho(z_1, z_2). \quad (7)$$

The Blaschke functions form a group with the operation of function-composition that is called Blaschke group. Using the concept of the Blaschke function and Blaschke group with the metric (6) a hyperbolic-type geometry can be

built in the unit-disc that conforms with the Poincaré unit-disc model of the hyperbolic geometry. Hence the Blaschke group can also be referred as the hyperbolic group.

The Blaschke product associated to the sequence $\mathbf{a} = (a_n, n \in \mathbb{N})$ is defined as $B_{\mathbf{a}|n}(z) = \prod_{j=0}^{n-1} B_{a_j}(z)$. The corresponding orthonormal system of functions in \mathcal{H}_2 is

$$\phi_0^{\mathbf{a}}(z) = \frac{\sqrt{1-|a_0|^2}}{1-\bar{a}_0 z} \quad \phi_n^{\mathbf{a}}(z) = \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z} B_{\mathbf{a}|n}(z), \quad (8)$$

which is called Malmquist-Takenaka (MT) system generated by \mathbf{a} . Necessary and sufficient condition of the completeness is $\sum_{n=0}^{\infty} (1-|a_n|) = \infty$.

A useful class of MT systems is generated by periodic sequences and it is termed as a generalized orthonormal basis (GOBF) of \mathcal{H}_2 . In this case the sequence \mathbf{a} is obtained by the periodic repetition of a finite number of parameters $a_0, a_1, \dots, a_{N-1} \in \mathbb{D}$, i.e., $a_n = a_k$ if $n = lN + k$. The system ϕ_n generated by the periodic sequence is of the form

$$\phi_n = \phi_k B_{\mathbf{a}|lN}^l \quad n = lN + k, k = 0, \dots, N-1. \quad (9)$$

For convenience we will also use $B_{\mathbf{a}|N} = B_{\mathbf{a}}$. In the particular case when $N = 1, a_0 = a$ the discrete Laguerre system

$$L_n^a(z) = \Phi_a(z) B_a^n(z), \quad \Phi_a(z) = \frac{d_a}{1-\bar{a}z}, \quad (10)$$

with $d_a = \sqrt{1-|a|^2}$, while if $N = 2, a_0 = a, a_1 = \bar{a}$ the Kautz system is obtained, Schipp and Bokor (1998).

For a given MT system let us consider the n dimensional subspace $X_n^{\mathbf{a}} = \text{span}\{\phi_k^{\mathbf{a}} : 0 \leq k < n\}$. For a fixed $p \in \mathbb{D}$ we are interested in how well the function $R_p(z)$ can be approximated by the elements of $X_n^{\mathbf{a}}$, i.e., to estimate the quantity $\text{dist}(R_p, X_n^{\mathbf{a}})$.

Let $S_n^{\mathbf{a}} f$ denote the partial sum $S_n^{\mathbf{a}} f = \sum_{k=0}^{n-1} \langle f, \phi_k^{\mathbf{a}} \rangle \phi_k^{\mathbf{a}}$. In what follows we only consider periodic parameter sets.

Lemma 1. Either on \mathcal{H}_2 or on \mathcal{H}_{∞} we have the bound

$$\|R_p - S_{N_m}^{\mathbf{a}} R_p\| \leq |B_{\mathbf{a}}(p)|^m \|R_p\|. \quad (11)$$

Proof. Applying the Cauchy-formula we get

$$\langle f, R_p \rangle = f(p) \quad (12)$$

and we have

$$\begin{aligned} R_p(z) - (S_{N_m}^{\mathbf{a}} R_p)(z) &= \\ &= \sum_{k=0}^{N-1} \bar{\phi}_k(p) \phi_k(z) \sum_{l=m}^{\infty} \bar{B}_{\mathbf{a}}^l(p) B_{\mathbf{a}}^l(z) = \\ &= \sum_{k=0}^{N-1} \bar{\phi}_k(p) \phi_k(z) \frac{\bar{B}_{\mathbf{a}}^m(p) B_{\mathbf{a}}^m(z)}{1 - \bar{B}_{\mathbf{a}}(p) B_{\mathbf{a}}(z)}. \end{aligned}$$

Recall now the Christoffel-Darboux formula, see Lorentz et al. (1996):

$$\sum_{k=0}^{N-1} \bar{\phi}_k(p) \phi_k(z) = \frac{1 - \bar{B}_{\mathbf{a}}(p) B_{\mathbf{a}}(z)}{1 - \bar{p}z}$$

to obtain the following identity:

$$R_p(z) - (S_{N_m}^{\mathbf{a}} R_p)(z) = \bar{B}_{\mathbf{a}}^m(p) B_{\mathbf{a}}^m(z) R_p(z). \quad (13)$$

Thus the assertion follows.

Given a $f_{\lambda} \in \mathcal{R}_P$ from the identity (13) follows that

$$f_{\lambda}(z) - (S_{N_m}^{\mathbf{a}} f_{\lambda})(z) = B_{\mathbf{a}}^m(z) \tilde{f}_{\lambda}(z), \quad (14)$$

with $\tilde{f}_{\lambda}(z) = \sum_{p \in P} \bar{B}_{\mathbf{a}}^m(p) r_p(z)$.

Theorem 2. For any $f \in \mathcal{B}_P^q$ we have that

$$\text{dist}_q(f, X_{N_m}^{\mathbf{a}}) = \inf_{g \in X_{N_m}^{\mathbf{a}}} \|f - g\|_q \leq C_q \rho_{\mathbf{a}}^m, \quad (15)$$

where $\rho_{\mathbf{a}} = \max_{p \in P} |B_{\mathbf{a}}(p)|$ and $1 \leq q \leq \infty$.

Proof. Observe that by an application of (14) we have

$$\|f_{\lambda} - S_{N_m}^{\mathbf{a}} f_{\lambda}\|_q \leq \rho_{\mathbf{a}}^m \left\| \sum_{p \in P} \nu_p r_p(z) \right\|_q,$$

where $\nu_p = B_{\mathbf{a}}(p)/\rho_{\mathbf{a}}$ and thus $|\nu_p| \leq 1$. If we denote by

$$C_q = \sup_{|\nu_p| \leq 1, \|f\|_q \leq 1} \left\| \sum_{p \in P} \nu_p r_p(z) \right\|_q$$

the constant that is independent of the choice of \mathbf{a} , the assertion follows.

Remark 3. We are mainly interested in the cases $q = 2, \infty$, If $q = 2$, then

$$\left\| \sum_{p \in P} \nu_p r_p(z) \right\|_2 \leq \sum_{p \in P} \|r_p(z)\|_2 \leq (\max_{p \in P} d_p) \|\lambda\|_2,$$

with $\|\lambda\|_2 = (\sum_{p \in P} |\lambda_p|^2)^{1/2}$. Using condition $\|f\|_q \leq 1$, we have $\|\lambda\|_2 \leq C_P$, where C_P is a constant that depends only on P . Thus, C_q is a constant depending only on P . Actually, for practical purposes one can consider the set

$$\mathcal{B}_{2,P} = \{f_{\lambda} \in \mathcal{R}_P, \|\lambda\|_2 \leq 1\}.$$

On this set we have the bound

$$\text{dist}_2(f, X_{N_m}^{\mathbf{a}}) \leq (\max_{p \in P} d_p) \rho_{\mathbf{a}}^m. \quad (16)$$

Analogously, for $q = \infty$ it is convenient to consider the set

$$\mathcal{B}_{\infty,P} = \{f_{\lambda} \in \mathcal{R}_P, \|\lambda\|_1 = \sum_{p \in P} |\lambda_p| \leq 1\},$$

with the uniform bound

$$\text{dist}_{\infty}(f, X_{N_m}^{\mathbf{a}}) \leq \rho_{\mathbf{a}}^m. \quad (17)$$

Since we are interested in how well these functions can be approximated by the elements of $X_{N_m}^{\mathbf{a}}$ the so-called Kolmogorov-width is considered, i.e.,

$$\begin{aligned} w_m(\mathcal{B}_P^q, \mathcal{X}_{N_m}) &= \inf_{X \in \mathcal{X}_{N_m}} \sup_{f \in \mathcal{B}_P^q} \text{dist}(f, X) = \\ &= \inf_{\mathbf{a} \in \mathbb{D}^N} \sup_{f \in \mathcal{B}_P^q} \text{dist}(f, X_{N_m}^{\mathbf{a}}). \end{aligned} \quad (18)$$

Applying Theorem 2 we have

Theorem 4. Let us denote by $\rho_N = r(\mathbf{a}, P)$. Then

$$w_m(\mathcal{B}_P^q, \mathcal{X}_{N_m}) \leq C \rho_N^m. \quad (19)$$

Recall that $\rho_N = \inf_{\mathbf{a} \in \mathbb{D}^N} \rho_{\mathbf{a}}$,

For $N = 1$ the infimum is explicitly known: ρ_1 is the radius of the hyperbolic circle covering the set P , and \mathbf{a}^* , the optimal choice for \mathbf{a} , is the centre of this circle. Moreover, one can construct efficient algorithms – also based on hyperbolic geometrical ideas – to compute this radius. The case $N \geq 1$ is more involved and an iterative search is needed to obtain a fair approximation of ρ_N and the corresponding \mathbf{a}^* , the optimal pole configuration.

3. MODEL REDUCTION AND HYPERBOLIC DISKS

The optimal \mathcal{H}_2 model reduction problem gain a lot of interest in the context of modelling of large scale systems with moderate complexity. Due to the numerical difficulties encountered in the solution of Lyapunov equations the \mathcal{H}_∞ methods, or even the state space based \mathcal{H}_2 model reduction approaches, are not competitive in this area. Current solution proposals are based on an interpolation framework, where the key step is the selection of the poles of the reduced model, see, e.g., Antoulas et al. (2001), or, for a recent adaptive approach Mi et al. (2012).

A basic fact concerning the optimality condition of the reduced model is the following, see Gugercin et al. (2008): for a fixed set of simple poles A with $r = |A|$ let us consider the set \mathcal{R}_A . Then G_r solves the optimal \mathcal{H}_2 model reduction problem constrained to the subspace \mathcal{R}_A , i.e.,

$$\|G - G_r\|_2 = \min_{g \in \mathcal{R}_A} \|G - g\|_2$$

if and only if $\langle G - G_r, g \rangle = 0$ for all $g \in \mathcal{R}_A$. However, applying (12) and (14) we have that if $r = Nm$ and $G_r = S_{Nm}^a G$ then $\langle G - G_r, g \rangle = 0$ for all $g \in \mathcal{R}_A$, as expected.

Lemma 5. For all $m \geq 1$ and $r = |A|m$ we have

$$\|G - S_{Nm}^a G\|_2 = \min_{g \in \mathcal{R}_A} \|G - g\|_2.$$

Thus, the entire problem of the optimal \mathcal{H}_2 model reduction problem revolves around the question how to select optimal pole configuration A such that

$$\|G - G_r\|_2 = \inf_a \|G - S_r^a G\|_2.$$

The most widespread approach to provide a solution, i.e., the prototype of the interpolation-based model reduction methods for LTI systems, is the iterative rational Krylov algorithm (IRKA). For a given system G

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = C^T x(t)$$

and with a prescribed reduced system order N , the goal of the algorithm is to find a local minimizer \hat{G} for the \mathcal{H}_2 model reduction problem. For the class \mathcal{R}_p the first order necessary conditions for a local minimizer imply that $\hat{G}(z) = \hat{C}^T(zI - \hat{A})^{-1}B$ is a Hermite interpolant of $G(z)$ at its reflected system poles, Meier and Luenberger (1967), i.e.,

$$G(p) = \hat{G}(p), \quad G'(p) = \hat{G}'(p), \quad p \in P.$$

The IRKA algorithm iteratively updates the projection subspaces until interpolation at the reflected reduced system poles is ensured, see, e.g., Flag et al. (2012):

- (1) Make an initial selection of P which is closed under conjugation and fix a convergence tolerance ϵ_{tol} .
- (2) while $d(P, \hat{P}) > \epsilon_{\text{tol}}$ repeat: choose V and W such that $W^T V = I$ and

$$\text{Ran}(V) = \{(\zeta_1 I - A)^{-1}B, \dots, (\zeta_N I - A)^{-1}B\},$$

$$\text{Ran}(W) = \{(\zeta_1 I - A^T)^{-1}C, \dots, (\zeta_N I - A^T)^{-1}C\},$$

with $\zeta_i = \frac{1}{\bar{p}_i}$ and $p_i \in P$.

- (3) let

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = C V,$$

and $\hat{P} = \{1/\text{eig} \hat{A}\}$.

- (4) let $P = \hat{P}$.

While there are no rigorous convergence proofs, numerous experiments have shown that the algorithm often converges rapidly.

Another approach, initiated in Mi et al. (2012). Considering the n^{th} stage of the Malmquist-Takenaka expansion generated by \mathbf{a} , i.e.,

$$G = G_n(z) + G_{n+1}(z)B_{\mathbf{a}|n}(z),$$

the next pole a in A is selected by using the criteria

$$\max_a |\langle G_{n+1}(z), R_a(z) \rangle|.$$

All these approaches assume, however, that the transfer function G is completely known. To relax this assumption in this paper the following uncertainty model is considered: let us suppose that the poles of the nominal model G are partitioned in K different clusters. Each cluster P_i is covered by a hyperbolic circle H_i with centre p_i and radius ρ_i ($2K_c$ centres are symmetric to the real line while K_r centres are on the real line, i.e., $K = 2K_c + K_r$). In each cluster the number of poles is given, i.e., by an abuse of the notation $|P_i| = |H_i|$. Notice that neither the centre, nor the radius of H_i and that of the minimal covering hyperbolic circle associated to the set P_i are supposed to be identical.

Accordingly, the uncertain $G_u(z)$ can be written as

$$G_u(z) = \sum_{i=1}^K G_{u,i}(z), \quad G_{u,i} \in \mathcal{B}_{H_i}.$$

Instead of \mathcal{B}_{H_i} one might consider \mathcal{B}_{2,H_i} or \mathcal{B}_{∞,H_i} as well. Knowing the nominal G we would like to give a reduced model which is also good by considering the uncertain one.

In view of Theorem 2 and Lemma 5 a reasonable choice is to consider the Kolmogorov n -width approach, and to use as a reduce order model

$$G_r(z) = \sum_{i=1}^K \hat{G}_i(z), \quad \hat{G}_i = S_{|p_i|m_i}^{p_i} G_i,$$

where $r = \sum_{i=1}^K |p_i|m_i$ and $\mathbf{p}_i = p_i$ in the real case (Laguerre) while $\mathbf{p}_i = [p_i, \bar{p}_i]$ in the complex one (Kautz).

Theorem 6. Having a nominal model $G = \sum_{i=1}^K G_i$, the reduced model as above, and under the assumptions made on the uncertainty set we have the following bound:

$$\|G_u - G_r\|_q \leq \sum_{i=1}^K C_{q,i} \rho_i^{m_i}. \quad (20)$$

Proof. We can use (15) of Theorem 2. See also Remark 3.

We emphasize that this error bound reflects the worst-case paradigm of the n -width concept. E.g., if P denotes the set of reversed poles for G , one would be tempted to obtain a configuration satisfying

$$\mathbf{a}^* = \inf_{\mathbf{a} \in \mathbb{D}^r} \max_{p \in P} |B_{\mathbf{a}}(p)|.$$

Since the parameters of this \mathbf{a}^* does not coincide with the centres of the covering hyperbolic circles, the corresponding reduced model might produce bigger errors in the worst

case than the choice of the theorem. Analogously, different reduced order models might produce better bound on the nominal model while failing on the uncertainty set.

Note that the bound of Theorem 6 is also valid for the \mathcal{H}_∞ norm for the entire uncertainty set, while the reduced models obtained by the classical approaches based on the nominal model are tuned only for the \mathcal{H}_2 norm.

At the end of this section we summarize the main steps of the proposed algorithm. The input data is the nominal plant $G = C(zI - A)^{-1}B + D$ with its inverse pole set

$$P = \{p_i = 1/\bar{\lambda}_i, \lambda_i \in \text{eig}A\}.$$

An uncertainty description provided by the partitioning hyperbolic circles H_i and corresponding sets \mathcal{B}_{H_i} is also given. As a first step of the algorithm we need a quasi modal decomposition that fit to this partitioning. E.g., this can be performed by using a block Schur decomposition and a suitable permutation of the eigen-blocks. Note, that the eigen-decomposition is numerically not reliable in the context of large scale systems. Once we have the transfer functions G_i we apply a Laguerre or a Kautz expansion parameterized by p_i , the centre of the circle H_i . The value for the expansion length m_i can be selected based on the radius ρ_i of the hyperbolic circle, which determines the worst case decay rate. If other information is available there is a room for a weighted selection depending on the role of the term G_i (e.g., value of its gain relative to the norm of G). Finally the reduced model is given by

$$G_r(z) = \sum_{i=1}^K (S_{|p_i|}^{m_i} G_i)(z).$$

4. NUMERICAL EXAMPLES

The aim of this section is to illustrate the proposed algorithm in the two basic configurations, i.e., when the centre of the uncertainty ball is a real one and also the case when this pole is complex (two clusters with conjugate complex centres). In the first case we are dealing with a Laguerre expansion, while in the second a Kautz series is involved.

In both of the cases we consider a nominal transfer function G with its poles that determines pole clusters that corresponds to the two assumptions.

Thus in the first case we have the plant

$$G(z) = \frac{0.4071 \cdot 10^{-3}}{z^5 - 3.9567z^4 + 6.2637z^3 - 4.9593z^2 + 1.9638z - 0.3112}, \quad (21)$$

having the nominal pole set

$$P = \{0.8633, 0.8200 \pm 0.0755i, 0.7267 \pm 0.0591i\}.$$

In the second case the nominal plant is

$$G(z) = \frac{0.0890}{n_1(z)n_2(z)}, \quad (22)$$

$$n_1(z) = z^4 - 2.0134z^3 + 2.2188z^2 - 1.2171z + 0.3643,$$

$$n_2(z) = z^6 - 2.5732z^5 + 3.6882z^4 - 3.1812z^3 + 1.8304z^2 - 0.6331z + 0.1213,$$

having the nominal pole sets

$$P_1 = \{0.4800 + 0.6234i, 0.5267 + 0.5578i, 0.4033 + 0.6234i, 0.4833 + 0.5020i, 0.4000 + 0.5414i\},$$

$$P_2 = \{0.4800 - 0.6234i, 0.5267 - 0.5578i, 0.4033 - 0.6234i, 0.4833 - 0.5020i, 0.4000 - 0.5414i\}.$$

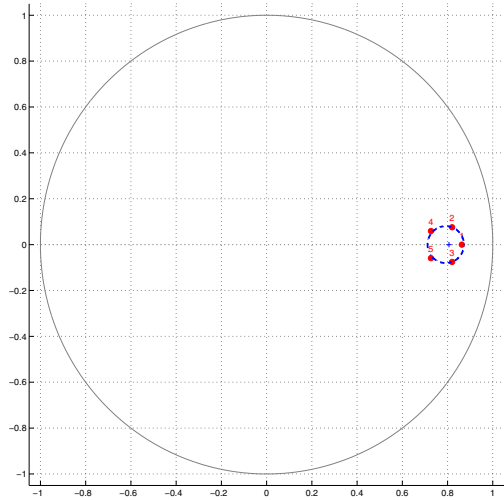


Fig. 1. One uncertainty circle with real centre (Laguerre case)

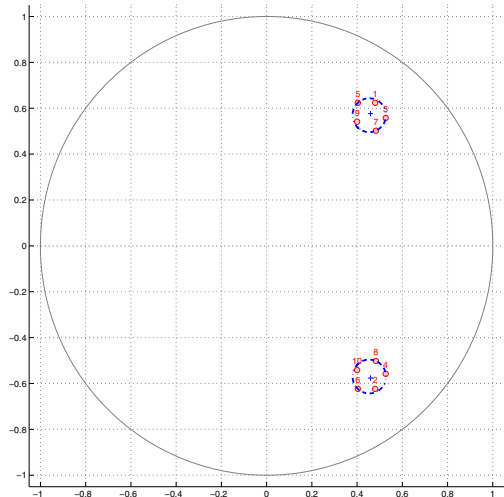


Fig. 2. Two uncertainty circles with conjugate complex centres (Kautz case)

Figure 1 and 2, respectively, show the minimal covering circles \tilde{H}_i , with centres 0.8064 and $0.4592 \pm 0.5764i$, that contains the nominal poles. In our problem setting the definition of the uncertainty set, i.e., the hyperbolic circles H_i is also needed. Since the covering circles associated to the nominal poles are contained in these uncertainty sets, the hyperbolic radius $\tilde{\rho}_i$ of \tilde{H}_i , i.e., 0.2228 and 0.1593 , respectively, gives the lowest rates in the error bounds (20). For simplicity, in this academic example we assume that the centre of the hyperbolic circles H_i and \tilde{H}_i , describing the pole-uncertainty set, coincides.

On Figure 3 and 4 we show the Bode plot of the reduced model corresponding to (21) and (22), respectively. To illustrate the effect of the uncertainty, Bode plots of the random plants with the poles in these sets are also

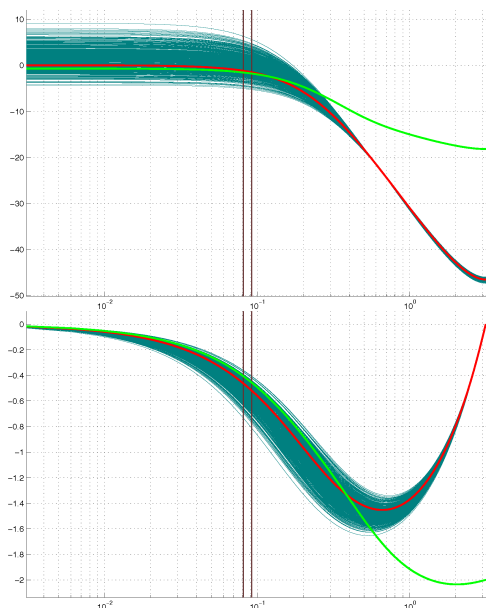


Fig. 3. Laguerre approximation $m = 3$ vs. 5 original poles depicted. Red line denotes the nominal model while green line represents the reduced model.

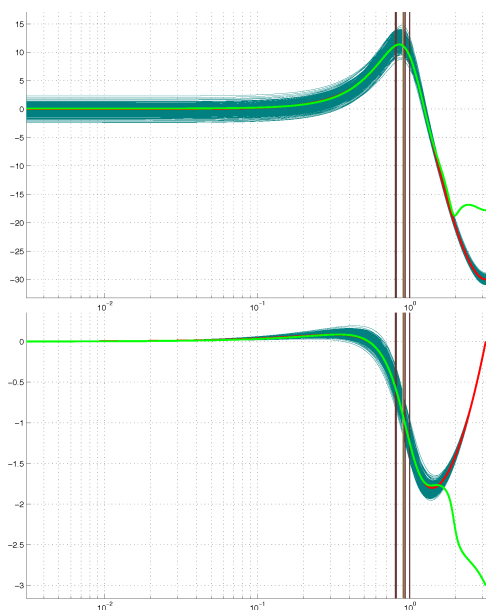


Fig. 4. Kautz approximation $m = 3$ vs. 10 original poles

5. CONCLUSION

This paper considered the model reduction problem of large scale systems formulated on the uncertainty set defined by a predefined collection of pole clusters. These clusters are specified as hyperbolic circles through their centre and radius. For the sake of simplicity only systems with simple poles are considered and the possible uncertainty of the gains is quantified by a norm condition on the residuals. The general case can be developed along the same ideas leading only to more complicated formulae.

The proposed approach is based on a results of the Kolmogorov n -width theory: useful bounds are given for

the worst case approximation error – both \mathcal{H}_2 and \mathcal{H}_∞ – in terms of the hyperbolic distance related to the sets of uncertain poles. The model reduction strategy uses only this a priori information. Moreover, the reduced model is computed as a combination of truncated Laguerre and Kautz expansions parameterized by the centres of the uncertainty sets.

REFERENCES

- Ahlfors, L. (1973). *Conformal invariants*. McGraw-Hill, New York.
- Antoulas, A., Sorensen, D.C., and Gugercin, S. (2001). A survey of model reduction methods for large scale systems. In *Contemporary Mathematics*, volume 280, 193–219. AMS Publications.
- Bui-Thanh, T., Willcox, K., and Ghattas, O. (2007). Model reduction for large-scale systems with high-dimensional parametric input space. *SIAM J. Sci. Comput.*, 30(6), 3270–3288.
- Dolgin, Y. and Zeheb, E. (2005). Model reduction of uncertain systems retaining the uncertainty structure. *System & Control Letters*, 54, 771–779.
- Duren, P.L. (1970). *Theory of \mathcal{H}_p spaces*. Academic Press.
- e Silva, T.O. (1996). A n -width result for the generalized orthonormal basis function model. In *Preprints of the 13th IFAC World Congress*, 375–380.
- Flagg, G., Beattie, C., and Gugercin, S. (2012). Convergence of the iterative rational krylov algorithm. *Systems & Control Letters*, 61(6), 688–691.
- Garnett, J. (1981). *Bounded Analytic Functions*. Academic Press.
- Goncalves, E., Palhares, R., Takahashi, R., and Chasin, A. (2009). Robust model reduction of uncertain systems maintaining uncertainty structure. *International Journal of Control*, 82(11), 2158–2168.
- Gugercin, S., Antoulas, A.C., and Beattie, C. (2008). \mathcal{H}_2 model reduction for large-scale linear dynamical systems. *SIAM. J. Matrix Anal. & Appl.*, 30(2), 609–638.
- Heuberger, P.S.C., Van den Hof, P.M.J., and Wahlberg, B. (2005). *Modeling and Identification with Rational Orthonormal Basis Functions*. Springer-Verlag.
- Lorentz, G., v. Golitschek, M., and Makovoz, Y. (1996). *Constructive Approximation*. Springer-Berlin.
- Meier, L. and Luenberger, D. (1967). Approximation of linear constant systems. *IEEE Transactions on Automatic Control*, 12(5), 585–588.
- Mi, W., Qian, T., and Wan, F. (2012). A fast adaptive model reduction method based on Takenaka–Malmquist systems. *Systems and Control Letters*, 61(1), 223–230.
- Schipp, F. and Bokor, J. (1998). Identification in Laguerre and Kautz Bases. *Automatica*, 34, 463–468.
- Tóth, R. (2008). *Modeling and Identification of Linear Parameter-Varying Systems, an Orthonormal Basis Function Approach*. Ph.D. thesis, Delft University of Technology.