Elsevier Editorial System(tm) for Discrete Mathematics
Manuscript Draft

Manuscript Number: DM 17980R1

Title: Distance domination versus iterated domination

Article Type: Special Issue: FrenchCC 2010

Keywords: dominating set; distance domination; class of connected graphs; forbidden induced subgraph

Corresponding Author: Dr. Gabor Bacso,

Corresponding Author's Institution: Hungarian Academy of Sciences

First Author: Gabor Bacso

Order of Authors: Gabor Bacso

Abstract: A k-dominating set in a graph G is a set S of vertices such that every vertex of G is at distance at most k from some vertex of S. Given a class D* of finite simple graphs closed under connected induced subgraphs, we completely characterize those graphs G in which every connected induced subgraph has a connected k-dominating subgraph isomorphic to some D in D*. We apply this result to prove that the class of graphs hereditarily D*-dominated within distance k is the same as the one obtained by iteratively taking the class of graphs hereditarily dominated by the previous class in the iteration chain. This strong relation does not remain valid if the initial hereditary restriction on D* is dropped.

Revision notes

All comments of the referees have been applied.

Distance domination versus iterated domination *

Gábor Bacsó and Zsolt Tuza †

Computer and Automation Institute
Hungarian Academy of Sciences
H–1111 Budapest, Kende u. 13–17, Hungary

Latest update on 29 Aug 2010

Abstract

A k-dominating set in a graph G is a set S of vertices such that every vertex of G is at distance at most k from some vertex of S. Given a class $\mathcal{D}$ of finite simple graphs closed under connected induced subgraphs, we completely characterize those graphs G in which every connected induced subgraph has a connected k-dominating subgraph isomorphic to some $D \in \mathcal{D}$. We apply this result to prove that the class of graphs hereditarily $\mathcal{D}$-dominated within distance k is the same as the one obtained by iteratively taking the class of graphs hereditarily dominated by the previous class in the iteration chain. This strong relation does not remain valid if the initial hereditary restriction on $\mathcal{D}$ is dropped.

* Research supported in part by the Hungarian Scientific Research Fund, grant OTKA 81493.

† Also affiliated with the Department of Computer Science and Systems Technology, University of Pannonia, Veszprém, Hungary

1 Introduction

A k-dominating set in a graph G is a set S of vertices such that every vertex of G is at distance at most k from some vertex of S. Although the bulk of literature on graph domination concentrates on the case $k = 1$ (that means thousands of papers), there are many interesting theorems on general k, too. We cite the survey [5] for a nice collection of results and references.

Here we are interested in the *structure* of k-dominating subgraphs. That is, we look for conditions under which a graph surely admits a k-dominating set that induces a subgraph belonging to a prescribed graph class $\mathcal{D}$. Among the requirements, connectivity will play a central role both for dominating subgraphs and for the graphs to be dominated.

The main result of this paper is Theorem 3, dealing with graph classes closed under connected induced subgraphs. It states that distance domination can be equivalently characterized with the recursive application of an operator ('*Dom*', to be defined later). An auxiliary result along the proof, that we call 'Legged Cycle Lemma' (Lemma 1), provides a necessary and sufficient condition in terms of forbidden induced subgraphs, and hence may be of interest in its own right, too.

1.1 Definitions and notation

We consider finite, simple graphs only. As usual, for a graph G we denote by $V(G)$ and $E(G)$ the vertex set and the edge set, respectively. Moreover, P_n and C_n denote the chordless path and cycle, respectively, on n vertices.

In this paper we shall deal with *induced* subgraphs. In this context a graph G is said to be H*-free* if it does not contain H as an induced subgraph. For a set $\mathcal{H}$ of graphs, the class of graphs which are H-free for every $H \in \mathcal{H}$, will be denoted by $Forb\,\mathcal{H}$.

A vertex set $S \subseteq V(G)$ is called k*-dominating* if for each $v \in V(G) \setminus S$ there exists a $w \in S$ such that the distance of v and w is at most k. "Dominating" and "1-dominating" mean the same. An induced subgraph D of G is k*-dominating* if its underlying vertex set $V(D)$ is k-dominating.

Let $\mathcal{D}$ be a class of graphs. We say that $\mathcal{D}$ is *compact* if it is closed under taking connected induced subgraphs. Moreover, $\mathcal{D}$ is *concise* if it is compact and contains connected graphs only.

A graph G is *minimal not-in-$\mathcal{D}$* if it is connected, $G \notin \mathcal{D}$, and all of its proper connected induced subgraphs are in $\mathcal{D}$.

A graph G is *$\mathcal{D}$-dominated* if there exists a dominating connected induced subgraph $D \in \mathcal{D}$ in G. A graph G is *hereditarily dominated by $\mathcal{D}$* if each of its connected induced subgraphs is $\mathcal{D}$-dominated.

The class $Dom_k\,\mathcal{D}$ consists of the graphs G for which every connected induced subgraph H of G is k-dominated by some connected graph $D \in \mathcal{D}$. This Dom_k is an operator acting on graph classes. Subscript '1' will be omitted, i.e. $Dom_1\,\mathcal{D} = Dom\,\mathcal{D}$.

A *connected* graph G is *minimal non-$\mathcal{D}$-dominated* if it is not $\mathcal{D}$-dominated but all of its proper connected induced subgraphs are.

To attach (to put) a leaf to a given vertex v of graph H means to add a new vertex v' to H such that v is the only neighbor of v' in H. The *leaf-graph* of a connected graph H is the graph obtained from H by attaching a leaf to each of its *non-cutting* vertices. The leaves will be pairwise non-adjacent, by definition. The leaf-graph of H will be denoted by $F(H)$. For example, $F(K_1) = K_2$ and $F(P_n) = P_{n+2}$ if $n \geq 2$.

Here we remark that, in general, the class $Dom\,\mathcal{D}$ contains disconnected graphs, too; but the definition of the operator Dom allows this, and its repeated application $Dom\,Dom\,\mathcal{D}$ will be well-defined.

For any operator Φ, operating on a set X and having its values in X, for arbitrary $x \in X$ and integer $k \geq 1$, the notation $\Phi^k(x)$ means the element obtained from x by applying Φ k times. We may also write $\Phi^0(x) = x$. For example, if H is a connected graph and $k \geq 0$ is any integer, $F^k(H)$ is the graph obtained from H by attaching a pendant path of length k to each non-cutting vertex of H.

Let us denote by $\mathcal{M}_k\mathcal{D}$ the set of minimal connected forbidden induced subgraphs for the class of graphs $Dom^k(\mathcal{D})$ (also here, we simply write $\mathcal{MD}$ for $\mathcal{M}_1\mathcal{D}$). This $\mathcal{M}_k\mathcal{D}$ is well-defined because membership in $Dom^k(\mathcal{D})$ is an additive induced-hereditary property for all $\mathcal{D}$ and all $k \geq 1$. (For the general theory of hereditary properties, see the survey [4].)

1.2 Some earlier results

Here we cite the main results of the papers [3], [1] and [7]. These will give the base for the proof of *Legged Cycle Lemma*, and through that, for *Theorem 3*, which will be stated in Section 2 and proved in Section 3.

The original problem, solved by these theorems was the following:

> *Given a (concise) class $\mathcal{D}$ of graphs, which are the minimal non-$\mathcal{D}$-dominated graphs?*

The non-2-connected case of this problem was solved about a decade ago:

Theorem 1 (Cut-point Lemma [3]) *Let $\mathcal{D}$ be a concise and nontrivial class of graphs. A graph G with at least one cut-point is minimal non-$\mathcal{D}$-dominated if and only if it is isomorphic to a leaf-graph $F(L)$, where $L \neq K_1$ is a graph minimal not-in-$\mathcal{D}$.*

For the 2-connected case, recently both [1] and [7] gave a solution, independently. To state the result, we need some further definitions.

For any class $\mathcal{C}$ of graphs, we denote by $\Theta(\mathcal{C})$ the minimum element of the set $\{j : P_j \notin \mathcal{C}\}$ if it is nonempty. The classes can be grouped into two types from our point of view:

Type 1 All chordless paths are elements of $\mathcal{D}$.

Type 2 Some path is not in $\mathcal{D}$.

Theorem 2 (Main Theorem in [1, 7]) *Let $\mathcal{D}$ be a nonempty concise class of graphs, different from the class of all connected graphs.*

(*i*) *If $\mathcal{D}$ is of Type 1 then there is no 2-connected minimal non-$\mathcal{D}$-dominated graph.*

(*ii*) *If $\mathcal{D}$ is of Type 2 then the only* 2*-connected minimal non-$\mathcal{D}$-dominated graph is the chordless cycle C_{t+2}, where $t = \Theta(\mathcal{D})$.*

Remark For non-compact classes, the original problem is also solved in [1] and [7] (the former dealing with the 2-connected case only), but here we do not use the general solution.

2 The results

In [6] it was asked whether the following equation is true for every $\mathcal{D}$:

$$Dom_2\,\mathcal{D} = Dom\,Dom\,\mathcal{D} \quad ? \tag{1}$$

We may ask the same question for general k: For what classes $\mathcal{D}$ of graphs is

$$Dom_k\,\mathcal{D} = Dom^k(\mathcal{D}) \tag{2}$$

valid?

For arbitrary $\mathcal{D}$, even (1) is not true, a counterexample is given in [2] (Proposition 1, page 127). However, we shall prove

Theorem 3 *If $\mathcal{D}$ is compact then the equation* (2) *is valid for all k.*

We will derive Theorem 3 from the results mentioned in Section 1.2, using the following statement, which is of interest in itself, too. To state it, we need the notation

$$\mathcal{F}_k := \{F^k(M) : M \text{ minimal not-in-}\mathcal{D}\}.$$

Recall that Types 1 and 2 have been defined before Theorem 2. The next assertion characterizes the class $\mathcal{M}_k\mathcal{D}$ of minimal connected forbidden induced subgraphs of $Dom^k(\mathcal{D})$.

Lemma 1 (Legged Cycle Lemma) *Let $\mathcal{D}$ be a compact class.*

(i) If $\mathcal{D}$ is of Type 1 then $\mathcal{M}_k\mathcal{D} = \mathcal{F}_k$.

(ii) Let $\mathcal{D}$ be of Type 2 and $\theta := \Theta(\mathcal{G})$, where $\mathcal{G}$ is the class of connected graphs in $Dom^{k-1}(\mathcal{D})$. Then

$$\mathcal{M}_k\mathcal{D} = \mathcal{F}_k \cup \{F^i(C_{\theta+2-2i}) : 0 \le i \le k-1\}.$$

In the concluding section we will show that the condition of compactness cannot be omitted, for any $k \ge 2$.

3 The proofs

Proof of the Legged Cycle Lemma

We apply induction on k. The class $\mathcal{D}$ is assumed to be compact; thus, by the results mentioned above, the Lemma is true for $k = 1$. That is,

$\mathcal{M}_1\mathcal{D} = \{F(M) : M$ minimal-not-in-$\mathcal{D}\}$ if $\mathcal{D}$ is of Type 1 and

$\mathcal{M}_1\mathcal{D} = \{C_{\tau+2}\} \cup \{F(M) : M$ minimal-not-in-$\mathcal{D}\}$ if $\mathcal{D}$ is of Type 2 and $\tau := \Theta(\mathcal{D})$.

We now suppose that the Lemma is true for some $k \geq 1$, and will prove it for $k + 1 =: l$. We shall use the following simple observation, whose proof is omitted.

Claim 1 *Let $\mathcal{D}$ be compact. Then for any $k \geq 1$, all of $Dom_k\mathcal{D}$, $Dom^k(\mathcal{D})$ and $\mathcal{D}$ have the same type and, for Type 2, $\Theta(Dom_k\mathcal{D}) = \Theta(Dom^k(\mathcal{D})) = \Theta(\mathcal{D}) + 2k$.* □

Let now $\mathcal{E} := Dom^k(\mathcal{D})$. For the proof of the Lemma, we argue depending on the type of $\mathcal{E}$.

I. Suppose $\mathcal{E}$ is of Type 1.

By the induction hypothesis, $\mathcal{M}_{l-1}\mathcal{D} = \mathcal{M}_k\mathcal{D} = \mathcal{F}_k$.

Obviously, $Dom^l(\mathcal{D}) = Dom\,\mathcal{E}$. Let us observe that $\mathcal{E}$ is also compact, thus, by the results quoted in Section 1.2,

$$\mathcal{M}_l\mathcal{D} = \mathcal{M}\mathcal{E} = \{F(L) : L \text{ minimal not-in-}\mathcal{E}\} = \{F(L) : L \text{ minimal not-in-}Dom^{l-1}(\mathcal{D})\} = \{F(L) : L \in \mathcal{M}_{l-1}\mathcal{D}\}.$$

By the induction hypothesis, this is equal to

$$\{F(L) : L \in \{F^{l-1}(M) : M \in \mathcal{M}\mathcal{D}\}\} = \{F^l(M) : M \in \mathcal{M}\mathcal{D}\} = \mathcal{F}_l.$$

Hence, we are done if $\mathcal{E}$ (and $\mathcal{D}$) is of Type 1.

II. Suppose $\mathcal{E}$ is of Type 2, and let $\nu := \Theta(\mathcal{E})$.

Similarly as in **I**, $Dom^l(\mathcal{D}) = Dom\,\mathcal{E}$. Referring again to Section 1.2, we begin with the first equation in the proof for Type 1 and continue with a set, adding a single cycle to the original one. In this way we derive

$$\mathcal{M}_l\mathcal{D} = \mathcal{ME} = \{F(L) : L \text{ minimal not-in-}\mathcal{E}\} \cup \{C_{\nu+2}\} = \{F(L) : L \text{ minimal not-in-}Dom^{l-1}(\mathcal{D})\} \cup \{C_{\nu+2}\} = \{F(L) : L \in \mathcal{M}_{l-1}\mathcal{D}\} \cup \{C_{\nu+2}\}.$$

This is equal to

$$\{F(L) : L \in \{F^{l-1}(M) : M \in \mathcal{MD}\} \cup \{F^i(C_{\theta+2-2i}) : 0 \le i \le k-1\}\} \cup \{C_{\nu+2}\}$$

where $\theta = \theta(Dom^{k-1}(\mathcal{D})) = \nu - 2$. Thus we obtain

$$\{F^l(M) : M \in \mathcal{MD}\} \cup \{F^j(C_{\nu-2j}) : 1 \le j \le k\} \cup \{C_{\nu+2}\} = \mathcal{F}_l \cup \{F^j(C_{\nu-2j}) : 0 \le j \le k\}.$$

Using Claim 1, we obtain just the list in the *Legged Cycle Lemma* and we are done for **II**, as well.

Proof of Theorem 3

a) First, we show: $Dom_k\,\mathcal{D} \supseteq Dom^k(\mathcal{D})$.

We can use induction on k again. For $k = 1$ there is nothing to prove. Let us consider a graph G in $Dom^{k+1}(\mathcal{D})$. By assumption, G has a dominating connected induced subgraph $D \in Dom^k(\mathcal{D})$. Using the induction hypothesis, D is in $Dom_k\,\mathcal{D}$, that is, D has some k-dominating subgraph $H \in \mathcal{D}$. This H will be $(k+1)$-dominating in G, and the assertion of **a)** follows.

b) Thus, for the proof of *Theorem 3*, it remains to show: $Dom_k\,\mathcal{D} \subseteq Dom^k(\mathcal{D})$.

To prove this, we need the Legged Cycle Lemma. If a graph is not in $Dom^k(\mathcal{D})$ then it contains some minimal forbidden subgraph with respect to it, namely some graph G from the list of the Legged Cycle Lemma. We state that no such G is in $Dom_k\,\mathcal{D}$.

For a $\mathcal{D}$ of Type 1, it is enough to refute the membership in $Dom_k\,\mathcal{D}$ for graphs in $\mathcal{F}_k$. Consider any $G = F^k(M)$, where $M \notin \mathcal{D}$. A k-dominating subgraph Δ in G has to k-dominate every leaf and thus it has to contain at least one vertex from each of the k-paths attached to the non-cutting vertices of M. Hence, if Δ is connected, then $M \subseteq \Delta$. But this implies $\Delta \notin \mathcal{D}$ since $\mathcal{D}$ is compact.

For a $\mathcal{D}$ of Type 2, the argument for graphs in $\mathcal{F}_k$ is the same as above, therefore it is enough to deal with a graph $G = F^i(C_\eta)$ where $\eta = \theta - 2i + 2$ and $0 \le i \le k-1$. We see by Claim 1 that $\eta \ge \Theta(\mathcal{D}) + 2$. Let us consider a minimal k-dominating connected induced subgraph Δ in G. The lower bound on η implies $C_\eta \not\subseteq \Delta$, and so connectivity yields that $V(\Delta) \cap V(C_\eta)$ induces a path, say P. Take those vertices on the cycle, which are farthest from P. (This means two vertices if $|V(C_\eta) \setminus V(P)|$ is even, and just one vertex otherwise.) Consider the leaves at the ends of the pendant i-paths attached to them. Since P k-dominates those leaves,

$$|V(P)| \ge \eta - 2(k-i) = (\theta - 2i + 2) - 2(k-i) = \theta - 2(k-1) = \Theta(\mathcal{D})$$

must hold, the last step implied by Claim 1. Consequently, $P \notin \mathcal{D}$ and hence Δ is not in $\mathcal{D}$ either.

This refutes the membership in $Dom_k\,\mathcal{D}$ for every graph on the list, and completes the proof of Theorem 3. □

4 Concluding remarks

As we have mentioned above, for $k = 2$ the equation (2) is not true in general. Here we show that for any $k \ge 2$, there exists some class $\mathcal{D}$ of graphs for which (2) is not valid. By Theorem 3, such a class is not compact, of course.

For a given $k \ge 2$, let $G := F(C_{4k-1})$ and $\mathcal{D} = \mathcal{D}_k := \{H : diam(H) \le 2k-1\}$. With this notation we state

Proposition 1 $G \in Dom_k\,\mathcal{D} \setminus Dom^k(\mathcal{D})$.

Proof

First, we show $G \in Dom_k\,\mathcal{D}$.

Since $diam(C_{4k-1}) = 2k-1$, we have $C_{4k-1} \in \mathcal{D}_k$ and hence the cycle k-dominates (and even dominates) all subgraphs of G that contain C_{4k-1}. On the other hand, if a connected induced subgraph H of G misses a vertex v of C_{4k-1}, then because of connectivity it also misses the pendant neighbor v' of v. Now, keeping the path P induced by the $2k$ vertices of C_{4k-1} at distance at least k from v, we obtain a subgraph that k-dominates the entire

$G - v - v'$. Hence, $V(P) \cap V(H)$ k-dominates H, and the membership of G in $Dom_k\,\mathcal{D}$ is established.

Second, we show $G \notin Dom^k(\mathcal{D})$.

Suppose, for a contradiction, that $G \in Dom^k(\mathcal{D})$. Then G has a connected induced dominating subgraph Δ, being in $Dom^{k-1}(\mathcal{D})$. This Δ contains C_{4k-1} because each leaf v' has to be dominated, and connectivity implies that the neighbor v of v' on the cycle is necessarily contained in Δ.

Omitting a vertex from the cycle, we obtain a path $P \cong P_{4k-2}$. The shortest subpath $(k-1)$-dominating P would have length = diameter = $2k$. Consequently, Δ is not even in $Dom_{k-1}\,\mathcal{D}$, a contradiction which completes the proof of the second statement and *Proposition 1*, too. □

References

[1] G. Bacsó: Complete description of forbidden subgraphs in the structural domination problem, Discrete Math. 309 (2009), 2466–2472.

[2] G. Bacsó, A. Tálos, Zs. Tuza: Graph domination in distance two, Discuss. Math. Graph Theory 25 (2005), 121–128.

[3] G. Bacsó, Zs. Tuza: Structural domination in graphs, Ars Combinatoria 63 (2002), 235–256.

[4] M. Borowiecki, I. Broere, M. Frick, P. Mihók, G. Semanišin: Survey of hereditary properties of graphs, Discuss. Math. Graph Theory 17 (1997), 5–50.

[5] M. A. Henning: Distance domination in graphs, Domination in Graphs: Advanced Topics (T. W. Haynes et al., Eds.), Marcel Dekker, New York, 1998, 321–349.

[6] Attila Tálos: On 2-Dominating Subgraphs, Master Thesis, Eötvös L. University, Budapest, 1992. (2-domináns részgráfokról, in Hungarian)

[7] Zs. Tuza: Hereditary domination in graphs: Characterization with forbidden induced subgraphs, SIAM J. Discrete Math. 22 (2008), 849–853.