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RESEARCH ARTICLE

Approximate Relational Games For Green Influencer Marketing Optimization

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ABSTRACT The collaboration between celebrities, macro-influencers, and micro-influencers plays a pivotal role in amplifying the effectiveness and longevity of environmental sustainability messages. This study introduces the concept of a minimal coalition of influencers capable of efficiently promoting a sustainability message to the public. To address this, we propose a novel framework for Approximate Relational Games (ARG), which quantifies the degree of dependency of a follower on an influencer through a Dependency Decision Table (DDT). The DDT tracks follower reactions, categorized as ''Like only'' or ''Like and Share'', across a series of posts by the influencer. By setting a threshold for the degree of dependency, we focus on followers with stronger relational dependencies, eliminating those with weaker associations. This method enables the identification of a minimal winning coalition of influencers who can effectively disseminate the sustainability message despite resource constraints, both financial and non-financial. Utilizing combinatorial tools such as closure operators and choice functions, we present a method for determining minimal winning coalitions with polynomial time complexity. This approach offers a cost-effective solution for green organizations and governments aiming to maximize the impact of their campaigns. To validate the proposed model and algorithms, we conduct Python-based simulations using data from five real-world Facebook influencers, both macro and micro-tier, complemented by manual calculations. The results demonstrate the efficacy and computational efficiency of the proposed framework in optimizing influencer marketing strategies for promoting environmental sustainability.

INDEX TERMS Environmental sustainability, influencer marketing, algorithmic game theory, approximate relational game, decision table, minimal winning coalition.

I. INTRODUCTION

Environmental sustainability, an increasingly significant concept, focuses on preserving natural resources for future generations [1]. It emphasizes meeting present needs without compromising the ability of future generations to meet theirs. With the growing concern over climate change and its effects on consumption patterns, environmental sustainability has become a critical area of focus for both academia and industry. The depletion of natural resources, including deforestation, water scarcity, biodiversity loss, and air pollution, has spurred greater consumer awareness [2]. This has led to a heightened interest in sustainable consumption, with eco-labeling, packaging recycling, and organic products becoming essential considerations for consumers. As a result, society is gradually moving towards more environmentally harmonious consumption patterns [3].

In recent years, influencer marketing has gained significant traction as a means to promote sustainable consumer behavior [4]. Influencers effectively communicate sustainability messages through various channels, including conferences, mass media, and social networks. The widespread use of

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the internet and mobile devices has made platforms such as Facebook, Instagram, and TikTok integral to daily life, providing influencers with a powerful tool to influence public perceptions and drive behavioral changes [5], [6], [7]. Social media endorsements, with their higher levels of engagement, have proven to be more effective than traditional advertising methods [8], [9]. Influencers, perceived as more authentic and credible, face less resistance from their audiences, making them key players in shaping consumer decision-making. This trend has been reinforced by the rise of virtual influencers [10], further solidifying the role of social media influencers in shaping consumer behavior.

Despite the success of influencer marketing, identifying cost-effective and impactful influencer coalitions remains a complex challenge. While macro-influencers excel at raising broad awareness, their influence often culminates in mere consciousness-raising rather than fostering sustainable behaviors. In contrast, micro-influencers tend to cultivate authentic relationships with their followers, which can translate into meaningful behavior change. Research by Venciute et al. [11] further substantiates this notion, demonstrating that congruence between an influencer's persona and the values of their audience amplifies the impact on sustainable purchasing behaviors. Conversely, misalignment between the two can lead to counterproductive effects. Therefore, micro-influencers who embody sustainable lifestyles may serve as more potent drivers of ''authentic'' advocacy compared to their macro counterparts.

The increasing competition for natural resources and environmental amenities necessitates strategic decision-making in various management and utilization aspects. Since the inception of coalitional game theory by von Neumann and Morgenstern [12], the field has experienced periods of heightened interest and relative dormancy. In 1953, Nash [13] posited that solutions to cooperative games should be verifiable by demonstrating their equivalence to solutions of formally equivalent non-cooperative games. This perspective contributed to a temporary undervaluation of cooperative game theory. Nonetheless, cooperative game models have proven instrumental in enhancing efficiency and saving lives in various societal contexts.

Coalition formation has been extensively studied in game theory and economics [14], and has garnered increasing attention in artificial intelligence and data science. These disciplines explore the formation of coalitions among autonomous, self-interested agents collaborating to perform specific tasks [15], [16]. Game theory, particularly cooperative game theory, continues to be a significant force in driving social revolutions in the 21st century, as predicted by Larry Samuelson [17]. Building upon this trajectory, we introduce the Relational Dependency Coalitional Game (RDCG) model, which integrates the relational database model with coalitional games based on relational dependencies among players. Prior to our work, no research had merged coalitional games with the concept of functional dependencies in the relational database model proposed by Codd [18]. The primary advantage of incorporating relational dependencies into coalition games lies in the utilization of established knowledge from relational models, including combinatorial tools such as keys and anti-keys, as well as hypergraph tools like transversals and anti-transversals, and graph theory concepts like strongly connected and Hamiltonian graphs. These integrations provide powerful quantification tools to uncover new properties under both certain and uncertain conditions, as well as in scenarios involving complete and incomplete information. Additionally, by employing relational dependencies, we can apply Pawlak's rough set theory to decision tables [19], [20], which each player faces when forming coalitions under uncertainty.

In the context of influencer marketing, coalitional game theory offers a mathematical framework for identifying minimal influencer sets whose collective reach encompasses all followers whose dependency degree q exceeds a given threshold (e.g., 0.5). Simple games, as defined by Taylor and Zwicker [21], characterize winning coalitions through monotonicity and veto players, individuals whose presence is indispensable, and guarantee properties such as properness and strength. Recent advancements introduce closure operators that map any influencer set to the union of all followers dependent on it, encapsulating the notion of collective reach in intersection-closed coalition systems. Dual choice-function characterizations then facilitate the efficient identification of minimal influencer sets whose closures cover the dependent-follower universe in polynomial time.

Despite these theoretical advancements, real-world influencer marketing continues to rely on heuristics. Surveys indicate that 61% of marketers struggle to identify the right influencers, and fewer than 25% consider celebrity endorsements effective, highlighting the need for data-driven methods. The high fees associated with macro-influencers often exceed the budgets of many organizations, and challenges in discovering suitable micro-influencers further complicate campaign planning. To bridge this gap, we define an approximate relational dependency parameterized by q. Generally, two major approaches exist to measure the dependency of a follower on an influencer: homophily and heterophily. Homophily refers to the tendency of individuals to associate and bond with similar others, while heterophily pertains to associations with dissimilar others. In the context of influencer marketing, homophily has been found to positively influence customer value co-creation behavior and purchase intention [22]. Given the extensive research literature on homophily and for the purpose of simplification, we adopt the homophily approach and reserve the heterophily approach for future research.

This paper makes several key contributions to the field of influencer marketing optimization, particularly in the context of promoting environmental sustainability. These contributions are summarized as follows:

• We introduce Approximate Relational Games (ARG), a relational game-theoretic framework that models influencer-follower interactions within the context of environmental sustainability campaigns. This framework is based on coalitional game theory and incorporates the concept of approximate relational dependency, where the degree of dependency between influencers and their followers is quantitatively measured. By introducing Green Influencer Marketing Optimization (GIMO), this model provides a formal method for determining the minimal coalition of influencers necessary to promote sustainability messages effectively, considering both the influence of individual influencers and their engagement with followers.

• We propose new algorithms that leverage the relational game framework to identify minimal winning coalitions of influencers. These algorithms use combinatorial tools, such as closure operators and choice functions, to classify influencers into dependent and independent categories based on their relational dependencies with followers. By focusing on the subset of influencers whose followers exhibit the strongest dependency (i.e., those with high ''like and share'' interactions), the algorithms ensure that the selected influencers maximize the effectiveness of environmental messaging while minimizing costs. The algorithms are designed to run in polynomial time, demonstrating their computational efficiency and feasibility for practical implementation.

• The proposed algorithms are rigorously tested on real-world datasets to evaluate their practical effectiveness. We apply the algorithms to data from five prominent environmental influencers, both macro and micro-tier, and simulate their impact on followers in terms of engagement with sustainability content. The experimental results show that our approach significantly reduces the number of influencers required for a given sustainability message while maintaining a high level of follower engagement. These results validate the practical applicability of our model and algorithms, providing a valuable tool for organizations and governments seeking to optimize their influencer marketing strategies.

• This work advances the promotion of environmental sustainability by optimizing the allocation of resources in influencer marketing campaigns. By identifying and selecting the most impactful influencers with the highest relational dependency to their followers, this model minimizes the environmental cost associated with large-scale campaigns while maximizing their effectiveness. This ensures that sustainability messages are delivered efficiently and meaningfully, encouraging more widespread adoption of eco-friendly practices.

The structure of the paper is organized as follows:

• Section I provides an overview of the research, presenting the problem of optimizing influencer marketing strategies for promoting environmental sustainability.

• Section II offers a comprehensive review of relevant literature, including topics such as cooperative games in social networks, influencer marketing optimization, green communication strategies, influence maximization heuristics, and traditional coalition game models.

• Section III introduces the GIMO methodology, detailing the approach and its relevance to the study.

• Section IV presents fundamental concepts associated with relational games, essential for understanding the subsequent analyses.

• Section V explores the concept of Approximate Relational Dependency and its application within decision tables.

• Section VI describes the proposed algorithms, outlining their functionality and comparing their computational efficiency with existing methods.

• Section VII discusses the experimental results obtained from real-world data, providing insights into the practical application of the proposed methods.

• Section VIII addresses the limitations of the current model and outlines directions for future research.

• Section IX concludes the paper with a summary of findings and suggestions for future research directions.

II. RELATED WORKS

A. COOPERATIVE GAMES IN SOCIAL NETWORKS

The study of cooperative games within social networks has become an essential area of exploration, particularly when investigating how individuals or groups cooperate and form alliances to achieve shared goals. One of the foundational contributions to this field is the work of Gómez et al. [23], who explored the intersection of centrality and power in social networks through a game-theoretic lens. By applying cooperative game theory, they aimed to measure how central players in a network exert power and influence, leveraging mathematical models to quantify these interactions.

Zhou et al. [24] applied cooperative game theory to the problem of community detection in social networks. Their approach focuses on dividing a social network into communities by considering the interdependence of individuals. This method is particularly useful for understanding how groups form and interact in social contexts, providing a framework for detecting cohesive subgroups within large networks.

Molinero and Riquelme [25] connected cooperative game theory and social network analysis by presenting models of influence decisions in social networks. They examine how decision-making processes in networks can be understood through the lens of cooperative games, particularly highlighting the role of social influence and network effects in shaping individual decisions within a community.

Boruah et al. [26] introduced a multilayer network games approach, which blends cooperative game theory with multilayer networks to model complex relationships in real-world systems. This model facilitates a better understanding of how various types of relationships contribute to the overall cooperative behavior in a social network.

The authors in [27] presented a comprehensive network visualization of cooperative games, offering insights into how network structures can affect the outcomes of cooperative

interactions. Their work emphasizes the importance of visualizing network dependencies and interactions to better optimize cooperation in various social settings.

These studies collectively underscore the importance of applying cooperative game theory to social network analysis. By utilizing mathematical models and algorithms, researchers can uncover deeper insights into the dynamics of cooperation, coalition formation, and social influence in networks. This line of research has profound implications for optimizing various aspects of social network interactions, including marketing strategies and collaboration dynamics.

B. INFLUENCER MARKETING OPTIMIZATION

Influencer marketing has emerged as a pivotal strategy in digital advertising, leveraging the reach and credibility of social media influencers to promote products and services. The optimization of influencer marketing campaigns involves strategically selecting influencers, scheduling content, and managing budgets to maximize return on investment (ROI) and engagement.

Lagrée et al. [28] introduced a generic framework for influence maximization, addressing the challenge of selecting ''spread seeds'' from a set of influencers to optimize campaign reach. Their approach, known as Online Influencer Marketing with Persistence, assumes that once a node (influencer) activates others, it remains persistently active throughout the campaign. This model is particularly relevant when historical activation data is unavailable, as it relies on real-time estimations of influencers' remaining potential to reach new audiences. The proposed GT-UCB algorithm utilizes upper confidence bounds to efficiently estimate this potential, demonstrating high-quality spreads on both simulated and real datasets, even in large-scale online scenarios.

Mallipeddi et al. [29] developed a data-driven optimization framework to aid firms in conducting both short-horizon and long-horizon influencer marketing campaigns. Their models incorporate empirical data from social media platforms, such as Twitter, to estimate parameters like the multiple exposure effect and forgetting effect, which influence the effectiveness of repeated influencer interactions. By formulating the problem as a mathematical program, they provide polynomial-time procedures to select influencers and schedule content, thereby maximizing the firm's benefit. This framework underscores the importance of temporal dynamics and content scheduling in optimizing campaign outcomes.

Han et al. [30] addressed the challenge of cost-effectiveness in influencer marketing by proposing models that consider the trade-off between influencer costs and their potential impact. Their approach emphasizes the need to select influencers who offer the highest return relative to their cost, thereby optimizing the marketing budget. This cost-effectiveness perspective is crucial for firms aiming to maximize ROI while adhering to budget constraints.

López-Dawn and Giovanidis [31] tackled the problem of optimizing influencer marketing campaigns under budget constraints by formulating the problem as a convex program. They introduced an efficient iterative algorithm based on the Frank-Wolfe method, which converges to the global optimum and exhibits low computational complexity. Their approach allows for the selection of appropriate influencers and content types (e.g., text, image, video) to maximize impact metrics such as impressions, sales, or audience reach. The algorithm's scalability makes it suitable for large-scale campaigns with millions of social users.

Zhang et al. [32] proposed a joint optimization approach that integrates influencer selection with traffic promotion strategies. Their model considers both the direct impact of influencers on audience engagement and the indirect effects through traffic generation. By jointly optimizing these aspects, firms can enhance the overall effectiveness of their campaigns, ensuring that influencer efforts are complemented by adequate traffic support.

Yuan et al. [33] extended traditional assortment optimization by integrating influencer marketing strategies. Their two-stage optimization problem first selects influencers based on their potential impact and then personalizes product assortments to align with the preferences of the influencers' audiences. This approach ensures that marketing efforts are tailored to the target demographic, enhancing engagement and sales outcomes.

The optimization of influencer marketing campaigns is a multifaceted challenge that requires a comprehensive understanding of algorithmic strategies, budget management, and audience engagement. The reviewed works contribute to this understanding by providing models and algorithms that address various aspects of influencer marketing, from selection and scheduling to cost-effectiveness and dynamic simulation. Future research can build upon these foundations to develop more sophisticated models that incorporate additional factors such as influencer-follower relationships, content virality, and real-time feedback mechanisms.

C. GREEN COMMUNICATION STRATEGIES

Green communication strategies are pivotal in shaping consumer perceptions and behaviors toward sustainability initiatives. These strategies encompass the dissemination of environmental information through various channels, aiming to foster awareness, trust, and engagement among target audiences. Effective green communication not only highlights a brand's environmental efforts but also aligns with consumer values, thereby enhancing brand loyalty and promoting sustainable consumption patterns.

Lin et al. [34] investigated the impact of green brand communication on consumer attachment, focusing on dimensions such as self–brand connection and brand prominence. Their study revealed that green transparency and competence enhancement positively influence consumer perceptions, leading to stronger brand attachment. This attachment subsequently enhances brand loyalty and resistance to negative information, underscoring the importance of clear and credible environmental messaging in building enduring consumer relationships.

Lin et al. [35] explored how product transformation salience (PTS) and consumer traceability knowledge affect green purchase intentions. Their findings indicate that emphasizing the transformation process of products, coupled with providing traceability information, significantly boosts consumers' intentions to purchase green products. Additionally, perceived green brand coolness mediates this relationship, highlighting the role of brand perception in influencing consumer choices.

Wu and Long [36] examined the effectiveness of green communication on social media in promoting eco-friendly consumption intentions. Their research demonstrated that perceptions of information usefulness and green trust mediate the relationship between social media marketing and eco-friendly purchase intentions. This underscores the necessity for brands to provide valuable and trustworthy environmental information to leverage social media platforms effectively.

Crapa et al. [37] assessed the performance of green communication strategies across social media platforms within the large-scale retail industry in Italy. Their study found that transparent and consistent environmental messaging enhances consumer engagement and brand perception, leading to improved sales performance. This highlights the commercial benefits of integrating green communication into marketing strategies.

Akshita et al. [38] conducted a systematic review on the impact of green marketing communication in the Fast-Moving Consumer Goods sector. Their findings suggest that consistent and transparent green messaging enhances brand loyalty by fostering consumer trust and emotional engagement, particularly when aligned with the brand's core values.

The integration of effective green communication strategies is essential for brands aiming to promote sustainability and build lasting consumer relationships. By emphasizing transparency, providing traceability information, leveraging social media platforms, and ensuring active corporate governance, brands can enhance consumer trust and loyalty. These strategies not only contribute to environmental goals but also offer commercial advantages in an increasingly eco-conscious market.

D. INFLUENCE MAXIMIZATION HEURISTICS

The classical Influence Maximization (IM) problem seeks to identify a small subset of seed nodes within a social network that can maximize the spread of influence through a diffusion process. Prominent algorithms for this problem include the Greedy algorithm, introduced by Kempe et al. [39], and its more efficient variant, Cost-Effective Lazy Forward (CELF) [40]. These algorithms typically rely on the property of submodularity in influence functions, which guarantees that adding an element to a smaller set provides a greater or equal marginal gain than adding it to a larger set. This property allows the greedy algorithm to achieve a strong theoretical approximation guarantee $(1 - 1/e)$ for maximizing influence spread.

However, a significant challenge with these traditional IM approaches lies in their computational complexity. Both the greedy and CELF versions exhibit a worst-case time complexity of $O\left(E^k\right)$, where E represents the number of edges in the network and k is the size of the desired seed set. For large-scale social networks, this complexity can render these algorithms computationally prohibitive. Even optimized implementations can become prohibitively expensive on large graphs, with runtime scaling poorly in the number of edges and the desired seed set size. This computational burden substantially constrains the applicability of classical IM techniques to large-scale social networks.

The ARG framework addresses a related but distinct problem: identifying a minimal set of influencers that cover all highly dependent followers, rather than maximizing influence spread from a fixed budget. By leveraging the specific mathematical properties of relational dependencies and closure operators, the proposed algorithms achieve polynomial-time complexity. This computational advantage is crucial for practical applications in large social networks, where the exact solutions or even approximations of general IM problems might be intractable. The framework's design inherently bypasses the computational bottlenecks associated with traditional IM by reframing the problem around the concept of relational closure, offering a more scalable solution for its specific objective.

E. TRADITIONAL COALITION GAME MODELS

Coalitional games, particularly simple games and weighted voting games (WVGs), have been widely studied in the context of multi-agent systems and social choice theory. Simple games model situations where players form coalitions to achieve certain outcomes, with the key characteristic being that there is a list of minimal winning coalitions that can determine the outcome of the game. These minimal winning coalitions are subsets of players whose combined influence ensures the success of the coalition, while a ny proper subset of these players cannot achieve the same outcome.

The authors in [41] explored the computational complexities of solving coalition formation problems in the context of simple games and WVGs. They demonstrated that for WVGs with a constant number of weight values, a polynomial-time algorithm can compute an optimal coalition structure. However, when the number of weight values is not fixed, the problem becomes strongly NP-hard. This results in computational challenges, particularly when determining the optimal coalition structure for large numbers of participants.

Simple games that do not involve weighted voting are even more complex to approximate, as the problem of identifying minimal winning coalitions is not only NP-hard but also computationally intensive. In these games, the task of determining whether a given player is part of a winning coalition requires evaluating all possible coalitions, leading to exponential computational complexity as the number of players increases.

Consequently, algorithms for simple games are often infeasible for large-scale applications, such as influencer marketing optimization, without significant computational resources.

The ARG framework distinguishes itself by transforming the problem of finding minimal winning coalitions into a polynomially solvable category through the integration of relational dependencies and the application of closure operators. This fundamental shift in the underlying mathematical structure allows the proposed algorithms to operate with significantly lower computational complexity compared to general simple games or weighted voting games, which often face NP-hard challenges. The deliberate choice of this relational model enables the framework to overcome the computational limitations of more general game-theoretic approaches, making it a viable and efficient tool for large-scale social network analysis in influencer marketing.

III. METHODOLOGY

To achieve GIMO, we propose a structured framework comprising three distinct steps: (i) quantification of relational dependencies, (ii) filtering of active followers based on dependency thresholds, and (iii) identification of minimal winning coalitions of influencers. Each step is underpinned by specific quantitative tools and methodologies, selected for their relevance and efficacy in capturing the complexities inherent in influencer-follower interactions.

Details of the proposed method steps are described in Figure 1.

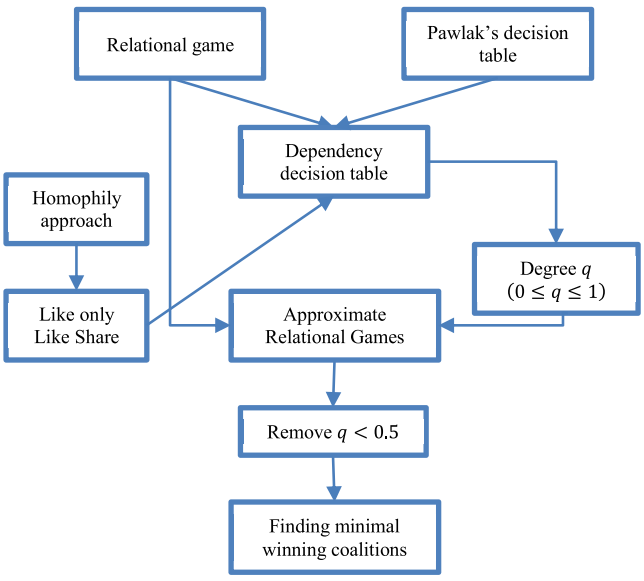

FIGURE 1. Basic flow chart of GIMO methodology.

A. STEP 1

In this initial phase, we aim to quantify the degree of dependency between followers and influencers based on their interactions over a defined period. Drawing upon Pawlak's concept of decision tables [19], we construct an approximate relational dependency decision table. In this table, each conditional column represents a topic associated with an influencer's posts, while the decision column captures the follower's reaction, categorized as either "Like and Share" or "Like only".

For simplification, we assume that, on average, an influencer posts once per day, aligning with recommendations for optimal engagement frequency [42]. In our model, a "1" denotes the presence of a post on a given day, and a "0" indicates its absence. This binary representation, while simplifying the analysis, introduces a limitation that we acknowledge. Future work will address this by distinguishing between different posts under the same topic, thereby enhancing the granularity of the relational dependency model.

Additionally, we incorporate Pawlak's notion of functional dependencies within decision tables [19]. This inclusion facilitates the identification of reducts, minimal subsets of attributes that preserve the decision-making capability of the table. The process of reducing a decision table is crucial, as it eliminates redundant rules and conditions, thereby simplifying complex logic and improving the efficiency of decision-making processes [43], [44].

B. STEP 2

Recognizing the vast number of potential follower-influencer interactions, we focus on active followers, those who engage with the influencer's posts. To manage this complexity, we introduce a relational dependency degree, denoted as q, and retain only those followers whose dependency degree meets or exceeds a given threshold (e.g., 0.5). This approach ensures that our analysis centers on meaningful interactions, enhancing the reliability and relevance of the identified influencer-follower relationships.

The rationale for this focus on active followers is grounded in the distinction between active and passive social media users. Active users, characterized by their engagement through actions such as posting content and interacting with others, are generally viewed as more adaptive. In contrast, passive users, who primarily browse and scroll without interacting, provide limited data, making it challenging to quantify their reactions accurately. Therefore, concentrating on active users allows for a more precise and actionable analysis.

C. STEP 3

The final step involves determining minimal winning coalitions of influencers capable of effectively promoting a given message. This is achieved by analyzing the refined relational game, which has been adjusted through the previous steps. To facilitate this process, we employ combinatorial tools such as closure operators and choice functions [45], [46]. These tools assist in classifying influencers as dependent or independent relative to others, thereby streamlining the identification of optimal coalitions.

The application of closure operators and choice functions serves to simplify the coalition identification process, making each step more deterministic. This deterministic approach is

advantageous, as it potentially classifies the problem into a polynomially solvable category, thereby enhancing the scalability and efficiency of the optimization process.

IV. PRELIMINARIES

This section presents some basic concepts pertinent to relational games, as discussed in prior works [46]. The main notations used in this paper are shown in Table 1.

TABLE 1. Key symbols used in the paper.

Symbol	Description
U	A nonempty finite set of agents
RD	Relational dependencies
F	The family of all relational dependencies
v	A real-valued characteristic function
q	Relational dependency degree $0 \leq q \leq 1$
P	The set of all coalitions
w	Winning coalition
l	Losing coalition
W	Minimal winning coalition
L	Closure operator
C	Choice function

A. RELATIONAL DEPENDENCIES, INDEPENDENCIES

We define our relational dependency game as follows:

Let $U = \{a_1, \ldots, a_n\}$ be a nonempty finite set of n agents. A *relational dependency* (RD) $A \rightarrow B$, where $A, B \subseteq U$, signifies that B depends on A. For instance, a group of doctors B depends on a particular doctor $\{a_i\}$ due to his superior expertise and role as a mentor to group B.

In relational coalitional games, predetermined relationships among agents are established. We categorize the relationships between two agents into three types:

(1) Agent A depends on agent B.
(2) Agent A does not depend on agent B, but agent B depends on A.
(3) Agent A and B have no relation.

If agent B depends on agent A, we denote this as $A \rightarrow B \in F$, where F represents the family of all relational dependencies (RDs) among agents. Consequently, we can define the family FI of relational independencies among agents: $A \rightarrow B \notin F$ if and only if $A \rightarrow B \in FI$, implying that either B does not depend on A or A and B have no relation.

Given a family F of RDs over U, there exists a unique minimal f-family F^+ that contains F. It can be observed that F^+ encompasses all RDs derivable from F by applying Armstrong's axioms [47]:

(1) $X \rightarrow X \in F$.
(2) If $X \rightarrow Y \in F$ and $Y \rightarrow Z \in F$ then $X \rightarrow Z \in F$.
(3) If $X \rightarrow Y \in F$ and $X \subseteq V$, $W \subseteq Y$ then $V \rightarrow W \in F$.
(4) If $X \rightarrow Y \in F$ and $V \rightarrow W$ then $X \cup V \rightarrow Y \cup W \in F$.

A family of RDs satisfying the above Armstrong's axioms is called an *f-family* F^+ over U.

Definition 1 (Coalition form): Let $n \geq 2$ represent the number of agents in a game, and let $U = \{a_1, \ldots, a_n\}$ be a nonempty finite set of n agents. A *coalition* S is defined as a subset of U, and $P(U)$ denotes the set of all coalitions. The empty set is also considered a coalition and is referred to as an empty coalition. The game involving all agents of U is termed the *grand coalition*.

Definition 2 (Coalition form and characteristics functions): A *coalition* $S =< U, F, v >$ and a coalition game $r =< U, F, v >$ with $G \subseteq U$ are defined as follow: F is a set of RDs over U, and a real-valued *characteristic function* or *valuation function* $v : P(U) \longrightarrow R$ and $v : P(G) \longrightarrow R$ for $r \rightarrow$ map the set of all possible coalitions of agents of U and G, respectively, to a set of payments that satisfies $v(\emptyset) = 0$. The quantitative $v(A)$ is a real number for each coalition $A \subset U$, representing the value (or worth, or power) of coalition A when its members collaborate as a team. $\emptyset$

Definition 3 (Winning coalition and losing coalition): Define w_k, a coalition of k agents in a game S over U, as a *winning coalition* if all agents of U have relational dependencies on k, implying $v(w_k) = v(U)$. Thus, all winning coalitions have the same payoffs as the grand coalition of all agents. This definition of winning coalitions is closely related to closure operators, as elaborated upon in subsequent sections.

A *losing coalition* l_k is a coalition when all agents of U do not have relational dependencies on k. Losing coalitions $l_{i1}, \ldots, l_{it}$ exhibit varying payoffs $v\left(l_{ij}\right)$, with larger losing coalitions yielding higher payoffs, such that $v(l_{i1}) \leq \ldots \leq v(l_{it})$ if $l_{i1} \subseteq \ldots \subseteq l_{it}$. The maximal losing coalition has a smaller payoff than the minimal winning coalition, $v(l_{it}) < v(w_k)$. This definition of losing coalition is intricately connected to choice functions, as discussed further below.

B. CLOSURE OPERATORS, CHOICE FUNCTIONS

To classify agents as dependent or independent, we introduce closure operators and choice functions. These concepts streamline the coalition formation process by making each step more deterministic, thereby enhancing the efficiency of the underlying algorithms and potentially allowing their execution to fall within the computationally feasible class P.

Definition 4: Let $A^+ = \left\{a : A \rightarrow \{a\} \in F^+\right\}$. A^+ is called the *closure* of A over U. It is clear that $A \rightarrow B \in F^+$ if and only if $B \subseteq A^+$.

Definition 5: Let U be a nonempty finite set of agents and $P(U)$ its power set, containing all subsets of U. A map $L : P(U) \rightarrow P(U)$ is called a *closure* over U if it satisfies the following conditions:

(1) Extensiveness: $A \subseteq L(A)$.
(2) Monotonicity: $A \subseteq B$ implies $L(A) \subseteq L(B)$.
(3) Closure: $L(L(A)) = L(A)$.

Set $L(A) = \left\{a : A \rightarrow \{a\} \in F^+\right\}$, and it can be observed that L is a closure over U.

Definition 6: Let $M \subseteq P(U)$. M is called a *Sperner system* over U if for all $A, B \in M$, neither $A \subseteq B$ nor $B \subseteq A$.

Definition 7: A map $C : P(U) \to P(U)$ is called a *choice function*, if every $A \in P(U)$ then $C(A) \subseteq A$.

Choice functions are widely studied in the context of rational behaviors of individuals and groups in social studies and economics [48], [49], [50]. In this framework, $P(U)$ represents the set of all alternatives. A is a set of alternatives presented to the decision-maker, and $C(A)$ is the selection of the best alternatives among A. The most important properties of choice functions are:

For $X, Y \subseteq U$

- Heredity (H): if $X \subseteq Y$ then $C(Y) \cap X \subseteq C(X)$.
- Concordance (C): $C(X) \cap C(Y) \subseteq C(X \cap Y)$.
- Out casting (O): if $C(X) \subseteq Y \subseteq X$ then $C(X) = C(Y)$.
- Monotonicity (M): if $X \subseteq Y$ then $C(X) \subseteq C(Y)$.

Let L be a closure operator. Based on existing research [51], [52], [53], [54], we have the following results:

Definition 8: Define two choice functions associated with L as follows:

- Choice Function type-I (CF-I): For $X \subseteq U, C(X) = U - L(U - X)$.
- Choice Function type-II (CF-II): For $X \subseteq U, C_L(X) = L(U - X) \cap X$.

Note that both choice functions related to closure operators above are uniquely determined by the closure L. In fact, $L(X) = X \bigcup C_L(U - X)$ and $L(X) = U - C(U - X)$.

For every $X \subseteq U$, the sets $C_L(X)$ and $C(X)$ form a partition of X, that means $C_L(X) \cup C(X) = X$ and $C_L(X) \cap C(X) = \emptyset$.

It is evident that CF-I satisfies the properties of (O) and (M), while CF-II satisfies the properties of (H) and (O). Based on the definitions above, we can deduce:

Proposition 1: There is a one-to-one correspondence between CFs-I and closure operators on U.

Proposition 2: There is a one-to-one correspondence between CFs-II and closure operators on U.

Proposition 3: The mapping $FI \to C_{FI}$ establishes a mutually inverse one-to-one correspondence between the full family FI and choice function I.

C. MINIMAL WINNING, MAXIMAL LOSING

In the context of relational dependency games, the concepts of minimal winning and maximal losing coalitions are pivotal for understanding the dynamics of coalition formation among agents. These coalitions are defined with respect to closure operators and choice functions, which provide a structured framework for analyzing agent dependencies and independencies.

Definition 9: A coalition $A \subseteq U$ is termed a *minimal winning coalition* if it satisfies two conditions:

- $L(A) = U$, indicating that the closure of A over U encompasses all agents, thereby ensuring that every agent has a relational dependency on A.
- For any proper subset $B \subset A, L(B) \neq U$, meaning that the closure of B does not include all agents, and thus, B alone is insufficient to form a winning coalition.

The set of all minimal winning coalitions is denoted as W. It is obvious that W forms a Sperner-system.

Lemma 1: A minimal winning coalition W is a special case of closure operator L over U :

$$W \subseteq L(W) = U$$

Definition 10: Let W be a Sperner system, we define the set of *maximal losing coalition* of W, denoted W^{-1} as following:

$$\mathcal{W}^{-1} = \{A \subset U : (B \in W) \text{ implying } (B \not\subseteq A) \text{ and } (A \subset C) \text{ implying } (\exists B \in W)(B \subseteq C)\}$$

It is easy to see that the maximal losing coalition $\mathcal{W}^{-1}$ of $\mathcal{W}$ are the subsets of U not containing the elements of $\mathcal{W}$ and which are maximal for this property and $\mathcal{W}^{-1}$ is a Sperner system too.

Lemma 2: If W is a minimal winning coalition of a closure operator over U, then maximal losing coalition W^{-1} contains a choice function CF-II.

Proof: Since for $W \subseteq U, W$ is a minimal winning coalition of L if $L(W) = U$ and $L(W) = U - C(U - W) = U$, that means the choice function CF-I C is an empty set in this case.

In the other case, W is a minimal winning coalition of L if $L(W) = U$ and $L(W) = W \bigcup C_L(U - W) = U$.

We could see that maximal losing coalition, denoted as W^{-1} does not contain any element of W so W^{-1} contains choice function $C_L(U - W)$. The proof is completed.

It is easy to have the following results from Lemma 2.

Proposition 4: There is a one-to-one correspondence between minimal winning coalitions and closure operators on U.

V. DEPENDENCY DECISION TABLE

In real-world interactions, relational dependencies often exhibit varying degrees of intensity. For instance, a child may rely on a parent for essential needs without necessarily aligning with all of the parent's views. Similarly, a social media follower may engage with an influencer's content without fully endorsing every opinion or recommendation. This nuanced understanding of dependency is crucial for accurately modeling influencer-follower relationships.

To capture this variability, we introduce the concept of Approximate Relational Dependency, formalized through a Decision Table. This framework quantifies the degree of dependency between an influencer and a follower based on their interactions over a specified period. Each entry in the table records the follower's response to the influencer's posts, categorized as ''Like and Share'' or ''Like only''. For example, consider an influencer who posts 200 times in a year. If a follower strongly agrees with 140 posts (indicated by ''Like and Share'') and agrees with 60 posts (indicated by ''Like only''), the Decision Table captures these interactions, leading to a calculated dependency level of 70% for that follower.

Formally, we define the relational dependency between an influencer (denoted as INFL) and a follower (denoted as FL)

as $INFL \rightarrow_q FL$, where q represents the degree of "Like and Share" interactions. Assuming that the influencer covers p major topics throughout the year within a specific domain, we construct a Decision Table to systematically capture and quantify the approximate relational dependency of FL on INFL. This table serves as a structured framework for assessing the strength and consistency of the follower's engagement with the influencer's content.

Definition 11: Consider a *dependency decision table* $DDT = (U, C \cup D, V, f)$ measuring the degree of Approximate Relational Dependency (ARD) of a follower FL on the influencer INFL: $INFL \rightarrow_q FL$

where:

• U is non-empty finite set of objects, called the universe, and C and D are two set of attributes such that: $C = \{c_1, \ldots, c_t\}$ is a set of all topics influencer INFL posts on a social media platform, D is a set of decision attributes that follower FL decide to either "Like only" or "Like and Share" the post made by INFL, and $C \cap D = \emptyset$.

• $V = \bigcup_{a \in C \cup D} V_a$, where V_a is the value range of the attribute $a \in C \cup D$.

• $f : U \times (C \cup D) \rightarrow V_a$ is an information function, where $\forall a \in C \cup D, u \in U, f(u, a) \in V_a$.

• p : number of "Like and Share".

• t : number of rows of $\{d\}$.

• $q = p/t, 0 \leq q \leq 1$. If $q = 1$ then FL is fully dependent on INFL; if $q = 0$ then FL is fully not dependent on INFL; and FL is approximately dependent on INFL if $0 < q < 1$.

For any $\forall a \in C \cup D, u \in U$, we will denote the value of the attribute a on the object u by $a(u)$ instead of $f(u, a) \in V_a$.

If $B = \{b_1, \ldots, b_k\} \subseteq C \cup D$ is a subset of attributes, then the set of $b_i(u)$ is denoted as $B(u)$. Therefore, if u and v are two objects in U, then $B(u) = B(v)$ if and only if $b_i(u) = b_i(v)$ for any $i = 1, 2, \ldots, k$.

Without loss of generality, suppose that D consists of only one decision attribute d. Therefore, from this time we consider the dependency decision table $DDT = (U, C \cup \{d\}, V, f)$, where $d \notin C$.

To illustrate the concept of approximate relational dependency, consider a green influencer, Messi, and a follower, FL1, on a social media platform over the course of a year. Assume Messi covers four major topics related to environmental sustainability throughout the year. In the first week of January 2024, FL1's interactions with Messi's posts are recorded in a Decision Table (Table 2), capturing the extent of FL1's engagement with Messi's content.

TABLE 2. Decision table of "Like only" / "Like and Share" reactions by one follower for five posts on four topics of Messi.

Date	Sustainable Fashion	Green Cosmetics	Sustainable Travel	Water Issues	Like and Share / Like only
1/1	0	1	0	0	0
1/1	1	0	0	0	1
1/3	0	0	1	0	1
1/5	0	0	0	1	0
1/6	0	0	1	0	1

The integrity constraint of that DDT above is:

$$C = Date \wedge SustainableFashion \wedge GreenCosmetics \wedge SustainableTravel \wedge WaterIssues \longrightarrow \{d\} = Like\ and\ Share\ /Like\ only$$

where:

• Conditional column: "Date", "SustainableFashion", "GreenCosmetics", "SustainableTravel" and "WaterIssues".

• Decision column: "Like and Share/Like only".

• "1" denoted as a message being posted by Messi on that date on that topic column while "0" means no post on that date.

In decision column, we denote "0" as a "Like only" (which includes three emoji "Like", "Love", and "Haha") and "1" as "Like and Share". During the first week of January 2024, Messi made five posts, with three posts being liked and shared, and two posts being liked only. Thus, he has 60% "Like and Share" and 40% "Like only". Therefore, the approximate relational dependency of FL1 on Messi for all topics is represented as:

$$Messi \rightarrow_{0.6} FL$$

Similarly, we have:

$$Messi_{SustainableFashion} \rightarrow_1 FL$$
$$Messi_{GreenCosmetics} \rightarrow_0 FL$$
$$Messi_{SustainableTravel} \rightarrow_1 FL$$
$$Messi_{WaterIssues} \rightarrow_0 FL$$

Next, based on approximate relational dependencies of followers on influencers, we find the minimal winning coalition of green influencers to save more cost for the government and organizations promoting environmental sustainability.

VI. PROPOSED ALGORITHMS

In this section, we propose two algorithms designed to efficiently identify minimal winning coalitions in relational games. Both algorithms leverage the power of closure operators to minimize computational complexity, ensuring that the coalition identification process is computationally feasible. The detailed steps and computational feasibility of each algorithm are discussed below.

A. ALGORITHM 1

Algorithm 1 begins by considering the entire set of agents, denoted as the grand coalition U. The algorithm then progressively reduces this set by systematically removing agents. The removal process continues until a minimal winning coalition is identified. In each iteration, the algorithm refines the coalition by eliminating agents whose inclusion does not contribute to the formation of a minimal winning coalition. This iterative approach guarantees that the remaining set is both minimal and winning, ensuring that no extraneous agents remain in the final coalition.

Algorithm 1

Input: Let S be a coalition game over $< U, F, v >$ with $U = \{a_1, \ldots, a_n\}$, $F = \{RD_1, \ldots, RD_p\}$, and given a degree of dependency q.
Step 1: Removing all approximate relational dependency ARD_i along with related agents a_j with degree of dependency less than given q, then
$$U_q = U - \{a_j; \rightarrow_k a_j \text{ and } k < q\}.$$
Step 2: $W(0) = U_q = \{a_1, \ldots, a_n\}$.
Step (q+1): For the $q = 0, 1, \ldots, n-1$ define:
$W(q+1) = W(q) - \{a_{q+1}\}$, **if** $W(q) - \{a_{q+1}\} \rightarrow U \in F^+$.
$W(q+1) = W(q)$, **otherwise**.
Output: $W(m)$ is minimal winning coalition of S.

Lemma 3: The sets $W(0), W(1), \ldots, W(n)$ are winning coalitions over U and $W(n)$ is the minimal winning coalition.

Proof: We prove this lemma by induction. It is obvious that $W(0)$ is a winning coalition. If $W(q)$ is a winning coalition, and $W(q+1) = W(q)$, then $W(q+1)$ is a winning coalition. If $W(q+1) = W(q) - \{a_{q+1}\} \rightarrow U \in F^+$, then $L(W(q+1)) = U$, that mean $W(q+1)$ is a winning coalition.

It is very obvious that $W(n)$ is the minimal winning coalition. The proof is completed.

It is clearly that the worst-case scenario running time of the above algorithm is $O(n^2|U|)$, where $|U| = n$.

B. ALGORITHM 2

Algorithm 2 adopts a different strategy, beginning by selecting agents whose closure has the highest cardinality. This selection indicates that these agents have significant relational dependencies with other agents, making them crucial for forming a winning coalition. From this initial set, the algorithm iteratively selects the smallest subset of remaining agents whose closure collectively covers the entire universe of agents U. This process continues until a minimal winning coalition is formed, ensuring that all agents within the coalition meet the necessary relational dependency criteria to qualify as a winning coalition.

Lemma 4: $SC = SC_0 \cup SC_1 \cup \cdots \cup SC_{v-1}$ is the minimal winning coalition of S.

Proof: It is very obvious that $SC = SC_0 \cup SC_1 \cup \cdots \cup SC_{v-1} \rightarrow U_q \in F^+$, that means $L(SC) = U_q$ or SC is a winning coalition.

Now we have to prove SC is a minimal winning coalition. By using contradiction method, we assume there exist A, a proper subset of SC such that $L(A) = U_q$. That means A is winning coalition with smaller cardinality than SC. Since $L(SC) = L(A) = U_q$ and $A \subset SC$, that means $\exists a \in SC - A$ such that $L(A) > L(B) \forall b \in SC$, that is contraction that SC contains all agents whose closure operator has largest cardinality.

The proof is completed.

Algorithm 2

Input: Let S be a coalition game over $< U, F, v >$ with $U = \{a_1, \ldots, a_n\}$, $F = \{RD_1, \ldots, RD_p\}$, $|\mathbf{L}(a_n)| \leq \cdots < |\mathbf{L}(a_1)|$ and condition $(*)$.
Step 1: Removing all approximate relational dependency ARD_i along with related agents a_j with degree of dependency less than given q, then
$$U_q = U - \{a_j :\rightarrow_k a_j \text{ and } k < q\}.$$
Step 2: Setting $SC_0 = \{a_1\}$.
Step 3: Determining $L(SC_0) = SC_0^+ = \{a : SC_0 \rightarrow a \in F\}$.
Step 4: Determining $Temp_1 = U_q - L(SC_0)$.
Step 5: One condition needed here:
$\exists! a_j | L(a_j) = max\{L(a_i) \forall a_i \in Temp_1\}(*)$.
Then determining $SC_1 = \{$only one agent with maximal number of relational dependencies (highest closure cardinality) in $U_q - L(SC_0)$, that is $a_j\}$.
Step 6: Determining $L(SC_1)$.
Step 7: Determining $Temp_2 = U_q - L(SC_0) \cup L(SC_1)$.
Step 8: Then determining $SC_2 = \{$by condition $(*)$, there is only one agent with maximal number of relational dependencies in $U_q - L(SC_0) \cup L(SC_1)\}$.
Step 9: Repeating Step 2 to 7 until $Temp_v = \emptyset$.
Output: $SC = SC_0 \cup SC_1 \cup \cdots \cup SC_{v-1}$ is the minimal winning coalition of S.

C. COMPUTATION TIME

The classical Influence Maximization problem aims to identify a small subset of seed nodes within a social network that can maximize the spread of influence through a diffusion process. A well-known algorithm for this problem is the Greedy algorithm, introduced by Kempe et al. [39]. In this study, we compare the computational performance of Algorithm 1 with the Greedy algorithm.

As shown in the Figure 2, the proposed algorithm consistently demonstrates significantly lower computation times, with time values remaining relatively constant as the input size increases. In contrast, the Greedy algorithm exhibits a notable increase in computation time, particularly as the problem size grows. This indicates that the proposed algorithm is more efficient in terms of time complexity, especially for larger datasets. These results highlight the superior scalability of the proposed algorithm compared to the Greedy algorithm, making it a more suitable choice for large-scale applications where computational efficiency is critical.

VII. SIMULATION ANALYSIS

It is well-known that animal protection and environmental sustainability are intrinsically linked because promoting and protecting animal welfare strongly contributes to a more sustainable planet by encouraging biodiversity, reducing environmental impact, and nurturing ethical practices.

In this simulation, we do not utilize the Wikipedia Election and soc-Epions datasets from the Stanford Large

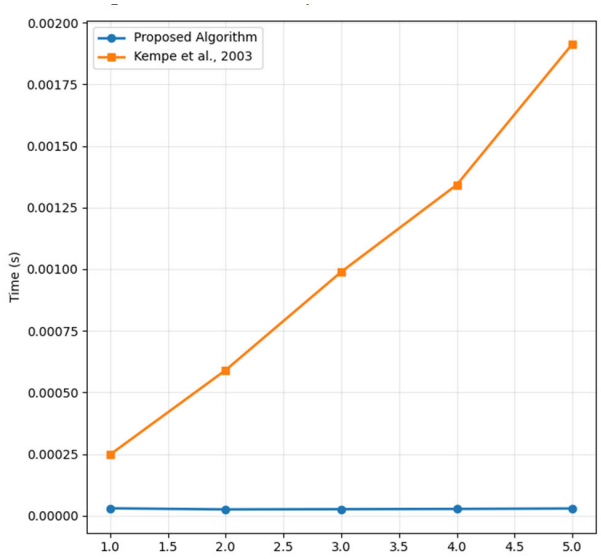

FIGURE 2. Comparison of computation time.

Network Dataset Collection as previous studies. The reason is that these datasets lack the necessary data components to determine the q value based on the dependency decision table between a follower and an influencer. Consequently, we employ real-world online social networks (OSN) data from Facebook to conduct our simulations. For data collection, we choose Vietnam for analysis because of the widespread popularity of the Facebook social network there and the availability of relevant first-hand data. We will implement the simulation in several steps. First, we choose five Organization Influencers (OIs) as follows:

- "Cat Vitamins" (893k followers)
- "CPAPS" (631k followers)
- "PETA Asia Vietnam" (243k followers)
- "Saigontime Dog & Cat Rescue Station" (178k followers)
- "Dog Yard" (113k followers)

Next, we will examine all of the posts of the above OIs in the first month of 2025. Then, we will select 7 other posts with the most interactions from each Organization Influencer, divided into 2 topics "Dogs" and "Cats". Finally, we will select the 10 most interactive followers for each Organization Influencer to test the algorithm.

In order to check the how long-lasting the impact of the influencer has on the follower, we could extend the data duration from one moth to two months. We examine all the post of above OIs in both Jan and Feb of 2025. Then we run simulation by going through three steps of Green Influencer Marketing Optimization (GIMO) mentioned in section II:

- Step 1: Setting up an approximate relational dependency decision table to quantify the degree of dependency of a follower upon the influencer.
- Step 2: By a given degree of dependency (often q is greater than or equal to 0.5), we remove all relational dependency of followers with dependency degree less than that given degree of dependency q.
- Step 3: Then, for a given message needed to promote and a given degree of dependency q, we find a minimal winning coalition of authentic influencers over that approximate relational game being refined by three above previous steps.

To verify the algorithm in the previous sections, we used two methods at the same time: manual calculation and programming in Python language. Using two methods at the same time helps to verify the accuracy of the algorithm.

To simplify the calculation and provide an initial validation, we conducted the simulations over a short period of time. The data used for this analysis was simulated for a limited time frame, which allowed us to keep the calculations manageable and focused on the core aspects of the proposed algorithms. However, given the constraints of the short simulation time, the data in this section should be considered preliminary and for reference purposes only. This is the limitation of the simulation section.

Below we will briefly present the results.

A. DATA COLLECTION

Based on real-world data collection for the simulation, we have the following tables (Tables 3 to 8).

TABLE 3. Top 10 most interactive followers in january and february 2025 for each influencers.

Influencer	Top 10 most interactive followers
Cat Vitamins	$A = \{a_1, a_2, \ldots, a_{10}\}$ $= \{Anh, Chi, Giang, Linh, Mai, Nhi, Nguyen, Phuc, Trang, Vy\}$
CPAPS	$B = \{b_1, b_2, \ldots, b_{10}\}$ $= \{Anh, Duy, Giang, Hoa, Kim, Linh, Mai, Nguyen, Thanh, Vy\}$
PETA Asia Vietnam	$C = \{c_1, c_2, \ldots, c_{10}\}$ $= \{An, Chi, Huy, Linh, Nam, Nguyen, Quang, Thao, Tran, Trang\}$
Saigontime Dog and Cat Rescue Station	$D = \{d_1, d_2, \ldots, d_{10}\}$ $= \{An, Duong, Giang, Huyen, Kim, My, Nam, Thanh, Tran, Vy\}$
Dog Yard	$E = \{e_1, e_2, \ldots, e_{10}\}$ $= \{Duong, Huyen, Mai, My, Nguyen, Nhi, Phuc, Quang, Thu, Tran\}$

In Python coding, to calculate degree of dependency q, we have following notation and formulas based on above sections:

- $\boldsymbol{a_j}$: Follower j^{th} of Influencer A ($j = 1, \ldots, 10$)
- $\boldsymbol{t^A}$: Total number of posts made by Influencer A
- $\boldsymbol{p_j^A}$: The number of "Like and Share" made by "a_j'' towards posts made by Influencer A
- $\boldsymbol{q_j^A}$: Degree of dependency of "a_j'' on Influencer A

$$q_j^A = p_j^A / t^A$$

- $\boldsymbol{t_i^A}$: Total number of posts belong to topic "i'' made by Influencer A ($i = 1, 2$)

TABLE 4. Decision table of "Like only" / "Like and Share" reactions by a set of followers $\{a_1, a_2, \ldots, a_{10}\}$ for 14 posts on two topics "Dogs" and "Cats" of Influencer A.

Cat Vitamins (Influencer A)												
Topic / Date	D	C	"Like only" / "Like and Share"									
			a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1/1	1	0	0	1	1	1	1	1	1	1	1	1
7/1	1	0	1	1	0	0	1	1	1	0	1	1
10/1	1	0	1	0	0	1	1	0	0	1	1	1
19/1	0	1	1	1	0	1	1	1	1	1	0	1
21/1	0	1	1	1	1	1	0	1	1	1	1	0
24/1	0	1	1	1	1	1	0	1	1	0	0	0
29/1	0	1	1	1	1	1	1	1	0	1	0	0
1/2	0	1	1	1	1	1	0	1	0	1	0	1
8/2	1	0	0	1	1	1	1	1	1	0	1	1
11/2	0	1	1	0	0	1	1	1	1	0	0	0
14/2	0	1	1	1	0	0	1	1	0	0	1	1
20/2	0	1	1	1	0	1	1	1	1	1	1	1
22/2	0	1	1	1	1	0	1	1	0	1	0	0
26/2	1	0	1	1	1	1	1	0	1	0	0	1
Total J	3	4	6	6	4	6	5	6	5	5	4	4
Total F	2	5	6	6	4	5	6	6	4	3	3	5

TABLE 5. Decision table of "Like only" / "Like and Share" reactions by a set of followers $\{b_1, b_2, \ldots, b_{10}\}$ for 14 posts on two topics "Dogs" and "Cats" of Influencer B.

CPAPS (Influencer B)												
Topic / Date	D	C	"Like only" / "Like and Share"									
			b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
1/1	1	0	1	0	1	1	1	1	1	1	1	0
2/1	0	1	1	1	1	1	1	1	1	1	1	0
9/1	0	1	1	0	0	1	0	1	0	0	1	1
17/1	0	1	1	1	0	0	1	1	1	1	0	0
19/1	1	0	0	1	1	0	1	1	1	1	1	1
22/1	0	1	1	0	1	1	0	0	0	1	0	1
26/1	0	1	1	0	0	1	1	1	1	1	0	0
5/2	1	0	1	0	1	0	0	1	1	1	1	1
6/2	1	0	1	1	1	1	1	1	0	1	0	1
9/2	0	1	0	1	0	0	0	1	0	1	1	0
11/2	1	0	1	0	0	0	1	1	1	1	0	0
20/2	1	0	1	1	0	1	1	0	1	1	0	0
23/2	1	0	1	0	1	1	1	1	1	1	1	0
27/2	0	1	0	0	0	1	1	1	1	0	1	0
Total J	2	5	6	3	4	5	5	6	5	6	4	3
Total F	5	2	5	3	3	4	5	6	5	6	4	2

TABLE 6. Decision table of "Like only" / "Like and Share" reactions by a set of followers $\{c_1, c_2, \ldots, c_{10}\}$ for 14 posts on two topics "Dogs" and "Cats" of Influencer C.

PETA Asia Vietnam (Influencer C)												
Topic / Date	D	C	"Like only" / "Like and Share"									
			c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
12/1	1	0	0	1	1	0	0	0	0	1	1	1
14/1	1	0	1	1	0	1	1	1	1	1	1	1
16/1	1	0	1	0	1	1	1	1	0	1	0	0
18/1	0	1	0	0	0	0	0	0	0	1	0	0
20/1	0	1	1	1	1	1	0	1	1	1	1	1
24/1	1	0	1	1	1	0	1	1	1	1	0	0
31/1	1	0	0	1	0	0	0	0	0	0	0	0
1/2	1	0	0	1	1	1	0	1	0	1	0	1
2/2	0	1	0	1	1	0	1	1	0	1	1	0
9/2	1	0	1	1	0	1	0	1	0	1	0	1
12/2	0	1	0	0	0	1	0	0	1	0	0	0
16/2	0	1	0	1	0	0	0	1	0	1	0	1
20/2	1	0	1	1	1	0	1	0	1	0	1	1
28/2	1	0	1	1	1	1	1	1	0	1	1	0
Total J	5	2	4	5	4	3	3	4	3	6	3	3
Total F	4	3	3	6	4	4	3	5	2	6	3	4

- $\boldsymbol{p_{ij}^A}$: The number of "Like and Share" made by "a_j'' towards posts made by Influencer A for topic "i''
- $\boldsymbol{q_{ij}^A}$: Degree of dependency of "a_j'' on Influencer A for topic "i''

$$q_{ij}^A = p_{ij}^A / t_i^A$$

TABLE 7. Decision table of "Like only" / "Like and Share" reactions by a set of followers $\{d_1, d_2, \ldots, d_{10}\}$ for 14 posts on two topics "Dogs" and "Cats" of Influencer D.

Saigontime Dog and Cat Rescue Station (Influencer D)												
Topic / Date	D	C	"Like only" / "Like and Share"									
			d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
2/1	1	0	1	1	1	0	1	0	0	1	1	1
7/1	1	0	1	1	0	1	1	1	1	1	1	0
15/1	1	0	0	0	1	1	1	1	1	1	0	0
19/1	1	0	0	0	0	1	0	1	1	0	0	1
22/1	0	1	1	0	1	0	0	0	0	1	1	1
28/1	1	0	1	1	0	1	1	0	1	0	1	1
29/1	1	0	1	0	0	0	0	0	0	1	1	1
5/2	1	0	1	0	0	1	1	0	0	1	0	1
6/2	0	1	0	0	1	0	1	0	0	1	0	1
10/2	1	0	1	0	1	0	1	1	1	1	1	0
16/2	1	0	1	1	0	1	0	1	1	1	1	1
19/2	0	1	1	0	0	0	0	0	0	1	1	1
24/2	1	0	1	0	1	1	1	0	1	1	1	0
27/2	1	0	1	1	0	1	0	0	1	0	0	1
Total J	6	1	5	3	3	4	4	3	4	5	5	5
Total F	5	2	6	2	3	4	4	2	4	6	4	5

TABLE 8. Decision table of "Like only" / "Like and Share" reactions by a set of followers $\{e_1, e_2, \ldots, e_{10}\}$ for 14 posts on two topics "Dogs" and "Cats" of Influencer E.

Dog Yard (Influencer E)												
Topic / Date	D	C	"Like only" / "Like and Share"									
			e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
1/1	1	0	1	1	0	1	1	1	0	0	1	1
5/1	1	0	1	0	1	0	1	0	0	1	1	0
6/1	1	0	0	1	1	1	0	0	1	1	0	1
18/1	1	0	0	0	0	0	0	0	1	0	0	0
21/1	0	1	1	1	0	1	1	1	1	1	0	0
26/1	1	0	1	1	1	0	1	1	0	0	1	1
30/1	0	1	1	1	0	0	1	0	0	0	1	1
2/2	0	1	1	1	0	1	1	0	0	1	1	1
5/2	1	0	0	0	0	0	1	1	0	0	0	1
8/2	0	1	1	1	0	0	1	1	0	1	1	0
17/2	1	0	0	0	0	1	0	0	0	1	0	0
22/2	0	1	1	1	1	0	1	1	0	0	1	1
26/2	1	0	0	0	1	1	1	1	1	0	0	1
28/2	1	0	0	1	0	0	0	0	1	0	0	1
Total J	5	2	5	5	3	3	5	3	3	3	4	4
Total F	4	3	3	4	2	3	5	4	2	3	3	5

Similarly, we have notation and formulas for all 4 other Influencers.

B. FINDING MINIMAL WINNING COALITION

For the purpose of simplification, we finding the minimal set of influencers by implementing two following steps:

Step 1. Finding the union five influencers' followers.

Based on manual calculations as well as Python programming, we have the following union:

$$\begin{aligned} V &= A \cup B \cup C \cup D \cup E \\ &= \{An, Anh, Chi, Duong, Duy, Giang, Hoa, Huy, \\ &\quad Huyen, Kim, Linh, Mai, My, Nam, Nguyen, Nhi, \\ &\quad Phuc, Quang, Thanh, Thao, Thu, Tran, Trang, Vy\} \\ &= \{An, Ah, Ci, Du, Dy, Ga, Ho, Hy, \\ &\quad He, Ki, Li, Ma, My, Na, Ne, Ni, \\ &\quad Pu, Qa, Ta, To, Tu, Tn, Tg, Vy\} \\ &= \{v_1, v_2, \ldots, v_{24}\} \end{aligned}$$

Step 2. Then, for a given topic or a set of topics, find the minimal winning coalition of influencers.

In this step, we will divide it into 3 parts as follows:

• Find the minimal winning coalition of influencers, for a given set of all topics.

• Find the minimal winning coalition of influencers, for a given topic 1 of ''Dogs''.

• Find the minimal winning coalition of influencers, for a given topic 2 of ''Cats''.

Applying Algorithm 1 to do manual calculation and remove $q < 0.5$ as well as coding into Python, we get the following results (Tables 9 to 15).

TABLE 9. Degree "q''_j of 24 followers on 5 influencers after removing all degree lower than 0.5 (Jan 2025).

Jan 2025 (All)	An	Ah	Ci	Du	Dy	Ga	Ho	Hy	He	Ki	Li	Ma
A		$\frac{6}{7}$	$\frac{6}{7}$			$\frac{4}{7}$					$\frac{6}{7}$	$\frac{5}{7}$
B		$\frac{6}{7}$				$\frac{4}{7}$	$\frac{5}{7}$			$\frac{5}{7}$	$\frac{6}{7}$	$\frac{5}{7}$
C	$\frac{4}{7}$		$\frac{5}{7}$					$\frac{4}{7}$				
D	$\frac{5}{7}$								$\frac{4}{7}$	$\frac{4}{7}$		
E				$\frac{5}{7}$					$\frac{5}{7}$			

Jan 2025 (All)	My	Na	Ne	Ni	Pu	Qa	Ta	To	Tu	Tn	Tg	Vy
A			$\frac{5}{7}$	$\frac{6}{7}$	$\frac{5}{7}$						$\frac{4}{7}$	$\frac{4}{7}$
B			$\frac{6}{7}$				$\frac{4}{7}$					
C			$\frac{4}{7}$					$\frac{6}{7}$				
D		$\frac{4}{7}$					$\frac{5}{7}$			$\frac{5}{7}$		$\frac{5}{7}$
E			$\frac{5}{7}$						$\frac{4}{7}$	$\frac{4}{7}$		

TABLE 10. Degree "q''_{1j} of 24 followers on 5 influencers after removing all degree lower than 0.5 (Jan 2025).

Jan 2025 (Dogs)	An	Ah	Ci	Du	Dy	Ga	Ho	Hy	He	Ki	Li	Ma
A1		$\frac{2}{3}$	$\frac{2}{3}$								$\frac{2}{3}$	1
B1		$\frac{1}{2}$			$\frac{1}{2}$	1	$\frac{1}{2}$			1	1	1
C1	$\frac{3}{5}$		$\frac{4}{5}$					$\frac{3}{5}$				
D1	$\frac{2}{3}$			$\frac{1}{2}$					$\frac{2}{3}$	$\frac{2}{3}$		
E1				$\frac{3}{5}$					$\frac{3}{5}$			$\frac{3}{5}$

Jan 2025 (Dogs)	My	Na	Ne	Ni	Pu	Qa	Ta	To	Tu	Tn	Tg	Vy
A1			$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$						1	1
B1			1				1					$\frac{1}{2}$
C1		$\frac{3}{5}$	$\frac{3}{5}$					$\frac{4}{5}$				
D1	$\frac{1}{2}$	$\frac{2}{3}$					$\frac{2}{3}$			$\frac{2}{3}$		$\frac{2}{3}$
E1	$\frac{3}{5}$		$\frac{3}{5}$						$\frac{3}{5}$	$\frac{3}{5}$		

The simulation for the ''Dogs'' topic reveals that engaging just four of the five original influencers suffices to cover all high-dependency followers, yielding substantial cost savings for governmental and non-profit campaigns aimed at combating pet abuse. This finding highlights the pivotal roles of two factors: the number of active followers and each follower's degree of dependence on influencers, as quantified by decision tables tracking ''like share'' versus ''like only'' reactions. Focusing exclusively on followers with strong relational dependencies allows campaign managers to optimize both efficiency and cost.

TABLE 11. Degree "q''_{2j} of 24 followers on 5 influencers after removing all degree lower than 0.5 (Jan 2025).

Jan 2025 (Cats)	An	Ah	Ci	Du	Dy	Ga	Ho	Hy	He	Ki	Li	Ma
A2		1	1			$\frac{3}{4}$					1	$\frac{1}{2}$
B2		1					$\frac{4}{5}$			$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$
C2	$\frac{1}{2}$		$\frac{1}{2}$					$\frac{1}{2}$			$\frac{1}{2}$	
D2	1					1						
E2				1					1			

Jan 2025 (Cats)	My	Na	Ne	Ni	Pu	Qa	Ta	To	Tu	Tn	Tg	Vy
A2			$\frac{3}{4}$	1	$\frac{3}{4}$							
B2			$\frac{4}{5}$									
C2			$\frac{1}{2}$			$\frac{1}{2}$		1		$\frac{1}{2}$	$\frac{1}{2}$	
D2							1			1		1
E2			1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}$	$\frac{1}{2}$		

TABLE 12. Degree "q''_j of 24 followers on 5 influencers after removing all degree lower than 0.5 (Jan and Feb 2025).

Jan and Feb 2025 (All)	An	Ah	Ci	Du	Dy	Ga	Ho	Hy	He	Ki	Li	Ma
A		$\frac{6}{7}$	$\frac{6}{7}$			$\frac{4}{7}$					$\frac{11}{14}$	$\frac{11}{14}$
B		$\frac{11}{14}$				$\frac{1}{2}$	$\frac{9}{14}$			$\frac{5}{7}$	$\frac{6}{7}$	$\frac{5}{7}$
C	$\frac{1}{2}$		$\frac{11}{14}$					$\frac{4}{7}$			$\frac{1}{2}$	
D	$\frac{11}{14}$								$\frac{4}{7}$	$\frac{4}{7}$		
E				$\frac{4}{7}$					$\frac{9}{14}$			

Jan and Feb 2025 (All)	My	Na	Ne	Ni	Pu	Qa	Ta	To	Tu	Tn	Tg	Vy
A			$\frac{9}{14}$	$\frac{6}{7}$	$\frac{4}{7}$						$\frac{1}{2}$	$\frac{9}{14}$
B			$\frac{6}{7}$				$\frac{4}{7}$					
C			$\frac{9}{14}$					$\frac{11}{14}$			$\frac{1}{2}$	
D		$\frac{4}{7}$					$\frac{11}{14}$			$\frac{9}{14}$		$\frac{5}{7}$
E			$\frac{5}{7}$	$\frac{1}{2}$					$\frac{1}{2}$	$\frac{9}{14}$		

C. SENSITIVITY ANALYSIS

To assess the robustness and practical applicability of the model, a comprehensive sensitivity analysis was conducted by varying the dependency degree q. This analysis quantifies

TABLE 13. Degree "q''_{1j} of 24 followers on 5 influencers after removing all degree lower than 0.5 (Jan and Feb 2025).

Jan and Feb 2025 (Dogs)	An	Ah	Ci	Du	Dy	Ga	Ho	Hy	He	Ki	Li	Ma
A1		$\frac{3}{5}$	$\frac{4}{5}$			$\frac{3}{5}$					$\frac{4}{5}$	1
B1		$\frac{6}{7}$				$\frac{5}{7}$	$\frac{4}{7}$			$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$
C1	$\frac{2}{3}$		$\frac{8}{9}$					$\frac{2}{3}$			$\frac{5}{9}$	
D1	$\frac{9}{11}$								$\frac{7}{11}$	$\frac{7}{11}$		
E1												

Jan and Feb 2025 (Dogs)	My	Na	Ne	Ni	Pu	Qa	Ta	To	Tu	Tn	Tg	Vy
A1			$\frac{4}{5}$	$\frac{3}{5}$							$\frac{4}{5}$	1
B1			1				$\frac{4}{7}$					
C1		$\frac{5}{9}$	$\frac{2}{3}$					$\frac{7}{9}$			$\frac{5}{9}$	
D1		$\frac{8}{11}$					$\frac{8}{11}$			$\frac{7}{11}$		$\frac{7}{11}$
E1			$\frac{5}{9}$							$\frac{2}{3}$		

TABLE 14. Degree "q''_{2j} of 24 followers on 5 influencers after removing all degree lower than 0.5 (Jan and Feb 2025).

Jan and Feb 2025 (Cats)	An	Ah	Ci	Du	Dy	Ga	Ho	Hy	He	Ki	Li	Ma
A2		1	$\frac{8}{9}$			$\frac{5}{9}$					$\frac{7}{9}$	$\frac{2}{3}$
B2		$\frac{5}{7}$					$\frac{5}{7}$			$\frac{4}{7}$	$\frac{6}{7}$	$\frac{4}{7}$
C2			$\frac{3}{5}$									
D2	$\frac{2}{3}$					$\frac{2}{3}$						
E2				1					1			

Jan and Feb 2025 (Cats)	My	Na	Ne	Ni	Pu	Qa	Ta	To	Tu	Tn	Tg	Vy
A2			$\frac{5}{9}$	1	$\frac{2}{3}$							
B2			$\frac{5}{7}$				$\frac{4}{7}$					
C2			$\frac{3}{5}$					$\frac{4}{5}$		$\frac{3}{5}$		
D2							1			$\frac{2}{3}$		1
E2			1	$\frac{3}{5}$		$\frac{3}{5}$			$\frac{4}{5}$	$\frac{3}{5}$		

TABLE 15. Minimal winning coalition of influencers for respective topic(s) and time.

Time / Topic	Jan 2025	Jan and Feb 2025
All topics	A+B+C+D+E	A+B+C+D+E
Topic 1	A+B+C+D+E	A+B+C+D
Topic 2	A+B+C+D+E	A+B+C+D+E

the impact of changes in the required level of follower engagement on both the size and cost-effectiveness of the minimal winning coalition. The algorithms were re-executed for a range of q values (e.g., 0.5, 0.6, 0.7). For each value of q, the resulting minimal winning coalition size and the total number of followers covered were calculated. This analysis was carried out using datasets from a two-month period (January and February 2025), as well as for the "All Topics," "Dogs," and "Cats" topics where applicable (Table 16).

TABLE 16. Minimal coalition size and covered followers across varying q values (Jan and Feb 2025).

Topic	q	Minimal Coalition Size	Total Followers Covered
All	0.5	5 A+B+C+D+E	21
	0.6	4 B+C+D+E	14
	0.7	3 A+C+D	11
Dogs	0.5	4 A+B+C+D	18
	0.6	3 A+C+D	17
	0.7	3 B+C+D	13
Cats	0.5	5 A+B+C+D+E	19
	0.6	5 A+B+C+D+E	18
	0.7	4 B+C+D+E	12

This sensitivity analysis provides a quantitative overview of how the model's output (coalition size, reach) changes with the key input parameter q. The results reveal the critical trade-offs inherent in influencer selection: as q increases, the definition of "dependency" becomes more stringent, resulting in smaller and potentially more cost-effective coalitions. However, this also leads to a decrease in the total number of followers classified as "highly dependent," thus reducing the overall coverage. This analysis offers practical guidance for campaign managers in determining optimal values for q, balancing cost efficiency with the desired reach and engagement. For example, a significant reduction in coalition size for a slight increase in q might suggest a highly efficient threshold, whereas a modest reduction in coalition size accompanied by a substantial decrease in covered followers may indicate a point of diminishing returns. This sensitivity analysis highlights the model's flexibility, enabling it to adapt to varying strategic objectives and priorities in influencer marketing campaigns.

D. PERFORMANCE METRICS (COST-EFFECTIVENESS)

To explicitly quantify cost savings, a hypothetical uniform cost per influencer was assumed. The total cost of engaging the "grand coalition" (i.e., all five influencers) was compared to the cost of the identified minimal winning coalition across various q values and topics. For the purpose of this analysis, the assumed costs for influencers A, B, C, D, and E

were \$5000, \$4000, \$3000, \$2000, and \$1000, respectively (Table 17).

TABLE 17. Minimal coalition size and covered followers across varying q values (Jan and Feb 2025).

Topic	q	Grand Coalition Cost	Minimal Coalition Cost	Cost Savings (%)
All	0.5	\$15000	\$15000	0%
	0.6		\$10000	33.3%
	0.7		\$10000	33.3%
Dogs	0.5	\$15000	\$14000	6.7%
	0.6		\$10000	33.3%
	0.7		\$9000	40%
Cats	0.5	\$15000	\$15000	0%
	0.6		\$15000	0%
	0.7		\$10000	33.3%

The calculation and visualization of cost savings offer a compelling demonstration of the practical value proposition of the proposed framework. For example, for the ''Dogs'' topic, increasing the q value to 0.5 results in a 6.7% cost saving by engaging four influencers, while a q value of 0.7 can lead to a 40% cost saving with only three influencers, though at the cost of reaching fewer highly dependent followers. This analysis extends the validation of the framework beyond mere algorithmic efficiency, highlighting its tangible economic benefits. Such insights are of considerable relevance to organizations and governments working within resource constraints, offering a clear path to optimizing influencer selection while balancing budgetary limitations and desired campaign reach.

VIII. LIMITATIONS AND FUTURE RESEARCH

A. INCREMENTAL AND DYNAMIC RELATIONAL GAME

One of the key limitations of the ARG model lies in its static nature, particularly concerning the fixed number of OSN users and the static count of Approximate Relational Dependencies (ARD). This limitation restricts the model's adaptability to dynamic environments, where both the number of agents and their relational dependencies evolve over time. To address this, future research will introduce the concepts of incremental relational games and dynamic relational games, where relational dependencies and agents can be continuously updated.

In an incremental approach, the challenge lies in efficiently modifying the existing game structure when ARD change. Specifically, the recalculation of closures, winning coalitions, and minimal winning coalitions becomes necessary every time ARD are modified. This recalculation, in its current form, requires $O\left(|U|^3\right)$ time complexity for updating ARD closures and $O\left(2^{|U|}\right)$ for the computation of winning coalitions, where $|U|$ represents the number of users. These high computational costs make the incremental updates of large relational games infeasible, particularly in the context of OSN with extensive user bases.

To overcome this computational bottleneck, future work will focus on developing more efficient algorithms for incremental updates. Instead of recalculating the entire game structure from scratch, these algorithms will leverage previously computed data, reusing and updating only the affected components of the game. By adopting such an approach, we aim to significantly reduce computational overhead and enable the ARG model to handle large-scale, evolving OSN. This advancement will enhance the model's practical applicability, allowing it to support dynamic changes in user interactions and relational dependencies without sacrificing computational efficiency.

B. DETERMINISTIC REACTIONS TO INFLUENCER POSTS

The current model simplifies follower reactions to a binary outcome (''Like only'' or ''Like and Share'') and calculates the dependency degree q as a deterministic ratio (p/t) . This approach assumes a direct, predictable response to a post. However, real-world social media interactions are inherently probabilistic, shaped by a range of complex and dynamic factors that extend beyond the content of the post itself. These factors include variables such as the timing of the post, the quality of the content, the mood of the follower at the moment, external events, and individual psychological states, all of which can significantly influence engagement.

Future extensions of this model could replace the simplistic deterministic ratio with a more advanced probabilistic approach, one that estimates the likelihood of a ''Like and Share'' interaction based on a combination of contextual and individual factors. Probabilistic modeling techniques, particularly those employing Bayesian networks, have demonstrated utility in capturing user sentiment, brand perception, and the likelihood of conversion. These models allow systems to reason under uncertainty and to update predictions as new data becomes available. Additionally, stochastic modeling frameworks can be used to link user behavior to platform interface elements and user activity patterns, enabling predictions based on a user's progression through various states. Integrating such models would result in a probabilistic measure of q, providing a more nuanced and robust decision-making framework. This would allow for the modeling of uncertainty and variability in human behavior, offering a predictive approach that extends beyond a simplistic average, and more accurately reflects the complexities of social media interactions.

C. STATIC FOLLOWER-INFLUENCER RELATIONSHIPS

Over a fixed period, the approximate relationship between a follower and an influencer may remain static. However, in real-time updates, this relationship can become dynamic, as the value of q is continuously updated based on the ongoing reactions of the follower to the influencer's posts. Future research could extend the existing framework by incorporating temporal elements into the relational game model. This extension would involve integrating time-series analysis of q values, possibly using techniques such as recurrent neural networks (RNNs) or dynamic Bayesian networks to model the temporal strengthening or decay of follower dependencies. By doing so, the framework could facilitate

adaptive influencer selection strategies that respond to the changing dynamics of the network, ensuring the long-term effectiveness of marketing campaigns and providing a more accurate representation of social influence over time.

IX. CONCLUSION

In this paper, we have developed a rigorous mathematical framework for green influencer marketing by introducing the concept of Approximate Relational Games (ARG). By constructing decision tables that distinguish ''like share'' from ''like only'' reactions over defined time windows, we compute each follower's dependency degree q and focus on those with $q \geq 0.5$. Leveraging closure operators to expand any influencer subset into the full set of dependent followers and dual choice-function characterizations to identify minimal covering influencer sets, we transform the selection problem into one solvable in polynomial time. We realize this theory through two concrete algorithms: one that iteratively prunes the grand coalition to its minimal core and another that greedily seeds from the influencer with the largest reach before incrementally augmenting until full coverage. Both algorithms are shown to run in polynomial time in the number of influencers and followers. Although our current analysis assumes complete and consistent reaction data, it establishes a scalable foundation for future extensions to settings with noisy, partial, or adversarial information, thus offering nonprofits, governmental bodies, and corporate sustainability programs a principled tool for assembling cost-effective, high-impact influencer coalitions.

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