



## Technical communique

The most powerful unfalsified linear parameter-varying model<sup>☆</sup>Ivan Markovsky<sup>a,b,\*</sup>, Chris Verhoeck<sup>c</sup>, Roland Tóth<sup>c,d</sup><sup>a</sup> Catalan Institution for Research and Advanced Studies, Pg. Lluís Companys 23, 08010 Barcelona, Spain<sup>b</sup> Centre Internacional de Mètodes Numèrics en Enginyeria, Gran Capità, 08034 Barcelona, Spain<sup>c</sup> Control Systems Group, Eindhoven University of Technology, 5600 Eindhoven, The Netherlands<sup>d</sup> Systems and Control Lab, HUN-REN Institute for Computer Science and Control, 1111 Budapest, Hungary

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## ABSTRACT

The most powerful unfalsified model (MPUM), i.e., the least complex exact model for the given data, is well established for linear time-invariant (LTI) systems. It has not been generalized for linear parameter-varying (LPV) systems. In order to do this, we define the notions of complexity for LPV systems with shifted-affine scheduling dependence. The MPUM leads to identifiability conditions and a method for exact LPV system identification. The method is based on lifting the LPV system to a higher dimensional space and LTI embedding in the lifted space. It is made rigorous by proving a formal connection between the parameters of the LTI embedding and the original LPV system.

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## 1. Introduction

The *most powerful unfalsified model (MPUM)* of a vector time series, i.e., the least complex exact model in a given class of systems for data consisting of vector time series, laid a clear and rational foundation for system identification (Willems, 1986b). The problem of finding the MPUM is a generalization of the partial realization problem, when the data is a general trajectory of the system rather than the impulse response. The MPUM was defined originally for the class of *linear time-invariant (LTI)* systems and data being an infinite time series. Later on, it was used to generalize and give a system theoretic interpretation of the Berlekamp–Massey algorithm and was modified for the case of finite time series (Markovsky & Dörfler, 2023, Section IV) and time series with missing values (Alsalti et al., 2025; Markovsky, 2016). It has not been defined and used for other classes of systems beyond LTI, the main issue being the generalization of the notion of model complexity.

The problem of finding the MPUM is an exact (deterministic) identification problem. The algorithms proposed in Willems

(1986b) inspired the development of the N4SID subspace identification methods (Van Overschee & De Moor, 1996). In particular, the fact that a state sequence of a state-space realization of an unknown system can be obtained from a single trajectory of the system was proposed in Willems (1986b) as a method for computing the MPUM.

In this communique, we generalize the notion of the MPUM for the class of *linear parameter-varying (LPV)* systems, i.e., linear systems whose behavior is defined by a linear relation that depends on a signal, called the *scheduling signal* (Tóth, 2010). The LPV model class is a generalization of the LTI one and is an intermediate step towards general non-linear systems (Verhoeck, Koelwijn et al., 2023). The scheduling variable is often assumed measurable.

In this work, the laws describing the LPV system are assumed to have a representation with an affine dependence on the scheduling variable (see (2) and (5)). We call this subclass *shifted-affine LPV (SALPV)*. It has been shown that the SALPV model class has a minimal state-space realization with affine dependence (Tóth et al., 2012), which makes it directly applicable for control design as it has been also demonstrated in El-Dine et al. (2012). Furthermore, recent developments in direct data-driven LPV controller synthesis are based on the SALPV model class and are successfully applied in practice (Verhoeck, Abbas et al., 2023).

Our objective is to define the MPUM for the class of SALPV systems and find methods for its computation. To this end, we transform the SALPV identification problem to a problem that is similar to LTI system identification, by a lifting transformation: a Kronecker product of the manifest variable and the scheduling variable. The connection between the SALPV and the lifted LTI system is established via the kernel representations of the original and lifted systems.

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We use the behavioral approach, *i.e.*, we view dynamical systems as sets of trajectories (Markovsky et al., 2023; Willems, 2007). Although *per se* the behavioral approach is applicable to nonlinear as well as linear systems and to time-varying as well as time-invariant systems, there have been only a few attempts to develop behavioral systems theory for nonlinear and/or time-varying systems, *e.g.*, LPV systems (Tóth, 2010; Tóth et al., 2011), external representations of nonlinear systems (Nijmeijer & Schaft, 1990), and convex conical behaviors (Padoan et al., 2023). Next, we review the behavioral setting for LPV systems.

A parameter-varying system  $\mathcal{B}$  is a dynamical system, whose variables are partitioned into a *scheduling variable*  $\rho$  and a *manifest variable*  $w$ . Let  $\Pi_w \begin{bmatrix} \rho \\ w \end{bmatrix} := w$  be the projection on the manifest variable and  $\Pi_\rho \begin{bmatrix} \rho \\ w \end{bmatrix} := \rho$  be the projection on the scheduling variable. The manifest behavior  $\Pi_w \mathcal{B}$  for a fixed scheduling signal  $\rho$  is denoted by  $\mathcal{B}_\rho$ . If  $\mathcal{B}_\rho$  is linear for all  $\rho \in \Pi_\rho \mathcal{B}$ , the system  $\mathcal{B}$  is called LPV. For a given  $\rho$ , the projected behavior  $\mathcal{B}_\rho$  is *linear time-varying (LTV)*. We will refer to  $\mathcal{B}_\rho$  as the “behavior along  $\rho$ ”. We consider discrete-time LPV systems with *unrestricted scheduling variable*, *i.e.*,  $\Pi_\rho \mathcal{B} = (\mathbb{R}^{n_\rho})^\mathbb{T}$ . The time-axis  $\mathbb{T}$  is the one-sided infinite  $[1, \infty)$  or a finite  $[1, T]$  interval.

The contributions are: (i) a novel characterization of the finite-horizon behavior  $\mathcal{B}_\rho|_T$  of a general LPV system  $\mathcal{B}$  in terms of a kernel representation (Section 2.1), (ii) definition and computational method for the MPUM in the SALPV model class (Section 2.2), and (iii) identifiability condition for SALPV systems that is verifiable from the data and the complexity of the data-generating system (Theorem 2). The results lead to a method that estimates a kernel representation of the system. The maximum-likelihood SALPV identification problem in the errors-in-variables setting is reformulated as a structured low-rank approximation problem. This opens the path for the development of new LPV identification methods. The results are demonstrated on an example.

## 2. Scheduling dependent kernel representation

First, we present the kernel representation of a general LPV system. Then, we introduce the class of LPV systems with shifted-affine scheduling dependence.

### 2.1. General case

Consider a *finite-dimensional* LPV system  $\mathcal{B}$  with  $n_\rho$  scheduling variables and  $q$  manifest variables. Then, there is a natural number  $\ell$  and functions

$$R_i : (\mathbb{R}^{n_\rho})^\mathbb{N} \times \mathbb{T} \rightarrow \mathbb{R}^{q \times q}, \quad \text{for } i = 0, 1, \dots, \ell, \quad (1)$$

such that for any scheduling signal  $\rho \in (\mathbb{R}^{n_\rho})^\mathbb{N}$ , the manifest behavior  $\mathcal{B}_\rho$  admits a representation

$$\underbrace{R_0(\rho, t)w(t) + R_1(\rho, t)w(t+1) + \dots + R_\ell(\rho, t)w(t+\ell)}_{=:(R(\rho, \sigma)w)(t)} = 0, \quad (2)$$

where  $(\sigma w)(t) := w(t+1)$ , is the unit shift operator. The manifest behavior  $\mathcal{B}_\rho$  of  $\mathcal{B}$  is therefore given as the kernel of the difference operator  $R(\rho, \sigma)$ :

$$\mathcal{B}_\rho = \ker R(\rho, \sigma) := \{ w \in (\mathbb{R}^q)^\mathbb{T} \mid (2) \text{ holds} \}. \quad (3)$$

Vice versa, for any set of parameters (1), equation (3) defines an LPV system  $\mathcal{B} = \{ (\rho, w) \mid \rho \in (\mathbb{R}^{n_\rho})^\mathbb{N} \text{ and } w \in \mathcal{B}_\rho \}$ .

Consider a finite horizon  $\mathbb{T} := \{1, \dots, T\}$  and the restriction  $w|_T := (w(1), \dots, w(T))$  of  $w \in (\mathbb{R}^q)^\mathbb{N}$  to  $\mathbb{T}$ . For  $t \in \mathbb{T}$ , equation (2) is equivalent to  $\mathcal{M}_T(R, \rho)w = 0$ , where  $\mathcal{M}_T(R, \rho) \in \mathbb{R}^{g(T-\ell) \times qT}$

is the polynomial multiplication, defined in Fig. 1. In the LTI case,  $\mathcal{M}_T(R, \rho)$  additionally has a block-Toeplitz structure and is called the multiplication matrix as it represents polynomial multiplication (Markovsky & Dörfler, 2023, Section II.B). This leads us to the characterization of the finite horizon behavior  $\mathcal{B}_\rho|_T$  as the kernel of  $\mathcal{M}_T(R, \rho)$

$$\mathcal{B}_\rho|_T := \{ w|_T \mid w \in \mathcal{B}_\rho \} = \ker \mathcal{M}_T(R, \rho), \quad (4)$$

which is a generalization of the explicit matrix characterization of the finite horizon behavior of LTI systems in Markovsky and Dörfler (2023). Given (1) and  $\rho \in (\mathbb{R}^{n_\rho})^\mathbb{T}$ ,  $\mathcal{M}_T(R, \rho)$  can be evaluated, giving us a basis for  $\mathcal{B}_\rho|_T$ . The availability of  $\mathcal{B}_\rho|_T$ , in turn, opens the path of solving analysis, noise filtering, and control problems, using the methods presented in Markovsky (2024). The result is applicable for general LPV systems.

### 2.2. Shifted-affine scheduling dependence

Next, we consider two special cases: (1) LTI systems and (2) SALPV systems. In the LTI case, the parameters (1) are constant in time. Then, (3) becomes the classical kernel representation defined by a polynomial matrix  $R(z) := R_0 + R_1z + \dots + R_\ell z^\ell$  and  $\mathcal{M}_T(R, \rho)$  is the polynomial multiplication matrix for  $R(z)$ . In the SALPV case, considered later in the communicate, the parameters (1) are

$$R_i(\rho, t) = R_i^0 + R_i^1 \rho_1(t+i) + \dots + R_i^{n_\rho} \rho_{n_\rho}(t+i), \quad (5)$$

for  $i = 0, 1, \dots, \ell$ , where the dependence of  $w(t+i)$  on the scheduling signal is limited to  $\rho(t+i)$ , *i.e.*, the time dependence of the scheduling signal shifts along with the manifest variable. The class of SALPV systems with  $n_\rho$  scheduling variables and  $q$  manifest variables is denoted by  $\mathcal{P}^{n_\rho, q}$ . When the dimensions  $n_\rho$  and  $q$  are understood from the context, they will be skipped from the notation  $\mathcal{P}^{n_\rho, q}$ .

For an SALPV system  $\mathcal{B} \in \mathcal{P}^{n_\rho, q}$ , with  $\rho_{\text{ext}} := \begin{bmatrix} 1 \\ \rho \end{bmatrix}$ ,

$$R_i(\rho(t+i)) = \underbrace{\begin{bmatrix} R_i^0 R_i^1 & \dots & R_i^{n_\rho} \end{bmatrix}}_{R'_i} (\rho_{\text{ext}}(t+i) \otimes I_q), \quad (6)$$

where  $\otimes$  is the Kronecker product. We define the matrix  $R' := \begin{bmatrix} R'_0 & R'_1 & \dots & R'_\ell \end{bmatrix}$ , which completely characterizes (3) in the case of shifted-affine scheduling dependence.

## 3. Exact identification of SALPV systems

### 3.1. SALPV model complexity

In the behavioral setting, the complexity of a system  $\mathcal{B}$  characterizes the “size” of the set  $\mathcal{B}$ , *i.e.*, the larger the set, the more complex the system is. This intuitive idea is formalized for the case of LTI systems by the triple (Markovsky & Dörfler, 2023)

$$\mathbf{c}(\mathcal{B}) := (\mathbf{m}(\mathcal{B}), \mathbf{l}(\mathcal{B}), \mathbf{n}(\mathcal{B})), \quad (7)$$

where  $\mathbf{m}(\mathcal{B})$  is the number of inputs of  $\mathcal{B}$ ,  $\mathbf{l}(\mathcal{B})$  is the lag of  $\mathcal{B}$ , and  $\mathbf{n}(\mathcal{B})$  is the order of  $\mathcal{B}$ . (The number of inputs  $\mathbf{m}(\mathcal{B})$ , lag  $\mathbf{l}(\mathcal{B})$ , and order  $\mathbf{n}(\mathcal{B})$  are properties of the system  $\mathcal{B}$  and are invariant of its representations (Willems, 1986a).) The rationale for defining the complexity of  $\mathcal{B}$  via (7) is that the dimension  $\dim \mathcal{B}|_T$  of the finite horizon behavior for “large enough” horizon, namely  $T \geq \mathbf{l}(\mathcal{B})$ , is  $\mathbf{m}(\mathcal{B})T + \mathbf{n}(\mathcal{B})$ , so that it is determined by  $\mathbf{c}(\mathcal{B})$ .

As in the LTI case, for an LPV system  $\mathcal{B}$ , the functions  $\mathbf{m}(\mathcal{B})$ ,  $\mathbf{l}(\mathcal{B})$  and  $\mathbf{n}(\mathcal{B})$  are also well defined (Tóth, 2010). For a general LPV system, the complexity is determined by the triple  $(\mathbf{m}(\mathcal{B}), \mathbf{l}(\mathcal{B}), \mathbf{n}(\mathcal{B}))$  as well as the complexity of the functions in (1). The simplest nontrivial case of (1) is affine, *i.e.*, the SALPV class. Affine functions (1) have complexity that depends on the

$$\mathcal{M}_T(R, \rho) := \begin{bmatrix} R_0(\rho, 1) & R_1(\rho, 1) & \cdots & R_\ell(\rho, 1) \\ & R_0(\rho, 2) & R_1(\rho, 2) & \cdots & R_\ell(\rho, 2) \\ & & \ddots & \ddots & \ddots \\ & & & R_0(\rho, T-\ell) & R_1(\rho, T-\ell) & \cdots & R_\ell(\rho, T-\ell) \end{bmatrix} \in \mathbb{R}^{q(T-\ell) \times qT}.$$

Fig. 1. Definition of the multiplication matrix.

dimension  $n_\rho$  of the scheduling variable only. Thus, the complexity of  $\mathcal{B} \in \mathcal{P}^{n_\rho, q}$  is solely determined by the triple  $(\mathbf{m}(\mathcal{B}), \mathbf{l}(\mathcal{B}), \mathbf{n}(\mathcal{B}))$ . We define the complexity of an SALPV system by (7) and the subclass of SALPV systems  $\mathcal{P}^{n_\rho, q}$  with complexity bounded by  $(m, \ell, n)$  as

$$\mathcal{P}_{m, \ell, n}^{n_\rho, q} := \{ \mathcal{B} \in \mathcal{P}^{n_\rho, q} \mid \mathbf{c}(\mathcal{B}) \leq (m, \ell, n) \}.$$

### 3.2. MPUM in the class of SALPV systems

The MPUM for given data in a specified model class is the least complex system in the model class that fits the data exactly. The MPUM depends on the model class as well as the notion of complexity. For given data  $(\rho_d, w_d)$ , we denote by  $\text{mpum}(w_d)$  the MPUM (of  $w_d$ ) in the LTI model class  $\mathcal{L}^q$  and by  $\text{mpum}(\rho_d, w_d)$  the MPUM (of  $(\rho_d, w_d)$ ) in the SALPV model class  $\mathcal{P}^{n_\rho, q}$ .

For infinite data  $w_d \in (\mathbb{R}^q)^\mathbb{T}$ ,  $\text{mpum}(w_d) = \text{span}\{\sigma^t w_d \mid t \in \mathbb{N}\}$ , i.e.,  $\text{mpum}(w_d)$  is the span of  $w_d$  and all its shifts. Its construction can be viewed as an LTI embedding of the data  $w_d$ . More generally, given a set of infinite trajectories  $\mathcal{B} \subset (\mathbb{R}^q)^\mathbb{T}$ , its LTI embedding  $\text{LTI}(\mathcal{B}) := \text{span}\{\sigma^t w \mid t \in \mathbb{N} \text{ and } w \in \mathcal{B}\}$  is the smallest LTI system that includes  $\mathcal{B}$ .

LTI embedding is a procedure for construction of the MPUM in the class of LTI systems. It cannot be used in the SALPV case. One difficulty is that  $\mathcal{B}_{\rho_d}$ , for which there is data  $w_d$ , is not time-invariant. Another one is that there is no given data for other scheduling sequences. For a general LPV system, these difficulties make the problem of computing the MPUM ill-posed. For SALPV systems, however,  $\text{mpum}(\rho_d, w_d)$  is well-defined and therefore it can be constructed. Next, we present an algorithm for doing this.

### 3.3. The lifting operation

The key idea of finding  $\text{mpum}(\rho_d, w_d)$  is lifting the data. For signals  $\rho \in (\mathbb{R}^{n_\rho})^\mathbb{T}$  and  $w \in (\mathbb{R}^q)^\mathbb{T}$ , define the Kronecker product  $\rho \otimes w$ , point-wise in time:  $(\rho \otimes w)(t) := \rho(t) \otimes w(t) \in \mathbb{R}^{n_\rho q}$  and the “lifting” of  $(\rho, w) \in (\mathbb{R}^{n_\rho+q})^\mathbb{T}$  as

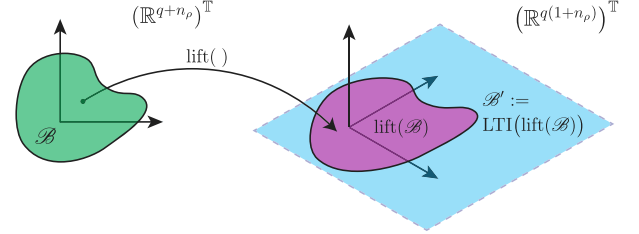
$$\text{lift}(\rho, w) := \begin{bmatrix} 1 \\ \rho \end{bmatrix} \otimes w \in (\mathbb{R}^{(1+n_\rho)q})^\mathbb{T}. \quad (8)$$

The word “lifting” in the name of  $\text{lift}(\rho, w)$  refers to the fact that it increases the dimension: the lifted signal  $\text{lift}(\rho, w)$  has dimension  $q' := (1 + n_\rho)q$ .

Applied on an LPV system  $\mathcal{B}$ , the “lift” operation acts on all signals in  $\mathcal{B}$ , producing another system:

$$\text{lift}(\mathcal{B}) := \{ \text{lift}(\rho, w) \mid \begin{bmatrix} \rho \\ w \end{bmatrix} \in \mathcal{B} \}.$$

The lifting and LTI embedding are used in the next section for the construction of  $\text{mpum}(\rho_d, w_d)$ , see Fig. 2 for the relationship between  $\mathcal{B}$ ,  $\text{lift}(\mathcal{B})$ , and the LTI embedding of  $\text{lift}(\mathcal{B})$ . Similar signal lifting, without formal characterization of the lifted behavior, has been considered extensively in the LPV subspace literature, see for example (Tóth et al., 2012; van Wingerden & Verhaegen, 2009; Verdult & Verhaegen, 2002).

Fig. 2. The LTI embedding of  $\mathcal{B} \in \mathcal{P}^{n_\rho, q} \subset (\mathbb{R}^{n_\rho+q})^\mathbb{T}$  is applied on the lifted system  $\text{lift}(\mathcal{B}) \subset (\mathbb{R}^{(n_\rho+1)q})^\mathbb{T}$ .

### 3.4. $\text{mpum}(\rho_d, w_d)$ computational method

The method for computing  $\text{mpum}(\rho_d, w_d)$  is: (1) construct the lifted signal  $w' := \text{lift}(\rho, w)$ , (2) find the LTI embedding  $\mathcal{B}' := \text{mpum}(w')$ , and (3) recover  $\mathcal{B} := \text{mpum}(\rho_d, w_d)$  from  $\mathcal{B}'$ . Step 3 relies on the bijective relation

$$R'(z) = R'_0 + R'_1 z + \cdots + R'_\ell z^\ell, \quad \text{with} \quad R'_i = [R_i^0 R_i^1 \cdots R_i^{n_\rho}]^T, \quad \text{for } i = 0, 1, \dots, \ell \quad (9)$$

between a kernel representation  $\mathcal{B}_\rho = \ker R(\rho, \sigma)$  of an SALPV system  $\mathcal{B}$  and a kernel representation  $\ker R'(\sigma)$  of the LTI embedding of the lifted system  $\text{lift}(\mathcal{B})$ .

**Lemma 1.** Consider an SALPV system  $\mathcal{B} \in \mathcal{P}_{(m, \ell, n)}^{n_\rho, q}$  with a kernel representation  $\mathcal{B}_\rho = \ker R(\rho, \sigma)$  and define  $R'$  via (9). Then,  $\text{LTI}(\text{lift}(\mathcal{B})) = \ker R'(\sigma)$ .

**Proof.** Substituting (6) in (2) and defining  $w'_d := \text{lift}(\rho_d, w_d)$ , we obtain the linear constant-coefficients-based difference equation

$$R'_0 w'_d(t) + R'_1 w'_d(t+1) + \cdots + R'_\ell w'_d(t+\ell) = 0, \quad (10)$$

which defines an LTI system  $\mathcal{B}' := \ker R'(\sigma)$ . By construction, for any  $(\rho_d, w_d) \in \mathcal{B}$ , the lifted signal  $w'_d \in \mathcal{B}'$ , so that  $\text{lift}(\mathcal{B}) \subseteq \ker R'(\sigma)$ . Moreover,  $\mathcal{B}'$  is the smallest LTI system containing  $\text{lift}(\mathcal{B})$ , so that  $\mathcal{B}' = \text{LTI}(\text{lift}(\mathcal{B}))$ .  $\square$

Step 3 of the method for computing  $\text{mpum}(\rho_d, w_d)$ , is thus the conversion (9) of the kernel parameters in  $R'$  defining  $\mathcal{B}'$  to the parameters (1) defining a kernel representation (3) of the MPUM. Although,  $\text{lift}(\mathcal{B})$  and  $\mathcal{B}'$  are not equal, they share the same kernel parameters  $R'$ , so that  $\mathcal{B}$  can be inferred from  $\mathcal{B}'$ , which in turn can be inferred from the lifted data by the LTI embedding procedure.

The relation (9) between the parameters of an SALPV system  $\mathcal{B}$  and the LTI embedding of  $\text{lift}(\mathcal{B})$  is the reason why subspace approaches using explicitly or implicitly lifting, such as the Ho–Kalman algorithm of Tóth et al. (2012), early LPV subspace methods in Verdult and Verhaegen (2002), and recent state-of-the-art subspace approaches (Cox & Tóth, 2021; van Wingerden & Verhaegen, 2009) work. The developed theory in this paper (combined with Petreczky et al., 2017) shows that such methods implicitly assume the shifted affine dependence on the scheduling variable and via lifting they embed the behavior of the LPV system in an LTI system, for which application of LTI realization

theory leads to a representation. By using the result of that representation, it is possible to perform a reformulation to obtain a representation of the LPV system with static affine dependence on the scheduling variable. Hence, our results show the core reasons why the extension of LTI subspace algorithms is possible to LPV systems and generalizes the implicit concepts of these papers towards a general theory of LPV behaviors and subspace identification methods.

**Note 1** (Discrepancy Between  $\text{lift}(\mathcal{B})$  and  $\mathcal{B}'$ ). Not every trajectory  $w' \in \mathcal{B}'$  is a trajectory of  $\text{lift}(\mathcal{B})$ , cf., Fig. 2. The discrepancy between  $\text{lift}(\mathcal{B})$  and  $\mathcal{B}'$  is due to the structure of (8). More specifically,  $\text{lift}(\mathcal{B}) = \mathcal{B}' \cap \mathcal{B}_{\text{str}}$ , where  $\mathcal{B}_{\text{str}} := \{\text{lift}(\rho, w) \mid \rho \in (\mathbb{R}^{n_\rho})^\mathbb{T} \text{ and } w \in (\mathbb{R}^q)^\mathbb{T}\}$ .

**Note 2** (Complexity of the LTI Embedding). The complexity of the LTI embedding  $\mathcal{B}'$  is

$$\mathbf{c}(\mathcal{B}') = \underbrace{(\mathbf{m}(\mathcal{B}) + n_\rho q, \mathbf{l}(\mathcal{B}), \mathbf{n}(\mathcal{B}))}_{m'} \quad (11)$$

Intuitively, the LTI embedding relaxes the  $n_\rho q$  variables  $\rho \otimes w$  of  $\text{lift}(\mathcal{B})$  by treating them as inputs (free variables). This is equivalent to removing the constraint  $w' \in \mathcal{B}_{\text{str}}$ .

For a known dimension  $n_\rho$  of the scheduling variable  $\rho_d$ , there is a one-to-one map between the complexity of an SALPV system and the complexity of its LTI embedding.

**Note 3.** In case of finite data, define the Hankel matrix

$$\mathcal{H}_\ell(w_d) := [w_d|_L \quad (\sigma w_d)|_L \quad \dots \quad (\sigma^{T_d-L} w_d)|_L].$$

Then, assuming that the lag  $\ell$  of the MPUM is a priori given, a kernel representation of  $\mathcal{B}'$  can be computed from the left null space of the Hankel matrix  $\mathcal{H}_{\ell+1}(w_d')$ . When the lag  $\ell$  is unknown, it can be found from  $w_d$  by a rank test as in the LTI case (Markovsky & Dörfler, 2023, Section IV).

### 3.5. Identifiability condition

Assuming that the data  $(\rho_d, w_d)$  is generated by a system  $\mathcal{B} \in \mathcal{P}_{m,\ell,n}^{n_\rho,q}$ , the *identifiability problem* is to find the data-generating system  $\mathcal{B}$  back from the data  $(\rho_d, w_d)$ . The following result gives conditions under which this is possible.

**Theorem 2.** Consider an SALPV system  $\mathcal{B} \in \mathcal{P}_{m,\ell,n}^{n_\rho,q}$  and let  $(\rho_d, w_d) \in \mathcal{B}|_{T_d}$  be a trajectory of  $\mathcal{B}$ . Under the generalized persistency of excitation condition

$$\text{rank } \mathcal{H}_{\ell+1}(\text{lift}(\rho_d, w_d)) = r := (n_\rho q + \mathbf{m}(\mathcal{B}))(\ell + 1) + \mathbf{n}(\mathcal{B}), \quad (12)$$

$\mathcal{B}$  is identifiable from  $(\rho_d, w_d)$ , i.e.,  $\text{mpum}(\rho_d, w_d) = \mathcal{B}$ .

**Proof.** By Markovsky and Dörfler (2023, Theorem 17), under the generalized persistency of excitation condition (12), the LTI embedding of the lifted data-generated system  $\mathcal{B}$  is identifiable, i.e.,  $\text{mpum}(\text{lift}(\rho_d, w_d)) = \text{LTI}(\text{lift}(\mathcal{B}))$ . Then, by Lemma 1, if  $\text{mpum}(\text{lift}(\rho_d, w_d)) = \ker R'(\sigma)$  and  $R$  is defined via (9), we have that  $\ker R(\rho, \sigma) = \mathcal{B}_\rho$ , for all  $\rho \in (\mathbb{R}^{n_\rho})^\mathbb{T}$ .  $\square$

Under the specified condition, the data generating system coincides with the MPUM, so that the result is constructive.

**Note 4.** Identifiability in the classical setting is defined as a property of the model structure. It characterizes distinguishability of the process and noise models (or the 1-step-ahead predictors associated with them) for each value of the parameter

vector (Gevers et al., 2009; Ljung, 1999). The classical notion of identifiability is therefore independent of data. In contrast, the notion of identifiability we use in the paper is a property of the data and the model class that guarantees unique recovery of the data-generating system. It is related to the conditions of the fundamental lemma (Willems et al., 2005) for a data-driven representation of an LTI system and the informativity concept of van Waarde et al. (2020).

## 4. Approximate identification of SALPV systems

The results developed so far assume exact data  $(\rho_d, w_d)$ . In this section, we apply them to the problem of SALPV system identification in the presence of noise on  $w_d$ .<sup>1</sup> More specifically, we consider the errors-in-variables (EIV) setup, where  $w_d$  is a true trajectory  $\bar{w}_d$  of a system  $\mathcal{B} \in \mathcal{P}_{m,\ell,n}^{n_\rho,q}$  plus additive noise  $\tilde{w}_d$ , which is zero mean, white, Gaussian (Söderström, 2007):

$$w_d = \bar{w}_d + \tilde{w}_d, \quad \text{where } \bar{w}_d \in \mathcal{B}_{\rho_d}|_{T_d} \text{ and } \tilde{w}_d \in \mathcal{N}(0, SI).$$

Given  $(\rho_d, w_d)$  and a complexity bound  $(m, \ell, n)$ , the SALPV maximum-likelihood estimation problem in the EIV setup is

$$\begin{aligned} &\text{minimize over } \hat{w} \text{ and } \hat{\mathcal{B}} \quad \|\bar{w}_d - \hat{w}\| \\ &\text{subject to } \hat{w} \in \hat{\mathcal{B}}_{\rho_d}|_{T_d} \text{ and } \hat{\mathcal{B}} \in \mathcal{P}_{m,\ell,n}^{n_\rho,q}. \end{aligned} \quad (13)$$

We choose the EIV setup because it has not been considered before in the LPV literature. The approach presented in this section, however, can be used also for other noise models besides the EIV one, see Lemmerling and De Moor (2001) and Willems (2004).

Using the lifting operation introduced in Section 3 and Theorem 2, we obtain the following problem equivalent to (13):

$$\begin{aligned} &\text{minimize over } \hat{w} \quad \|\bar{w}_d - \hat{w}\| \\ &\text{subject to } \text{rank } \mathcal{H}_{\ell+1}(\rho_{d,\text{ext}} \otimes \hat{w}) \leq r. \end{aligned} \quad (14)$$

Problem (14) is a *structured low-rank approximation (SLRA)* problem with Kronecker-Hankel structure of the data matrix  $\mathcal{H}_{\ell+1}(\rho_{d,\text{ext}} \otimes \hat{w})$ . The Kronecker-Hankel structure is specific for the SALPV model class. There are no specialized solution methods for (14), however, the structure is affine in  $\hat{w}$ , so that variables projection methods can be used (Markovsky & Van Huffel, 2007; Markovsky et al., 2005; Usevich & Markovsky, 2014). This opens the path for development of a new class of methods for SALPV system identification based on SLRA. Advantages of the SLRA approach are using data from multiple experiments, i.e., multiple trajectories of the system (Markovsky & Pintelon, 2015), dealing with missing and exact data (Markovsky & Usevich, 2013), and using convex relaxations (Dörfler et al., 2023; Markovsky, 2019).

Development of optimization methods that exploiting the Kronecker-Hankel structure of (14) will be presented elsewhere. Next, we briefly outline a relaxation method based on *unstructured* low-rank approximation (LRA). Due to the noise almost surely the matrix  $\mathcal{H}_{\ell+1}(\rho_{d,\text{ext}} \otimes \hat{w})$  is full rank and therefore has trivial left kernel. The proposed adaptation is to compute an approximate left kernel of the required dimension  $p$  via the SVD. This is the method for computing the MPUM, presented in Section 3, however, the data matrix is preprocessed by rank  $n_\rho q(\ell + 1) - (q - m)$  approximation using the SVD.

## 5. Simulation example

As a test example, we take a mass-spring-damper system with a varying spring constant, which acts as a scheduling variable  $\rho$ . The manifest variables are an external force  $u$  acting on the

<sup>1</sup> We make the standard simplifying assumption that the scheduling variable is known exactly.

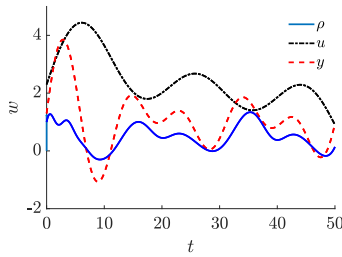


Fig. 3. Data used for identification of the system.

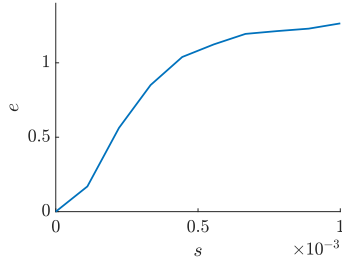


Fig. 4. Validation of the identified model from noisy data in the errors-in-variables setting.

mass and the displacement  $y$  of the mass. The system is defined by a second order differential equation:

$$\mathcal{B} = \left\{ w = \begin{bmatrix} u \\ y \end{bmatrix} \mid \frac{d^2}{dt^2}y + \frac{d}{dt}y + \rho y = u \right\}.$$

Simulating the system for 50 s with a multi-sine input and scheduling variable gives us the trajectories in Fig. 3.

The data satisfies the identifiability condition (12) and the exact identification method recovers  $\mathcal{B}$  from the data. This is an empirical confirmation of Theorem 2. In case of noisy data, using the LRA modification of the exact identification method, we obtain an approximate model  $\hat{\mathcal{B}}$ . The validation criterion used for the evaluation of the identified model is the angle  $e := \angle(\mathcal{B}_\rho|_T, \hat{\mathcal{B}}_\rho|_T)$  between the true finite-horizon behavior  $\mathcal{B}_\rho|_T$  and the estimated one  $\hat{\mathcal{B}}_\rho|_T$  for a new scheduling sequence  $\rho|_T$  (Markovsky, 2024). Fig. 4 shows the angle  $e$  as a function of the noise variance, averaged over 100 Monte-Carlo repetitions of the experiment.

## 6. Conclusions

We generalized the notion of the MPUM from LTI to LPV systems with shifted-affine scheduling dependence. The MPUM led us to an identifiability condition that is verifiable from the data and the complexity of the true system. The result is constructive and leads to a method for exact identification that yields a kernel representation of the MPUM. Using the results from approximate identification of SALPV systems as well as efficient solution methods exploiting the Kronecker-Hankel structure is a topic of current research. Comparison of the method proposed in the paper with classical LPV identification methods in terms of computational efficiency and robustness is also a topic of future work.

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