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Superpolynomial period lengths of the winning positions in the subtraction game

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Abstract

Given a finite set of positive integers, A, and starting with a heap of n chips, Alice and Bob alternate turns and on each turn a player chooses $x \in A$ with x less than or equal to the current number of chips and subtract x chips from the heap. The game terminates when the current number of chips becomes smaller than $\min\{A\}$ and no moves are possible. The player who makes the last move is the winner. We define $w^A(n)$ to be 1 if Alice has a winning strategy with a starting heap of n chips and 0 if Bob has a winning strategy. By the Pigeonhole Principle, $w^A(n)$ becomes periodic, and it is easy to see that the period length is at most an exponential function of $\max\{A\}$. The typical period length is a linear function of $\max\{A\}$, and it is a long time open question if exponential period length is possible. We consider a slight modification of this game by introducing an iitial seed S that tells for the few initial numbers of chips whether the current or the opposite player is the winner, and the game ends when the first such position is achieved. In this paper we show that the initial seed cannot change the period length of $w^A(n)$ if the size of A is 1 or 2, but it can change the period length with $|A| \geq 3$. Further, we exhibit a class of sets A of size 3 and corresponding initial seeds such that the period length becomes a superpolynomial function of $\max\{A\}$.

Keywords Combinatorial games · Subtraction game · Superpolynomial period length

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1 Introduction

Game Theory is the theory of interactive situations or games among rational decision-makers or players in which the decisions of each player are contingent on the decisions of the others. Combinatorial Game Theory considers games with perfect information and without elements of chance. That is, at all times during the game, players have perfect information about the state of the game, and further, the moves in the game are entirely decided by the players, there are no elements of chance once the game has begun. We further require that a combinatorial game must end with a clear winner.

An example of a two-player combinatorial game is the subtraction game. For a finite set $A \subset \mathbb{N}_+$, the ***A*-subtraction game** is a two-player combinatorial game which proceeds as follows. We begin with heap of n-chips. Players Alice and Bob alternate turns, and on each turn a player chooses $x \in A$ with $x \leq n$, and subtracts x chips from the heap, leaving $n - x$ remaining. The game terminates when $n < \min(A)$ and thus no moves are possible. The player who makes the last move is the winner. In general we consider a fixed A and ask for which values of n each player has a winning strategy. Note that when Alice makes a move x, Bob and Alice switch roles and we reduce to the $n - x$ game.

The subtraction game is also in the large class of combinatorial games called *impartial games*. An impartial game is a combinatorial game in which the allowable moves depend only on the position and not on which of the two players is currently moving, and where the payoffs are symmetric. It is also in *normal mode*, meaning that the winner is who can make the last possible move. The Sprague–Grundy theorem (Sprague 1936; Grundy 1939) says that any impartial game in normal mode is equivalent with a Nim game, which is the disjunctive sum of $\mathbb{N}_+$-subtraction games. Despite this reduction, we know little about the patterns of the winning positions of the subtraction game.

For any finite set $A \subset \mathbb{N}_+$, a dynamic programming recursion can compute which player has the winning strategy starting with a pile of size n. A simple reasoning by Pigeonhole Principle shows that the pattern of winning positions will eventually become periodic as n takes all possible positive integers, and the period length cannot be longer than $2^{\max(A)}$. It is a long time open question if exponential period lengths exist in the subtraction game. Althöfer and Bültermann conjectured that superpolynomial period lengths might exists if $|A| \geq 5$ (Althöfer and Bültermann 1995). Flammenkamp (Flammenkamp 1997) also had experimental evidences for exponential period length for some subtraction sets. He also gave non-trivial upper bounds on the period lengths (Flammenkamp 1997, page 62), see also Larsson (2024).

For some A, the subtraction game has a pre-period in the winning positions before becoming periodic, that is, a pattern of winning positions for small n that is never repeated. We also know little about for which A the subtraction game has a pre-period and for which A it is *purely periodic*, that is, has no pre-period.

In this paper, we generalize the subtraction game by modifying who is the winner for some small n. We call this initial pattern a *seed*. It is called "initial" since these are the initial values in the dynamic programming recursion to compute who has the winning strategy starting with a given value (see also Eq. (1)). On the other hand, the game ends when a seed position is reached. We give a complete analysis of subtraction games with seeds for $|A| = 1$ and 2. We prove that for all seeds the game has no pre-period if $|A| = 1$ or 2. For $A = \{a\}$, the period length is $2a$. For $A = \{a, b\}$, the period length can be any divisor of $a + b$, with a few exceptions. If a and b are relatively prime then the period can be any divisor except 1, 4, and 6. We also compute the number of possible distinct period lengths over all seeds. When $|A| = 3$, then there might be pre-periods. We give characterisations of period and pre-period lengths for a large class of possible sets A. Finally, we show that superpolynomial period lengths exist already when $|A| = 3$. The introduction of the seed is considered in Sects. 3 and 6, wheras Sects. 4 and 5 consider only the standard game.

2 Preliminaries

Definition 1 Let $w^A(n)$ be the winning indicator function of A, so $w^A(n) = 1$ if Alice (the first player) has a winning strategy and $w^A(n) = 0$ if Bob has a winning strategy.

For brevity, we may use $w(n)$ to refer to $w^A(n)$. By definition, w satisfies the recurrence relation

$$w(n) = \begin{cases} 1 & \text{if } w(n-x) = 0 \text{ for some } x \in A \\ 0 & \text{if } w(n-x) = 1 \text{ for all } x \in A \end{cases} = 1 - \min\{w(n-x) \mid x \in A\}. \quad (1)$$

This follows from the observation that for a particular n, Alice is in a winning position if she can subtract some x to give Bob a losing position. Otherwise, she will certainly move to a winning position for Bob, and he will win. We can then also describe w^A as the lexicographically least sequence in n such that for all $x \in A$, $w(n) = 0 \implies w(n + x) = 1$.

Note that it is natural to define $w(0)$ to be 0, because if a player has previously made a move to 0, then the next player will lose. Therefore, to satisfy recurrence relation (1), we shall define $w(n)$ to be 1 for all $n < 0$.

To describe an entire sequence $\{w^A(n)\}_{n=0}^{\infty}$ of winning positions, abbreviated as $\{w^A\}$, we use exponents to denote repeated values. A noted example in (Berlekamp et al. 2001, p. 86) is the set $\{2, 4, 7\}$. We find that $\{w^{\{2,4,7\}}\} = 0, 0, 1, 1, 1, 1, 0, 1, 1, 0, \ldots,$ which can be abbreviated to $\{w^{\{2,4,7\}}\} = 0^2 1^4 0 1^2 0 \ldots = 0^2 1^2 (1^2 0)^{\infty}$. In this example we have $w(0) = 0$, $w(1) = 0$, and $w(2) = 1$. This follows from the rule that if $n < 2$, Alice is unable to move, but at $n = 2$, Alice may subtract 2 chips and win the game. Similarly $w(6) = 0$ because for any move Alice makes, Bob can respond with a winning move. In general, if we present a prefix of $\{w^A\}$ of length ℓ, we use "$\underset{\ell}{\ldots}$" to indicate the continuation of the sequence and clarify the prefix's length for the reader. For example,

$\{w^{\{2,4,7\}}\} = 0^2 1^4 \underset{6}{\ldots}$ may indicate that the first 6 values of w are given and the rest are not yet derived. Throughout the paper, we refer to $\alpha = \max(A)$. The following example is an easy generalization of one in (Berlekamp et al. 2001, p. 83).

Proposition 1 *Suppose* $A = \{1, 2, \ldots, \alpha\}$. *Then* $w^A(n) = \begin{cases} 0 & (\alpha+1) \mid n \\ 1 & \text{otherwise} \end{cases}$, *so* $\{w^A\} = \left(01^\alpha\right)^\infty$.

Proof Suppose $n = k(\alpha + 1)$. We claim Bob has a winning strategy. If Alice subtracts x, then Bob can subtract $\alpha + 1 - x$, reducing to $(k-1)(\alpha+1)$ chips. Bob can repeat this until there are $(0)(\alpha+1)$ chips, winning the game. Suppose $n = k(\alpha+1) + y$. Then Alice has a winning strategy. She may subtract y, reducing the game to $k(\alpha+1)$, then play as Bob would by countering each of his moves x with $\alpha + 1 - x$. □

We define a notation for repeated concatenation of strings. By analogy to addition, for strings $w_1, w_2, \ldots w_k$, let

$$\sum_{i=1}^{k} w_i := w_1 \circ w_2 \circ \ldots \circ w_k.$$

This notation satisfies the equality $\left|\sum_{i=1}^{k} w_i\right| = \sum_{i=1}^{k} |w_i|$. Recall that sequence concatenation is not a commutative operation, unlike the usual summation of numbers. When we use a summation symbol for a concatenation of strings, we will always use an index that defines the order of the concatenation.

Definition 2 A sequence $\{w^A(n)\}$ is **periodic** over p if there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $w^A(n) = w^A(n+p)$. We say the **period** of $w^A(n)$ is the least such p and the **preperiod** is the least such N. We denote these by $\text{Per}(A)$ and $\text{PrePer}(A)$ respectively. A string X has **sub-period** Y if $X = Y^k$ for some $k > 1$.

In Example 1 we have $\text{Per}(\{1, \ldots, \alpha\}) = \alpha + 1$ and $\text{PrePer}(\{1, \ldots, \alpha\}) = 0$. We also observe that $\text{Per}(\{2,4,7\}) = 3$ and $\text{PrePer}(\{2,4,7\} = 4$, because for all $n \geq 4$ it holds that $w(n) = w(n+3)$.

Lemma 2 *For any finite set* $A \subset \mathbb{N}_+$, $\{w^A(n)\}$ *is periodic.*

Proof To prove this fact, we define a new tool called the vector of previous values. Given $A \subset \mathbb{N}_+$, let

$$v^A(n) := \langle w(n-\alpha), \ldots, w(n-2), w(n-1)\rangle = \langle v_1, \ldots, v_\alpha\rangle \in \mathbb{Z}_2^\alpha. \tag{2}$$

As shown above, $\mathbf{v}(n)$ will be an element of $\mathbb{Z}_2^\alpha$. Next, we use the recurrence relation to define a function $\mathcal{F} : \mathbb{Z}_2^\alpha \to \mathbb{Z}_2^\alpha$.

$$\mathcal{F}(\mathbf{v}) := \langle v_2, v_3, \dots, v_{\alpha-1}, 1 - \min\{v_{\alpha-x} \mid x \in A\}\rangle \tag{3}$$

Thus by Recurrence 1, we have $\mathbf{v}(n+1) = \mathcal{F}\big(\mathbf{v}(n)\big)$. Note that $\mathbb{Z}_2^{\alpha}$ has 2^{α} elements, so by the Pigeonhole Principle there must be distinct integers $0 \le N < M \le 2^{\alpha}$ such that $\mathbf{v}(N) = \mathbf{v}(M)$. Therefore, for all $n \ge N$ we have the equality

$$\mathbf{v}(n) = \mathcal{F}^{n-N}\big(\mathbf{v}(N)\big) = F^{n-N}(\mathbf{v}(M)) = \mathbf{v}(n + M - N).$$

Therefore, a single repeated vector guarantees periodicity of length $M - N$. □

Indeed this lemma often fails for infinite set games.

Proposition 3 *For some $k \ge 2$, suppose $A = \{n^k \mid n \in \mathbb{N}\}$. Then the sequence w^A is not periodic.*[1]

Proof We first show there must be infinitely many losing positions by contradiction. Suppose there are ℓ losing positions. Among all integers $\le (2\ell)^k$, there are 2ℓ elements of A. Because each winning position can be expressed as a losing position plus some $x \in A$, there can be combinatorially at most $2\ell^2$ distinct winning positions $\le (2\ell)^k$. This implies $\ell + 2\ell^2 \ge (2\ell)^k$, which contradicts finiteness of losing positions. Thus $\{w^A\}$ has infinitely many zeros. If we suppose w is periodic after some N with period p, then there is some $n > N$ with $w(n) = 0$. This implies that after p^{k-1} periods, we will have $w(n + p^k) = 0$, contradicting $p^k \in A$. □

The proof of Lemma 2 gives the result that for any finite A, $\mathrm{PrePer}\,(A) + \mathrm{Per}\,(A) \le 2^{\alpha}$. We can get a slightly tighter bound for dense sets, but no less than exponential in α.

Theorem 4 *For $A = \{a_1, a_2, \dots, \alpha', \alpha\}$, let $\beta = \min(\alpha, a_1 + \alpha')$. Then*

$$\mathrm{PrePer}\,(A) + \mathrm{Per}\,(A) \le (\beta + 1)2^{\alpha - |A|}. \tag{4}$$

Proof First, we show that there cannot be a string of ones longer than β in $\{w^A\}$. If some $w(n)$ is preceded by a string of α ones, then $w(n - x) = 1$ for all $x \in A$, so $w(n) = 0$. Alternately, if n is preceded by a string of $a_1 + \alpha'$ ones, this implies that in particular $w(n - a_1) = 1$, and $(n - a_1)$ is preceded by α' ones. This means that for all $a_i \in A$ *except for* α, we have $a_i \le \alpha'$ and therefore $w(n - a_1 - a_i) = 1$. Because $w(n - a_1) = 1$, by process of elimination we conclude that $w(n - a_1 - \alpha) = 0$. Therefore $w(n - \alpha) = 1$. Thus $w(n - x) = 1$ for all $x \in A$, so $w(n) = 0$. Hence β bounds the number of consecutive ones in $\{w^A\}$.

Now, let n_i be the i^{th} zero in $\{w^A\}$. By the proof above $n_i - n_{i-1} \le \beta + 1$, so $n_i \le i(\beta + 1)$. In order to have $w(n_i) = 0$, we require that within the vector $\mathbf{v}(n_i)$, the $\alpha - x^{\text{th}}$ entry $\mathbf{v}(n_i)_{\alpha - x} = 1$ for all $x \in A$. This fixes $|A|$ entries, so there are $2^{\alpha - |A|}$

[1] For $k = 2$, the losing positions of this sequence are in the Online Encyclopedia of Integer Sequences (OEIS Foundation Inc. 2023, A030193)

possibilities for $\mathbf{v}(n_i)$. By the Pigeonhole Principle, there is some repeated vector $\mathbf{v}_{n_N} = \mathbf{v}_{n_M}$ for some $N < M \leq 2^{\alpha-|A|}$, and $n_M \leq (\beta + 1)2^{\alpha-|A|}$. This proves the claim. □

A similar argument can give a bound of $(|A| + 1)2^{\alpha-|A|}$. Using the approach from Proposition 3, we can see that there are at most $(2^{\alpha-|A|} - 1) \cdot (|A|)$ ways to express a winning position less than $n_{2^{\alpha-|A|}}$, which yields this improved constant.

2.1 Initial seed

We can generalize the subtraction game by changing the end state of the game. In (Althöfer and Bültermann 1995, (ix)), Althöfer and Bülterman suggest that this variant "may be interpreted as simulations of certain computing devices," and pose open questions about this game. Through the analysis in Sects. 3, 4, and 6 we find that this generalization also provides insights into the original game. We begin with two motivating examples.

Example 1 The ***Miseré*** mode of the A-subtraction game is the same game except the player to make the last move is the loser.

Example 2 The ***Greedy*** mode of the A-subtraction game is the same game except a player may take $x > n$ chips, thereby making the heap negative. The game concludes when then heap is negative, and the player to make the last move is the winner. This is close to the version studied in Althöfer and Bültermann (1995).[2]

Notice that for $n \geq \alpha$, the recurrence relation for both of these games is the same as Eq. (1), but the ultimate sequence of winning positions may be different because of the players' final goals. We can account for this by adjusting negative values of $w(n)$, then allowing the recurrence relation to proceed for $n \geq 0$.

Definition 3 Define the **seed** S of a game to be the value of $\mathbf{v}(0)$, determining $w(n)$ for $n \in [-\alpha, 0)$. The recurrence relation follows, so $\mathbf{v}(n) := \mathcal{F}^n(S)$, with $\mathcal{F}$ defined in Eq. (3). We define $\{w^{A,S}(n)\}_{n=0}^{\infty}$ to be the sequence generated by seed S. Further define $\text{Per}(A, S)$ and $\text{PrePer}(A, S)$ to be the period and preperiod of $\{w^{A,S}\}$.

For ease of notation, we interpret S more generally as the negative values of $w^A(n)$. Ordinarily $|S| \leq \alpha$, and if not then $\mathbf{v}^{A,S}(0)$ is taken to be the last α elements of S. Similarly, if $|S| < \alpha$, then we presume S to be preceded by infinitely many 1's, so let $w^{\{A,S\}} := w^{\{A,1^{\alpha-|S|}S\}}$. It follows from this convention and Definition 2 that

[2] The exact game studied in Althöfer and Bültermann (1995) is the same but has $w(0) := 0$; this cannot be generated from a seed as there is no function $w : \mathbb{Z} \to \{0, 1\}$ satisfying recurrence 1 for all $n \in \mathbb{N}$. For example, if $A = \{1, 3\}$, we desire $w(0) = 0$, $w(1) = 1$, and $w(2) = 1$. Any choice of $w(-1)$ leads to a contradiction.

$\text{Per}(A) = \text{Per}(A, 1^\alpha)$, and thus in general we refer to a game with seed 1^α as having 'no seed'.

In Example 1, we observe that $S = 0^{\min(A)}1^{\alpha-\min(A)}$ generates the sequence of winning positions for the Miseré mode of the subtraction game, and in Example 2, we see that $S = 0^\alpha$ generates the sequence for the Greedy game. By considering all seeds in $\mathbb{Z}_2^\alpha$ we describe a larger class of games. Some seeds generate games which are similar or identical to the original, while some are dramatically different. For example, the Miseré and Greedy modes cause only a translation in $w(n)$ by $\min(A)$ and α respectively.

Notice that Results 2–4 consider the recurrence in generality and therefore hold for all seeds. In order to characterize $w^{A,S}$ over all seeds, we provide the following notation for the set of all winning sequences and their periods.

$$\mathcal{W}^A := \left\{ \{w^{A,S}(n)\}_{n=0}^{\infty} \mid S \in \{0,1\}^\alpha \right\} \tag{5}$$

$$\mathcal{P}^A := \left\{ \text{Per}(A, S) \mid S \in \{0,1\}^\alpha \right\} \tag{6}$$

Thus $\mathcal{W}^A$ is the set of all (A, S) games, i.e. all sequences satisfying recurrence relation 1, and $\mathcal{P}^A$ is the set of their periods. For general A, a natural open question is to find $\max(\mathcal{P}^A)$.

2.2 Properties of subtraction games

If the elements of A are not coprime, we can interpret the sequence w as multiple games $(w)_i$ proceeding in parallel. Formally, choose finite $A \subset \mathbb{N}_+$ with maximum α and gcd 1, and denote $kA = \{kx \mid x \in A\}$. Then given some seed S with $|S| = k\alpha$, where $S = \langle S(m) \mid m \in \{0, \dots, k\alpha - 1\}\rangle$, we break S into classes modulo k by defining S_i for $i \in \{0, \dots k-1\}$ such that for all $m \in \{0, \dots \alpha - 1\}$, we have $S_i(m) := S(mk + i)$. Thus $|S_i| = \alpha$ and S can be decomposed into S_i's. By analogy, similarly let $(w^{kA,S})_i(m) := w^{kA,S}(mk + i)$. We observe that $\{(w^{kA,S})_i(m)\}_{m=0}^{\infty}$ depends only on S_i.

Proposition 5 (Multiplicative Linearity) *For any* $k \in \mathbb{N}$, *set* A, *and seed* S *with* $|S| = k\alpha$, *then*

$$(w^{kA,S})_i(m) = w^{A,S_i}(m) \tag{7}$$

So if $S_i = S_0$ *for all* $i \in \{0, \dots, k-1\}$, *then* $\text{Per}(kA, S) = k\,\text{Per}(A, S_0)$ *and* $\text{PrePer}(kA, S) = k\,\text{PrePer}(A, S_0)$. *This condition holds for* $S = 1^{k\alpha}$, *implying that* $\text{Per}(kA) = k\,\text{Per}(A)$ *and* $w^{kA}(n) = w^A(\lfloor n/k \rfloor)$. □

Proposition 5 follows by applying the recurrence relation to $(w^{kA,S})_i(m)$.

Definition 4 (Extension) Choose set $A \subset \mathbb{N}_+$ and seed S. We say $b \in \mathbb{N}\backslash A$ is an **extension** of (A, S) if $w^{A\cup\{b\},S} = w^{A,S}$.

Proposition 6 (*Better Definition of Extension*) *For* $b \in \mathbb{N} \backslash A$ *is an extension of* (A, S) *if and only if for all* $n \in \mathbb{N}$ *it holds that* $w^{A,S}(n) = 0 \implies w^{A,S}(n-b) = 1$.

Proof Let $B = A \cup \{b\}$. Suppose it holds that $w^{A,S}(n) = 0 \implies w^{A,S}(n-b) = 1$, but $w^{B,S} \neq w^{A,S}$, and choose the least $n \in \mathbb{N}$ where they differ. If $w^{A,S}(n) = 1$, then for some $x \in A$, $w^{A,S}(n-x) = w^{B,S}(n-x) = 0$, so $w^{B,S}(n) = 1$. If instead $w^{A,S}(n) = 0$, then $w^{A,S}(n-b) = w^{B,S}(n-b) = 1$, so indeed $w^{B,S} = 0$, contradicting that $w^{B,S} \neq w^{A,S}$.

In the other direction, suppose $w^{B,S} = w^{A,S}$ but there is some n with $w^{B,S}(n) = w^{A,S}(n) = 0$ and $w^{B,S}(n-b) = w^{A,S}(n-b) = 0$. This contradicts the recurrence relation for B. □

This means that we can identify redundant elements of a set A if they are extensions of the other elements. The following proposition is a digression, but

Proposition 7 *For all finite or infinite sets* $A \subseteq \mathbb{N}_+$, *if* $\mathrm{PrePer}(A) = 0$ *and* $\mathrm{Per}(A) = p$, *then* $w^A = w^{A \cap \{1,\dots,p\}}$, *so every element of* A *greater than* p *is redundant as an extension of* $A \cap \{1, \dots, p\}$. *There is no analogous statement for sequences with preperiods; for example* $\{w^{\{1,4,7,\dots\}}\} = 0(101)^\infty$, *but no other set generates that sequence.*

Proof For the first statement, let $B = A \cap \{1, \dots, p\}$. Suppose for the sake of contradiction there is some least n such that $w^A(n) \neq w^B(n)$. By the recurrence relation it must be that $w^A(n) = 1$ and $w^B(n) = 0$, and $w^A(n-a) = 0$ for some $a \in A \backslash B$. By our definitions $a > p$ so $n > p$. Then $w^B(n-p) = w^A(n-p) = 1$ so there is some $x \in B$ such that $w^B(n-p-x) = 0$. Therefore $w^B(n-x) = 0$, so $w^B(n) = 1$, a contradiction. For the second statement, suppose $w^A = 0(101)^\infty$. Suppose $x \equiv 0 \pmod 3$. Then $w^A(2) = w^A(2+x) = 0$, so $x \notin A$. Suppose $x \equiv 2 \pmod 3$. Then $w^A(0) = w^A(x) = 0$, so $x \notin A$. Therefore $A \subseteq 3\mathbb{N}+1 = \{1, 4, 7, \dots\}$. It is easy to see that $A = 3\mathbb{N}+1$ is a valid choice. Suppose $A \subsetneq 3\mathbb{N}+1$, and $3k+1$ is the least element of $3\mathbb{N}+1 \backslash A$. Then surely $w^A(n) = w^{3\mathbb{N}+1}(n)$ for all $n < 3k+1$. However, for all $3j+1 \in A$, $j < k$, $w^A(3k+1-(3j+1)) = w^A(3(k-j)) = 1$, so $w^A(3k+1) = 0$, a contradiction. Thus $3\mathbb{N}+1$ is the only set generating this sequence. □

The following proposition gives another way to identify these extensions, which we use later in the paper.

Proposition 8 *If* $\{w^A\}$ *is periodic over* p *and* $\mathrm{PrePer}(A) = 0$, *then for all* $x \in A$ *and* $k \geq 1$, $b = kp + x$ *is an extension of* A.

Proof Suppose $\mathrm{Per}(A) \mid p$ and $\mathrm{PrePer}(A) = 0$. Choose any $x \in A$ and $k \geq 1$, and $n \in \mathbb{N}$ such that $w^A(n) = 0$. By the recurrence $w^A(n-x) = 1$. If $n - x - kp \geq 0$, then by periodicity $w^A(n-x-kp) = 1$. Otherwise, $w^A(n-x-kp) = 1$ because we have no seed. Thus by the better definition b is an extension of A. □

The following elementary example can be found in (Ho 2015, thm 1).

Example 3 If $A = \{1\}$, then $\{w^A\} = (01)^\infty$, so $\mathrm{Per}(A) = 2$ and $\mathrm{PrePer}(A) = 0$. Because $1 \in A$, any odd number $2k + 1$ is an extension of A.

From Example 1, if $\{1, \dots, k-1\} \subseteq B$ and $B \cap k\mathbb{N} = \emptyset$, then $\{w^B\} = (01^{k-1})^\infty$ and $\mathrm{Per}(B) = k$. The set $B = \{1\} \cup \{p \mid p \text{ prime}\}$ is an example for $k = 4$.

We must be careful to note that Proposition 8 does not hold for all seeds. If we do allow for initial seeds $|S| \leq \alpha$, the induction step fails for $n \in [p + x - \alpha, p)$. This margin leads to many counterexamples; we provide two. Let $A = \{3, 5\}$ and $S = 0110$. We find $\mathrm{Per}(A, S) = 8$ and $\mathrm{PrePer}(A, S) = 0$, so let $b = 8 + 3 = 11$. Then:

$$\{w^{A,S}\} = (01^5 0^2)^\infty \text{ and } \{w^{A \cup \{b\}, S}\} = 01^5 0101^3 (01)^\infty$$

These are vastly different and we even caused a preperiod of length 12. As another example:

$$\{w^{\{3,6,8\}, 0^2 1^3}\} = (011^3 (01)^3)^\infty \text{ and } \{w^{\{3,6,8,12\}, 0^2 1^3}\} = (01^4 01^3 001^3)^\infty.$$

We also note that the converse of Proposition 8 does not hold, though it appears to hold if $|A| \leq 3$. For example, let $A = \{1, 2, 6, 11\}$. Then $\{w^A\} = 0\big(1^2 01^3 (01^2)^2\big)^\infty$, so $\mathrm{Per}(A) = 12$, $\mathrm{PrePer}(A) = 1$. However it is true that for all $k \geq 1$ and $x \in A$, $b = 12k + x$ is indeed an extension of A. Also see the counterexample $\{1, 8, 13, 16\}$.

Contrasting these complex examples with Example 3, we see that sequences are easiest to examine with no seed and no preperiod. The following Lemma can simplify the process of checking whether a sequence is in this form.

Lemma 9 (Translating zeros) *Given a set A, then for any $p \in \mathbb{N}$, w^A is periodic over p and has no preperiod if and only if it holds for all $x \in A$ and $m < x$ that*

$$w^A(m) = 0 \implies w^A(m + p - x) = 1.$$

We call this "translating zeros" because $w(m) = 0 \implies w(m + p - x) = 1$ suggests that $w(m + p) = 0$, but does not discuss the translation of the 1's. Note that it does not *a priori* imply that the zeros translate, since only $m < x$ is considered.

Proof Suppose for the sake of contradiction that the premise $w(m) = 0 \implies w(m + p - x) = 1$ holds but there is some least $m \in \mathbb{N}$ such that $w(m) \neq w(m + p)$. There are two cases.

(i) If $w(m) = 1$, then $w(m + p) = 0$, so there is some $x \in A$ such that $w(m - x) = 0$ but $w(m + p - x) = 1$. Because there is no seed this implies $m - x \geq 0$, and because $w(m - x) \neq w(m - x + p)$ this contradicts the minimality of m.

(ii) If $w(m) = 0$, then $w(m + p) = 1$, so there is some $x \in A$ with $w(m + p - x) = 0$ but $w(m - x) = 1$. If $m - x \geq 0$, then this would contradict the minimiality of n. Otherwise, $m < x$ and $w(m) = 0$, so the premise implies that $w(m + p - x) = 1$, a contradiction.

Either case leads to a contradiction, so for all $m \geq 0$, we have $w(m) = w(m+p)$. In the other direction, if w is periodic with no preperiod then the condition follows by definition. □

Corollary 9.1 *If* $A = \{a, b, c\}$ *for any* $a < b, c$, *then* $\mathrm{Per}(A) \mid (b+c)$ *and* $\mathrm{PrePer}(A) = 0$ *if and only if* $w^A(b+c-i) = 1$ *for all* $i \in [1, a]$.

Proof Apply Lemma 9 for $p = b + c$. For all $x \in A$, if $x = b$, then $w(m) = 0 \implies w(m+p-b) = w(m+c) = 1$ by the recurrence. If $x = c$, then $w(m) = 0 \implies w(m+p-c) = w(m+b) = 0$. If $x = a = \min(A)$, then we note that for all $m < a$, $w(m) = 0$. Therefore it suffices to check that for $m \in [0, a-1]$, we have $w^A(b+c+m-a) = 0$. Simplify by substituting $i = a - m \in [1, a]$. □

This Corollary will likely help in proving Conjecture 3.

Corollary 9.2 *If* $A = \{1, b, c\}$, *then* $\mathrm{Per}(A) \mid (b+c)$ *and* $\mathrm{PrePer}(A) = 0$ *if and only if* $w^A(b+c-1) = 1$. □

These corollaries give some insight to the usefulness of the translating zeros Lemma. For Corollary 9.2 we can determine the period and preperiod by checking one value of the sequence! This will be used in Sect. 4.

Lemma 10 *The following statements are equivalent.*

(i)*w is periodic over p with no preperiod*
(ii)For all $n \geq 0$, $w(n) = w(n+p)$
(iii)For all $n \geq \alpha$, $\mathbf{v}(n) = \mathbf{v}(n+p)$
(iv)$\mathbf{v}(\alpha) = \mathbf{v}(\alpha + p)$
(v)*For all $x \in A$, for each $m < x$, we have $w(m) = 0 \implies w(m+p-x) = 1$.* □

3 The $\{a, b\}$ case

We first inspect the case when $|A| = 1$. If $A = \{a\}$, then the recurrence relation in Eq. 1 gives that $w^A(n) = 1 - w^A(n-a)$ for all $n \in \mathbb{N}$. This gives us an immediate characterization of the periodicity for all possible seeds.

Proposition 11 *Let* $A = \{a\}$. *For all seeds S, let $\overline{S}$ be the string exchanging* 0'*s and* 1'*s. Then* $\{w^{A,S}\} = \left(\overline{S}\,S\right)^\infty$, *so* $\mathrm{Per}(A, S) \mid 2a$ *and* $\mathrm{PrePer}(A, S) = 0$. □

This gives the result that $\mathcal{W}^{\{a\}} = \left\{(\overline{S}\,S)^\infty \mid S \in \{0,1\}^a\right\}$ and thus $\left|\mathcal{W}^{\{a\}}\right| = 2^a$.

Proposition 12 *Let* $A = \{a\}$, *and let* $a = 2^k c$, *where* c *is odd. Then* $\mathcal{P}^A = \{2^{k+1} d : d \mid c\}$.

Proof First, we show that $p = \mathrm{Per}(A, S)$ must be of the form $2^{k+1}d$. We know that $\{w^{A,S}\}$ will always be periodic over $2a$, so $p \mid 2a$. Additionally, we know $w^{A,S}(n) \neq w^{A,S}(n+a)$ for all n, so $p \nmid a$. Therefore $2^{k+1} \mid p$. We now show that for any $d \mid c$ we have $2^{k+1}d \in \mathcal{P}^{\{a\}}$. Because c/d is odd let $c = (2x+1)d$. Then let $S = \left(1^{2^k d}0^{2^k d}\right)^x 1^{2^k d}$, so we conclude $\{w^{A,S}\} = \left(\left(1^{2^k d}0^{2^k d}\right)^x 1^{2^k d} \left(0^{2^k d}1^{2^k d}\right)^x 0^{2^k d}\right)^\infty = \left(1^{2^k d}0^{2^k d}\right)^\infty$ and $\mathrm{Per}(\{a\}, S) = 2^{k+1}d$. □

These initial results are apparent at first inspection of the problem. With more work will get similarly general results for $|A| = 2$.

Theorem 13 *Let* $A = \{a, b\}$. *For all seeds* S, $\mathrm{Per}(A, S) \mid a+b$ *and* $\mathrm{PrePer}(A, S) = 0$.

Proof Let $A = \{a, b\}$. For all $n \in \mathbb{N}$, in the case where $w(n) = 0$ we have $w(n+a) = w(n+b) = 1$, which implies that $w(n+a+b) = 0$. Alternately, if $w(n) = 1$, then $w(n-a) = 0$ or $w(n-b) = 0$, so by the 0-case we know $w(n+b) = 0$ or $w(n+a) = 0$ respectively. This means $w(n+a+b) = 1$. □

The following Theorem on 2-move games with no seed is well known and can be found in (Berlekamp et al. 2004, p. 530).

Theorem 14 *Let* $A = \{a, b\}$ *with* $a < b$, *and let* $b = qa + r$ *with* $0 \le r < a$. *Then*

$$\{w^A\} = \begin{cases} (0^a1^a)^{q/2}0^r1^a & q \text{ is even.} \\ (0^a1^a)^{\lceil q/2 \rceil}1^r & q \text{ is odd.} \end{cases} = \Big(\underbrace{0^a1^a0^a1^a \ldots 1^a}_{b}\Big)^\infty \tag{8}$$

so $\mathrm{Per}(A) = 2a$ *if b is an odd multiple of a and* $\mathrm{Per}(A) = a + b$ *otherwise.*

Proof For $n < b$, we note that by having no seed $w^A(n-b) = 1$. This implies $w^A(n) = w^{\{a\}}(n)$. This yields the sequence $\{w^A\} = \underbrace{0^a1^a0^a1^a \ldots}_{b}$. For $n \in [b, a+b)$, we note that $n - b \in [0, a)$ so $w^A(n-b) = 0$ and thus $w^A(n) = 1$. Theorem 13 implies that this period repeats. □

3.1 A preliminary tool: studying $\{1, b\}$

Theorem 14 characterizes the period structure for the default seed. Our next goal is to describe the sequence under all seeds. One corollary of the following theorems is that for any a, b coprime, $4, 6 \notin \mathcal{P}^{\{a,b\}}$. At present, it seems like it should be possible to find a seed which generates a period of 4 or 6 as long as $4 \mid a+b$ or $6 \mid a+b$, but in fact it is not. We will give a full characterization of all period structures and lengths which will make this fact obvious. To do this, we first analyze the special case where $a = 1$.

Theorem 15 *Suppose we have $A = \{1, b\}$ and a string X with no sub-period. Then $X^\infty \in \mathcal{W}^A$ if and only if $|X| \Big| (1+b)$ and X is some concatenation of* 01 *and* 011 *under some rotation.*

Proof $\Longrightarrow$ Suppose that $\{w^{A,S}\} = X^\infty$. By the assumption that X has no sub-period, $\mathrm{Per}\,(A) = |X|$, so by Theorem 13 $|X| \Big| (1+b)$. Next we will show that X has no two consecutive 0's or three consecutive 1's. By the recurrence relation, $w(n) = 0$ implies $w(n+1) = 1$. Additionally, suppose for some sufficiently large n that $w(n) = w(n-1) = 1$ This implies $w(n-b) = 0$, so by Theorem 13 $w(n-b+b+1) = w(n+1) = 0$. These two restrictions mean that X is a concatenation of 01 and 011 under some rotation. $\Longleftarrow$ Suppose that $k|X| = (1+b)$ for $k \geq 1$ and X is some concatenation of 01 and 011. For the seed S we can simply choose X^k, so it suffices to show that X^∞ satisfies the recurrence relation. Choose $n \in \mathbb{N}$. First, suppose $X^\infty(n-1) = 0$. Because there are no consecutive 0's, $X^\infty(n) = 1$, which satisfies the recurrence relation. Now suppose $X^\infty(n-1) = 1$ and $X^\infty(n+1) = 0$. Because there are no consecutive 0's, $X^\infty(n) = 1$. Additionally because X^∞ is periodic over $|X|$, $X^\infty(n+1-k|X|) = X^\infty(n-b) = 0$, so indeed $X^\infty(n) = 1$ satisfies the recurrence. Now consider the last case $X^\infty(n-1) = X^\infty(n+1) = 1$. Because there are no three consecutive 1's, this implies $X^\infty(n) = 0$. Additionally $X^\infty(n+1-k|X|) = X^\infty(n-b) = 1$ and by assumption $X^\infty(n-1) = 1$, so $X^\infty(n) = 0$ satisfies the recurrence. □

Recall from Theorem 13 that $\{w^{A,S}\}$ can never have a preperiod, so Theorem 15 completely characterizes all possible sequences for $\{1, b\}$. We also note that instead of considering strings X with $|X| \Big| (b+1)$ and no sub-period, we can equivalently consider all X with $|X| = (b+1)$ and allow for any sub-period of length $p \mid (b+1)$. Let the sets

$$\mathcal{Q} := \{X \in \{0,1\}^\ell \mid \ell \in \mathbb{N},\ X \text{ is a concatenation of 01 and 011 under some rotation}\},$$

and $\mathcal{Q}(\ell) := \{X \in \mathcal{Q} \mid |X| = \ell\}$. Sequences of this form can be generated iteratively by hand or by computer. For example, $\mathcal{Q}(2) = \{01, 10\}$, $\mathcal{Q}(3) = \{011, 101, 110\}$, and $\mathcal{Q}(6) = \{010101, 101010, 011011, 101101, 110110\}$. Theorem 15 proves that $\mathcal{W}^{\{1,b\}} = \{X^\infty \mid X \in \mathcal{Q}(b+1)\}$. We can also provide an explicit enumeration of the possible sequences.

Theorem 16 *Let ϕ be the plastic constant and $z, \bar{z}$ be the other two complex roots of $x^3 - x - 1$, and let $Q(\ell) := \phi^\ell + z^\ell + \bar{z}^\ell$. Then the number of distinct sequences $\{w^{\{1,b\},S}\}$ over all seeds S is*

$$\left|\mathcal{W}^{\{1,b\}}\right| = |\mathcal{Q}(1+b)| = Q(1+b). \tag{9}$$

Proof To find a recurrence relation, we consider the four possible forms for the period structure X to take, and how each can be reduced to a different form. We will then add the sequences which count each of these forms.

$Q_1(\ell)$	$X = 0 \dots 01Q_1(\ell-2) + Q_2(\ell-2)$	by removing the last two numbers
$Q_2(\ell)$	$X = 0 \dots 11Q_1(\ell-1)$	by removing the last number
$Q_3(\ell)$	$X = 1 \dots 01Q_2(\ell)$	by rotating one character left
$Q_4(\ell)$	$X = 1 \dots 10Q_1(\ell) + Q_2(\ell)$	by rotating one character right

$$Q(\ell) = Q_1(\ell) + Q_2(\ell) + Q_3(\ell) + Q_4(\ell) = 3Q_2(\ell) + 2Q_1(\ell) = 3Q_1(\ell-1) + 2Q_1(\ell).$$

We note $Q_1(\ell) = Q_1(\ell-2) + Q_1(\ell-3)$, so to follows that $Q(\ell) = Q(\ell-2) + Q(\ell-3)$. This is a linear recurrence relation, meaning $Q(\ell)$ is a linear combination of the form $c_1\phi^\ell + c_2 z^\ell + c_3\bar{z}^\ell$, where ϕ, z, and $\bar{z}$ are the roots of $x^3 - x - 1$. We give initial conditions $Q(1) = 0$, $Q(2) = |\{01, 10\}| = 2$, and $Q(3) = |\{011, 101, 011\}| = 3$.

To find the coefficients we compute that $Q(0) = 3$, so $c_1 + c_2 + c_3 = 3$. Additionally, Q is real so $c_2 = c_3$. Finally, we know that $x^3 - x - 1 = (x-\phi)(x-z)(x-\bar{z}) = x^3 - (\phi + z + \bar{z})x^2 + \dots$, so $\phi + z + \bar{z} = 0 = Q(1)$. Therefore $c_1 = c_2 = c_3 = 1$. □

$Q(\ell)$ is in the OEIS, known as the Perrin sequence (OEIS Foundation Inc. 2023, A001608).

3.1.1 Distinct periodicities

If we analyze $\mathcal{W}^{\{1,5\}}$, or equivalently $\mathcal{Q}(6)$, we find that all of the sequences 'look like' some form of $(01)^\infty$ or $(011)^\infty$, with some initial shift that we call a 'rotation'. This is true for many sequences, motivating a formal notion of similarity between period structures.

Definition 5 We call two strings X_1, X_2 **similar** if they have a common sub-period, i.e. $X_1^{k_1} = X_2^{k_2}$ for $k_1, k_2 \geq 1$, or if X_2 is some **rotation** of X_1, i.e. $|X_1| = |X_2| = \ell$ and $X_2(n) = X_1(n + x \pmod{\ell})$. We define the equivalence relation $\simeq$ to be the transitive closure of these two criteria. We say X and Y are distinct if $X \not\simeq Y$.

Definition 6 We denote $\mathcal{Q}/\simeq$ as the set of equivalence classes of $\mathcal{Q}$ under $\simeq$. The length of a class $[Y]$ is the length of its smallest element, i.e. the smallest sub-period of Y or the period length of Y^∞.

Recall that $\mathcal{Q}(\ell)$ is the set of all strings in $\mathcal{Q}$ with length ℓ, which is in bijection with $\mathcal{W}^{\{1,\ell-1\}}$. Therefore $\mathcal{Q}(\ell)/\simeq$ is the set of distinct periodicites in $\mathcal{Q}(\ell)$, so we have proven that

$$\mathcal{W}^{\{1,b\}}/\simeq \;=\; \left\{[X^\infty], [X] \in \mathcal{Q}(\ell)/\simeq\right\} \tag{10}$$

Definition 7 We define $(\mathcal{Q}/\simeq)(\ell)$ to be set of distinct periodicities of $\mathcal{Q}/\simeq$ with length ℓ. Similarly $(\mathcal{W}^A/\simeq)(\ell)$ is the set of distinct periodicites in $\mathcal{W}^A/\simeq$ with length ℓ.

It follows that

$$(\mathcal{W}^{\{1,b\}}/\simeq)(\ell) \quad = \quad \left\{[X^\infty], [X] \in \Leftarrow\mathcal{Q}/\simeq)(\ell)\right\}. \tag{11}$$

This implies that $p \in \mathcal{P}^{\{1,b\}}$ if and only if $p \mid (1+b)$ and $|(\mathcal{Q}/\simeq)(p)| > 0$, as means there is some non-periodic string $X \in \mathcal{Q}$ with length $|X| = p$ and thus $X^\infty \in (\mathcal{W}^{\{1,b\}}/\simeq)(p)$.

Example 4 If we want to find the distinct periodicities of length three, six, and eight, we observe that $(\mathcal{Q}/\simeq)(3) = \{[011]\}$, $(\mathcal{Q}/\simeq)(6) = \emptyset$, and $(\mathcal{Q}/\simeq)(8) = \{[01101101]\}$. In contrast, if we wish to know all distinct periodicites of $\{1,2\}$, $\{1,5\}$, and $\{1,7\}$, we instead consult $(\mathcal{Q}(3)/\simeq) = \{[011]\}$, $(\mathcal{Q}(6)/\simeq) = \{[(01)^3], [(011)^2]\}$, and $(\mathcal{Q}(8)/\simeq) = \{[(01)^4], [01101101]\}$.

It follows that because the periodicites of $\mathcal{Q}(\ell)$ must have some length $d \mid \ell$, we can enumerate $\mathcal{Q}(\ell)/\simeq$ by partitioning over period length. The following is a natural result of Eqs. 10 and 11.

$$|\mathcal{W}^{\{1,\ell-1\}}/\simeq| = \sum_{d|\ell} |(\mathcal{W}^{\{1,\ell-1\}}/\simeq)(d)| \quad = \quad |\mathcal{Q}(\ell)/\simeq| = \sum_{d|\ell} |(\mathcal{Q}/\simeq)(d)| \tag{12}$$

In the rest of this section we will enumerate the sets $\mathcal{W}^{\{a,b\}}$, $\mathcal{W}^{\{a,b\}}/\simeq$, and $(\mathcal{W}^{\{a,b\}}/\simeq)(\ell)$ for all $\{a,b\}$. Recall that Theorem 16 gives $|\mathcal{W}^{\{1,b\}}| = Q(n)$.

Proposition 17 *Let*

$$N'(\ell) := \frac{Q(\ell) - \sum_{d!|\ell} d \cdot N'(d)}{\ell} \qquad N(L) := \sum_{\ell|L} N'(\ell), \tag{13}$$

where $d!|\ell$ represents proper divisors of ℓ. Then the number of distinct periodicities of $\{1,b\}$ of length $\ell \mid (1+b)$ is $|(\mathcal{W}^{\{1,b\}}/\simeq)(\ell)| = N'(\ell)$, and the total number of distinct perioditicites is $|\mathcal{W}^{\{1,b\}}/\simeq| = N(b+1)$.

Proof We will show $N'(\ell) = |(\mathcal{Q}/\simeq)(\ell)|$. First we consider all strings $X \in \mathcal{Q}(\ell)$. In particular suppose X has period p, so $\ell/p = k$ and $X = Z^k$ for some Z having no sub-period, i.e. $[Z] \in (\mathcal{Q}/\simeq)(p)$. Because Z has no sub-period, it has p different rotations and therefore p representatives $X_1, \dots, X_p \in \mathcal{Q}(\ell)$. Since p can be any divisor of ℓ, we conclude $|\mathcal{Q}(\ell)| = \sum_{p|\ell} p|(\mathcal{Q}/\simeq)(p)|$. We can rearrange this to conclude $|(\mathcal{W}^{\{1,b\}}/\simeq)(\ell)| = |(\mathcal{Q}/\simeq)(\ell)| = N'(\ell)$ as written in Eq. 13. Note that we do not need a base case to compute N' explicitly because 1 has no proper divisors. Equation 12 implies $|\mathcal{W}^{\{1,b\}}/\simeq| = N(b+1)$. □

Note that $N'(\ell)$ does not depend on b, and in fact the set of distinct periodicities of length ℓ is equal for any b as long as $\ell \mid (1+b)$. The OEIS contains sequences $N'(\ell)$ (OEIS Foundation Inc. 2023, A113788) and $N(L)$ (OEIS Foundation Inc. 2023, A127687), motivating an interesting bijection.

It is known that $Q(\ell)$ counts the maximal independent sets in vertex labeled cycles C_ℓ (see, for example, Example 1.2 in Füredi (1987)). In 2007, Bisdorff et. al demonstrated that $N(b+1)$ counts the number of unlabeled maximal independent sets of the cycle C_{b+1} Bisdorff and Marichal (2007). The correspondence between a binary sequence Y and a vertex set $S \subseteq V(C_{b+1})$ is simple; we include the i^{th} vertex in S exactly when $Y(i) = 0$. For example $\mathcal{Q}(10) \ni (01)^2(011)^2 \mapsto \{1,3,5,8\} \subseteq V(C_{10})$. The conditions for a string to be valid are equivalent to those for a maximal independent set. We cannot have two adjacent vertices of S by independence, and Y cannot have two consecutive zeros by Theorem 15. We also cannot have three adjacent non-vertices in S by maximality, or else we could add the middle vertex to S. Similarly Y cannot have three consecutive ones. By our construction, it is clear that N counts these sets in C_{b+1} up to rotation, but not reflection. For example, the sequence $(01)(011)(01)^2(011)(01)^3(011) \not\simeq (01)^3(011)(01)^2(011)(01)(011)$ because one cannot be rotated to the other, so they belong to different equivalence classes of $\mathcal{Q}/\simeq$. Thus, $N(21)$ will count them both. However a reflection automorphism of C_{21} would map the corresponding independent sets to each other. Moving forward, Table 1 shows the sequences $Q(\ell)$, $N'(\ell)$, and $N(\ell)$ for $\ell \in \{0,\dots,20\}$.

3.2 Generalizing to $\{a, b\}$

We will use a simple multiplicative permutation of $w^{A,S}$ to reduce every $\{a,b\}$ case to a version of $\{1,b'\}$. This will induce the strict structure of Theorem 15 on the seemingly complex periodicities of the $\{a,b\}$ case. We will generally assume that a, b are coprime. If not, we can divide out $g = \gcd(a,b)$ and then find g parallel copies of sequences $(w)_i$ for the coprime set $A = \{a/g, b/g\}$ using Multiplicative Linearity Proposition 5.

Table 1 $Q(\ell)$ counts the possible sequences $\{w^{A,S}\}$ over all S for $A = \{a,b\}$ if a, b coprime and $a+b=\ell$

ℓ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Q(\ell)$	0	2	3	2	5	5	7	10	12	17	22	29	39	51	68	90	119	158	209	277
$N'(\ell)$	0	1	1	0	1	0	1	1	1	1	2	2	3	3	4	5	7	8	11	13
$N(\ell)$	0	1	1	1	1	2	1	2	2	3	2	4	3	5	6	7	7	11	11	16

$N'(\ell)$ counts the number of distinct periodicities of length ℓ. $N(\ell)$ counts the number of distinct periodicities of any length for $A = \{a,b\}$

Definition 8 Given some a, b coprime, define the permutation $\sigma_{a,a+b} : \{0,1\}^{a+b} \hookrightarrow\!\!\!\!\rightarrow \{0,1\}^{a+b}$ by $\sigma_{a,a+b}(Y) = X$, where $a+b-1\}.$

$X(n) = Y(an \pmod{a+b})$ for all $n \in \{0, \dots a+b-1\}$.

Because a and $a+b$ are coprime, $\sigma_{a,a+b}(Y)$ is a permutation of the string Y for all $Y \in \{0,1\}^{a+b}$. This means $\sigma_{a,a+b}$ is invertible, so it is a permutation of the collection $\{0,1\}^{a+b}$.

Theorem 18 (Bijection) *Suppose* $A = \{a,b\}$ *with* a, b *coprime and let* $A' = \{1, a+b-1\}$. *For any string* Y *with* $|Y| = a+b$, *let* $X = \sigma_{a,a+b}(Y)$. *Then* $Y^\infty \in \mathcal{W}^A$ *if and only if* $X^\infty \in \mathcal{W}^{A'}$.

Proof It suffices to show that Y^∞ obeys the recurrence relation if and only if X^∞ does, as we could then use the seeds $Y^{(a+b)/|Y|}$ and $X^{(a+b)/|X|}$ respectively. Let a^{-1} be the multiplicative inverse of $a \pmod{a+b}$. This means $Y^\infty(n) = X^\infty(a^{-1}n \pmod{a+b})$, so $Y = \sigma_{a^{-1},a+b}(X)$. By leveraging the periodicity of Y^∞ and X^∞ over $a+b$, and noting that $b \equiv -a \pmod{a+b}$, we get

$$\begin{aligned}
Y^\infty(n) & & &= X^\infty(a^{-1}n) \\
Y^\infty(n-a) &= X^\infty(a^{-1}n - a^{-1}a) & &= X^\infty(a^{-1}n - 1) \\
Y^\infty(n-b) &= X^\infty(a^{-1}n - a^{-1}(-a)) = X^\infty(a^{-1}n + 1) & &= X^\infty(a^{-1}n - (a+b-1))
\end{aligned}$$

Therefore Y^∞ follows the recurrence relation of $A = \{a,b\}$ if and only if X^∞ follows the recurrence relation of $A' = \{1, a+b-1\}$. □

Theorem 18 implies that the set of all period structures of $\{a,b\}$ can be obtained by applying the inverse permutation $\sigma^{-1}_{a,a+b} = \sigma_{a^{-1},a+b}$ to each period structure of $\{1, a+b-1\}$. Because $\sigma_{a,a+b}$ is a permutation of $\{0,1\}^{a+b}$, it bijects the set $\mathcal{W}^{\{a,b\}}$ of all $\{w^{\{a,b\},S}\}$ sequences with the set $\mathcal{W}^{\{1,a+b-1\}}$. In particular, we conclude precisely that if a and b are coprime, then

$$\mathcal{W}^{\{a,b\}} = \{Y^\infty \mid Y = \sigma_{a^{-1},a+b}(X),\ X \in \mathcal{Q}(a+b)\}, \tag{14}$$

or informally $\mathcal{W}^{\{a,b\}} = \sigma_{a^{-1},a+b}\left[\mathcal{W}^{\{1,a+b-1\}}\right]$. Therefore Theorem 18 implies that we can generalize the results of Sect. 3.1.

Corollary 18.1 *Choose any* $A = \{a,b\} = \{\tilde{a}g, \tilde{b}g\}$, *where* $g = \gcd(a,b)$. *Then* $|\mathcal{W}^A| = (Q(\tilde{a}+\tilde{b}))^g$. □

The power of g follows from Linearity Proposition 5. Each of the independent parallel sequences $(w^{A,S})_i(m)$ for $i \in \{0,\dots,g-1\}$ is equal to some $\{\tilde{a},\tilde{b}\}$ game. Thus we can count g-tuples of strings in $\sigma_{\tilde{a}^{-1},\tilde{a}+\tilde{b}}[\mathcal{Q}(a+b)]$ to arrive at the total, which we formalize in Theorem 19. Next, we find that $\sigma_{a,a+b}$ also bijects the classes of distinct periods.

Corollary 18.2 *Let a, b be coprime and $A = \{a, b\}$. Then $|\mathcal{W}^A/\simeq(\ell)| = N'(\ell)$ is the number of distinct periodicities of length $\ell \mid (a+b)$, and $|\mathcal{W}^A/\simeq| = N(a+b)$ is the total number of distinct periodicities of A.*

Proof It suffices to show that $\sigma_{a,a+b}$ preserves rotational symmetries and sub-periods. Suppose some sequences Y and Y' are rotated copies of each other, i.e. there is some r such that $Y(n) = Y'(n + r \pmod{\ell})$ for all n. This implies $(\sigma Y)(n) = (\sigma Y')(n + ar \pmod{\ell})$, so σY and $\sigma Y'$ are rotated copies of each other with shift ar. Additionally, suppose Y has some sub-period, i.e. there is some $r \not\equiv 0 \pmod{\ell}$ such that $Y(n) = Y(n + r \pmod{\ell})$. This implies Y is a rotated copy of itself, so indeed σY is a rotated copy of itself with shift $ar \not\equiv 0 \pmod{\ell}$, since $\ell \mid (a+b)$ and $\gcd(a, a+b) = 1$. We conclude that for all X and Y, $X \simeq Y$ if and only if $\sigma X \simeq \sigma Y$, so equivalence classes of $\simeq$ are bijected by $\sigma_{a,a+b}$. □

If a and b are not coprime, it is more complicated to count the number of distinct periodicities, but we can use a generalization of Proposition 17.

Theorem 19 *Choose any $A = \{a, b\} = \{\tilde{a}g, \tilde{b}g\}$, where $g = \gcd(a, b)$. Define the following functions for all $g, \ell \in \mathbb{N}\backslash\{0\}$ and $L = g\ell$:*

$$N'(L, g) := \frac{Q(\ell)^g - \sum_{d!|L} d \cdot N'(d, \gcd(d, g))}{L} \qquad N(L, g) := \sum_{p|L} N'(p, \gcd(p, g)) \tag{15}$$

Then $|\mathcal{W}^A/\simeq(p)| = N'(p, \gcd(p, a, b))$ is the number of distinct periodicities of A with length p, and $|\mathcal{W}^A/\simeq| = N(a+b, g)$ is the total number of distinct periodicities of A.

Proof First we will justify that the total number of strings in $\mathcal{W}^{\{a,b\}}$ which are periodic over p is $\left(Q(\frac{p}{\gcd(p,g)})\right)^{\gcd(p,g)}$, not considering similarity under $\simeq$. Denote this set by $\mathcal{W}^{\{a,b\}}(p)$. Let $\ell = \tilde{a} + \tilde{b}$. We know by Theorem 13 that $\{w\}$ is always periodic over $g\ell = a + b$, so for all S, $\{w^{A,S}\} = Y^\infty$ for some Y with $|Y| = g\ell$. Using Linearity Proposition 5, we can write Y as a collection of parallel strings Y_i for $i \in \{0, \dots g-1\}$ where $|Y_i| = \ell$, and for all $m \in \{0, \dots, \ell - 1\}$, $Y_i(m) = Y(gm + i)$. Additionally, for each i, Y_i^∞ must satisfy the recurrence relation for $\{\tilde{a}, \tilde{b}\}$. Suppose that Y is also periodic over length $p \mid \ell$, so $Y^\infty \in \mathcal{W}^{\{a,b\}}(p)$. Using the division algorithm let $p = xg + r$, where $r < g$ and $x \in \mathbb{N}$. Additionally, let $\gamma = \gcd(p, g)$ and $\tilde{g} = g/\gamma$. We will prove that the following conditions are sufficient and necessary for this to occur.

(i) For all $i \in \{0,\dots,\gamma-1\}$, and all m, $Y_i(m) = Y_i(m+p/\gamma)$

(ii) For all $i \in \{\gamma,\dots,g-1\}$ and all m, $Y_i(m) = \begin{cases} Y_{i-r}(m-x) & i \geq r \\ Y_{i-r+g}(m-x-1) & i < r \end{cases}$

We first show necessity. Assuming Y is periodic over length p, this means for all m, i, we have

$$Y_i(m) = Y(mg+i) = Y(mg+i-p) = Y(mg+i-xg-r) = \begin{cases} Y_{i-r}(m-x) & i \geq r \\ Y_{i-r+g}(m-x-1) & i < r \end{cases}$$
$$= Y(mg+i+\tilde{g}p) = Y(mg+i+gp/\gamma) = Y_i(m+p/\gamma),$$

This proves stronger versions of (i) and (ii).

Next, we show sufficiency. Assume (i) and (ii) hold. For any n, use the division algorithm to get $n = mg+i$. We hope to show that $Y(n) = Y(n-p)$. Suppose that $i \in \{\gamma,\dots,g-1\}$. Then by condition (ii),

$$Y(n) = Y_i(m) = \begin{cases} Y_{i-r}(m-x) & i \geq r \\ Y_{i-r+g}(m-x-1) & i < r \end{cases} = Y(mg-gx+i-r-gx) = Y(n-p) \tag{16}$$

Alternately, suppose that $i \in \{0,\dots,\gamma-1\}$, and consider the set of indices $i, i+r, i+2r, \dots, i+(\tilde{g}-1)r$, taken modulo g. We denote these $I_j \equiv i+rj \pmod g$ for $j \in \{0,\dots,\tilde{g}-1\}$. Because $\gamma \mid r$, this means $I_j \equiv i \pmod \gamma$ for all j. Additionally, we know that the order of r in the group $\mathbb{Z}_g^+$ is $g/\gcd(g,r) = \tilde{g}$, which means that $I_j \neq i$ for $j \in \{1,\dots,\tilde{g}-1\}$. This means that only I_0 can be in the set $\{0,\dots,\gamma-1\}$ and for all $j > 0$ we must have $I_j \in \{\gamma,\dots,g-1\}$.

Now, let $n' = n-\tilde{g}p$, and consider the sequence of values $Y(n'), Y(n'+p), \dots, Y(n'+(\tilde{g}-1)p)$. We know that $Y(n') = Y_i(m-p/\gamma) = Y_i(m) = Y(n)$ by condition (i). We also observe that for all $j \in \{0,\dots,\tilde{g}-1\}$, we have $Y(n'+pj) = Y_{I_j}(m_j)$ for some m_j. Because we know $I_j \in \{\gamma,\dots,g-1\}$, Eq. 16 implies that $Y(n'+pj) = Y(n'+p(j-1))$. Therefore

$$Y(n) = Y(n') = Y(n'+p) = \dots = Y(n'+(\tilde{g}-1)p) = Y(n-p).$$

Therefore we have shown that conditions (i) and (ii) are equivalent to Y^∞ being periodic over length p, and we now enumerate the strings satisfying these conditions. To satisfy (i) we may choose any γ-tuple of sequences in $\mathcal{W}^{\{\tilde{a},\tilde{b}\}}$ which are periodic over p/γ for $Y_0,\dots,Y_{\gamma-1}$. To satisfy (ii) we must let $Y_\gamma,\dots,Y_{g-1}$ be the appropriate translations of these sequences, so they are fixed. Thus we find the total number of possibilities $\left|\mathcal{W}^{\{a,b\}}(p)\right| = (Q(p/\gamma))^\gamma$.

An argument similar to Proposition 17 shows that $\left|\mathcal{W}^{\{a,b\}}(p)\right| = \sum_{d|p} d\left|\mathcal{W}^A/\simeq(d)\right|$. Letting $N'(d,\gcd(d,g)) = \left|\mathcal{W}^A/\simeq(d)\right|$, these can be rearranged to Eq. 15. Note that we can also derive $N'(p,1) = N'(p)$ for all p, meaning that if $g=1$, we have reduced to the case of $\{a,b\}$ coprime. □

In the simpler case when a, b are coprime, refer to the sequences $Q(\ell), N'(\ell)$, and $N(\ell)$ in Table 1. As a result of the general enumeration we find that for all $g \mid L$, $N(L, g) = 0$ exactly when $L = g$ or $L = 4g$ or $(L, g) = (6, 1)$. Thus

$$\mathcal{P}^{\{a,b\}} = \left\{ p \;\middle|\; p|(a+b),\ p \not| \gcd(a,b),\ p \neq 4\gcd(p,a,b),\ (p, \gcd(p,a,b)) \neq (6,1) \right\}$$

We also see that for any coprime a, b and any $\ell \leq 10$, there is at most one distinct periodicity of a and b of length ℓ over all S. Given some $A = \{1, 11k-1\}$ for $k \geq 1$, the two periodicites of length 11 are $\Leftarrow\mathcal{Q}/\simeq)(11) = \left\{ \left[(01)^4(011)\right],\ \left[(01)(011)^3\right] \right\}$. This means that for any $\{a, b\}$, with $11 \mid a+b$, the two periodicities are $\sigma_{a^{-1},11}\left((01)^4(011)\right)$ and $\sigma_{a^{-1},11}\left((01)(011)^3\right)$.

Example 5 Consider the set $A = \{3, 11\}$. There are $N(14) = 5$ periodicites, where exactly $N'(14) = 3$ have period 14. First we write the 5 periodicities of $B = \{1, 13\}$, given by

$$\begin{aligned}\mathcal{Q}(14)/\simeq = \Big\{ &\left[(01)(011)^4\right],\ \left[(01)^4(011)^2\right],\ \left[(01)^3(011)(01)^1(011)\right],\\ &\left[((01)^2(011))^2\right],\ \left[(01)^7\right] \Big\}.\end{aligned}$$

Next, we permute these strings. $3^{-1} \equiv 5 \pmod{14}$, so we compute $Y = \sigma_{5,14}(X)$. For the first string:

n	**0**	**1**	**2**	**3**	**4**	5	6	7	8	9	10	11	12	13
X	**0**	**1**	**0**	**1**	**1**	0	1	1	0	1	1	0	1	1
$\sigma(X)$	**0**	0	1	**1**	1	0	**0**	1	1	**1**	0	1	**1**	1
$5n \pmod{14}$	**0**	5	10	**1**	6	11	**2**	7	12	**3**	8	13	**4**	9

This yields $\sigma_{5,14}\left(01(011)^4\right) = (0^2 1^3)^2(01^3)$. Repeating this procedure, we get the following:

$$\begin{aligned}\sigma_{5,14}\left(01(011)^4\right) &= &&= (0^2 1^3)^2(01^3)\\ \sigma_{5,14}\left((01)^4(011)^2\right) &= 01^3 0^3 1^3 (01)^2 &&\simeq 1^2 0^3 1^3 (01)^3\\ \sigma_{5,14}\left((01)^1 011 (01)^3 011\right) &= 0^2 1^3 0^3 1^5 0 &&\simeq 0^3 1^3 0^3 1^5\\ \sigma_{5,14}\left((01)^2 011)^2\right) &= 01^4 0^3 1^4 0^2 &&\simeq (0^3 1^4)^2\\ \sigma_{5,14}\left((01)^7\right) &= &&= (01)^7\end{aligned}$$

$$\mathcal{W}^{\{3,11\}}/\simeq\ = \left\{ \left[\left((0^2 1^3)^2(01^3)\right)^\infty\right], \left[\left(1^2 0^3 1^3 (01)^3\right)^\infty\right], \left[\left(0^3 1^3 0^3 1^5\right)^\infty\right], \left[(0^3 1^4)^\infty\right], \left[(01)^\infty\right] \right\}$$

Note that the period lengths of 2, 7, and 14 are conserved under the permutation, as shown in Corollary 18.2. This procedure yields all 5 periodicites of $\{3, 11\}$.

The asympotic behavior of these functions is generally well behaved. Because $|z| = |\bar{z}| < 1$, where z and $\bar{z}$ are the non-real roots of $x^3 - x - 1$, we have the convergence $Q(\ell) = \phi^\ell + z^\ell + \bar{z}^\ell \underset{\ell\to\infty}{\to} \phi^\ell$.[3] For the following analysis, let $A = \{a, b\} = g\{\tilde{a}, \tilde{b}\}$ with $g = \gcd(a, b)$, and assume that $\tilde{a} + \tilde{b} \geq 2.37 \ln g$, which excludes relatively few sets. Then $\left|\mathcal{W}^{\{a,b\}}\right| = Q(\tilde{a} + \tilde{b})^g \sim \phi^{a+b}$. The set of seeds $\{S \mid S \in \{0, 1\}^b\}$ has cardinality $2^b \geq \sqrt{2}^{a+b}$. Because $\phi \approx 1.32 < \sqrt{2}$ this means that the set of seeds grows faster than the sequences they generate, the number of seeds converging to the same sequence $\{w\}$ grows exponentially as $a + b$ increases. This approximation of $Q(\ell)$ also implies that $N'(\ell) \sim \phi^\ell/\ell$ and $N(\ell) \sim N(\ell)$. We can can also derive $\phi^L/L \sim N'(L, g) \sim N(L, g)$, giving a strong estimation of the of the number of possible period structures for large a, b.

4 The $\{1, b, c\}$ case

Note that we do not consider seeds in this section. Analogously to Sect. 3.1, a good starting point for understanding the $\{a, b, c\}$ game is the $\{1, b, c\}$ game. This case was studied by Ho in (Ho 2015, sec. 2), where he solves the game for $c < 4b$. We provide a full analysis by constructing $\{w^{\{1,b,c\}}\}$ for all b and c with no seed, where most importantly we specify the existence and structure of preperiods.

For the remainder of the paper we will use $x :\equiv y \pmod{z}$ to define x as the least non negative remainder of y modulo z.

Theorem 20 *Suppose we have some* $1 < b < c$, *and let* $A = \{1, b, c\}$. *Denote* $q := \lfloor c/(b + 1)\rfloor$, $r :\equiv c \pmod{b + 1}$. *This means* $c = q(b + 1) + r$. *Additionally let* $k := b/2$ *and* $\gamma := \frac{b-r-2}{2}$; *we will only use these when they take on integer values.*

Case	Conditions	Per (A)	PrePer (A)	$\{w^A\}$
i	b, c odd,	2	0	$[01]^\infty$
ii	b odd, c even,	$b + c$	0	$\left[(01)^{c/2}1^b\right]^\infty$
iii	b even, $c = b + 1$	$2b$	0	$\left[(01)^k 1^b\right]^\infty$
iv	b even, $r \in \{1, b\}$	$b + 1$	0	$\left[(01)^k 1\right]^\infty$
v	b even, $r > 1$ odd	$b + c$	0	$\left[((01)^k 1)^{q+1} 1^{r-1}\right]^\infty$
vi	b even, $r = b - 2$	$c + 1$	0	$\left[((01)^k 1)^q (01)^{k-1} 1\right]^\infty$
vii	b even, $c > b + 1$, $r < b - 2$ is even, $q > \gamma$	$c + 1$	$\gamma(b + c + 2) - b - 1$	Equation 17
viii	b even, $c > b + 1$, $r < b - 2$ is even, $q < \gamma$	$b + c$	$q(b + c + 2) - b - 1$	Equation 17
ix	b even, $c > b + 1$, $r < b - 2$ is even, $q = \gamma$	$b - 1$	$q(b + c + 2) - a - 1$	Equation 17

[3] For all $\ell \geq 10$, we have the exact equality $Q(\ell) = \mathrm{round}(\phi^\ell)$.

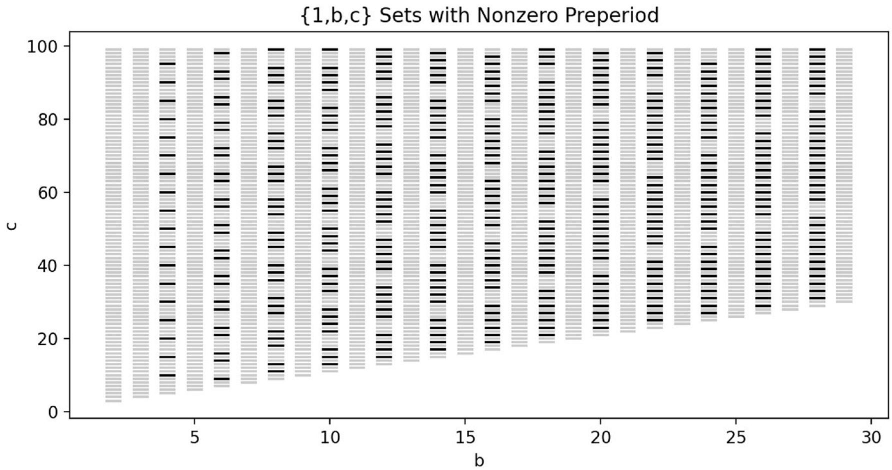

Fig. 1 Theorem 20 proves that $\mathrm{PrePer}(\{1, b, c\}) \neq 0$ exactly when b is even, $c > b + 1$, and $r < b - 2$ is even, with $r :\equiv c \pmod{b+1}$

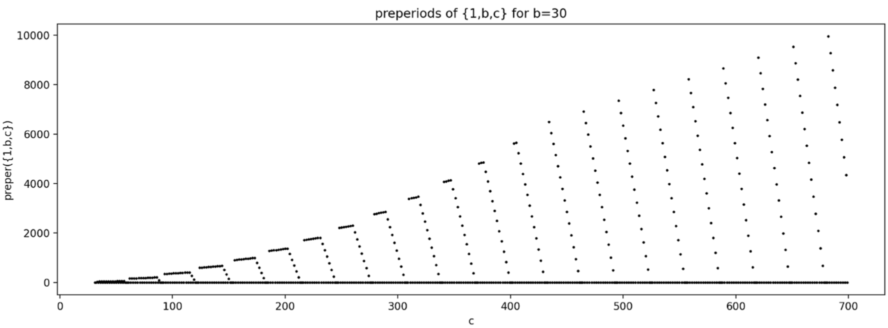

Fig. 2 Theorem 20 proves that $\mathrm{PrePer}(\{1, b, c\}) = \min\left(\left\lfloor \frac{c}{b} \right\rfloor, \frac{a-r-2}{2}\right)(b + c + 2) - b - 1$ whenever it is nonzero. This is approximately quadradic in c when c is close to b, and transitions to linear when $c \gg b$

The last three cases yield the most interesting results, providing an exact specification of the existence and structure of preperiods. These can be visualized in Figs. 1 and 2.

Proof (i) If b and c are odd, then they are both extensions of $\{1\}$ as in Example 3, so $\{w^A\} = \{w^{\{1\}}\} = (01)^\infty$. $\square$

Proof (ii) Suppose b is odd and c is even. For $n < c$, $w(n - c) = 1$, so $w(n) = 1 - \min\{w(n-1), w(n-b), 1\}$, and therefore $w^A(n) = w^{\{1,b\}}(n) = w^{\{1\}}(n)$, because b is an extension of $\{1\}$. Thus $\{w^A\} = (01)^{c/2}\underset{c}{\ldots}$ Now for all $n \in [c, c + b)$, we find that $n - b < c$ and $n - c < c$. Therefore if n is odd, then $n - b$ is even so

$w(n-b)=0$ and therefore $w(n)=1$. If n is even then $n-c$ is even so $w(n-c)=0$ and therefore $w(n)=1$. Thus:

$$\{w^A\} = (01)^{c/2}1^b\underset{b+c}{\ldots}$$

Because $w^A(b+c-1)=1$, Corollary 9.2 of the translating zeros lemma implies that this is the entire period with no preperiod. □

Proof (iii) Suppose $b=2k$ and $c=b+1$. For $n<c$ we have a single period of $\{1,b\}$, specifically $\{w^A\}=(01)^k1\underset{c}{\ldots}$ Next, for $n\in[b+1,2b)$, we find that $n-b<b+1$ and $n-c<b$. If n is odd, then $n-c$ is even so $w(n-c)=0$ and thus $w(n)=1$. If n is even, then $n-b$ is even so $w(n-b)=0$ and thus $w(n)=1$. This yields $\{w^A\}=(01)^k1^b\underset{b+c}{\ldots}$. Because $\mathbf{v}(b+c)=\mathbf{v}(0)=1^c$, this is the entire period with no preperiod, as implied by Lemma 10. □

Proof (iv) Suppose $b=2k$ and $r\in\{1,b\}$. Then c is an extension of $\{1,b\}$ by Proposition 8, so $\{w^A\}=\{w^{\{1,b\}}\}=\left((01)^k1\right)^\infty$. □

Proof (v) Suppose $b=2k$ and $r>1$ is odd. As above, for the first c elements of the sequence, $\{w^A\}$ is equal to $\{w^{\{}1,b\}\}$.

$$\{w^{\{1,2k\}}\} = \left((01)^k1\right)^q(01)^{\frac{r-1}{2}}0\underset{c}{\ldots}$$

Now let $n=q(b+1)+m$ for $m\in[r,b)$. This means that $n-b<q(b+1)$ and $n-c=m-r$. If m is odd, then $n-b$ is odd so $w(n-b)=0$, so $w(n)=1$. If m is even, then $w(n-1)=1$ and $w(n-b)=1$ and $w(n-c)=w(m-r)=1$, so $w(n)=0$. This extends the sequence to

$$\{w^A\} = \left((01)^k1\right)^q(01)^k\underset{q(b+1)+b}{\ldots}$$

We just showed $w(q(b+1))=0$, so $w(q(b+1)+b)=1$.

$$\{w^A\} = \left((01)^k1\right)^q(01)^k1\underset{(q+1)(b+1)}{\ldots}$$

Now let $n=(q+1)(b+1)+m$ for $m\in[0,r-1)$. If m is odd, then $n-b=q(b+1)+m+1$, which has an even remainder less than b, so $w(n-b)=0$ and therefore $w(n)=1$. If m is even, then $n-c=b+1-(r-m)$, which is even and less than $b+1$, so $w(n-c)=0$ and therefore $w(n)=1$. We then extend the sequence.

$$\{w^A\} = \left((01)^k1\right)^q(01)^k1^r\underset{b+c}{\ldots} = \left((01)^k1\right)^{q+1}1^{r-1}\underset{b+c}{\ldots}$$

Because $w^A(b+c-1)=1$, we may again apply Corollary 9.2 to claim that this is the complete period with no preperiod.

Note that if $r = 1$, this solution still works and has sub-period $b + 1$, which agrees with *(iv)*. □

Proof (vi, vii, viii, ix) Assume $b = 2k$, r is even, and $c > b + 1$, and recall $\gamma = \frac{b-r-2}{2}$. All four of the remaining cases are expressed in the following equation. Interestingly, the preperiod structure is the same for all cases.

$$\{w^A\} = \sum_{i=0}^{\min(q,\gamma)-1} \left(((01)^k 1)^{q-i}((01)^{k-1}1)^i (01)^{r/2+i} 1^{2(\gamma-i)+1} (01)^{k-(\gamma-i)} 1\right) \times \left(\begin{cases} ((01)^k 1)^{q-\gamma}((01)^{k-1}1)^{\gamma+1} & \gamma \le q \\ ((01)^{k-1}1)^q (01)^{r/2+q} 1^{2(\gamma-q)+1} (01)^{r/2+q} 1 & \gamma \ge q \end{cases}\right)^\infty \tag{17}$$

A proof of this Equation is given in Sect. 4.1. We simply check that the recurrence relation is satisfied.

In case *(vi)* where $r = b - 2$, this means $\gamma = 0$ so the summation is empty and there is no preperiod. Because we assume $q \ge 1$, this falls into the $\gamma \le q$ case of Eq. 17, which has length $q(b + 1) + (b + 1) - 2 = c + 1$.

In the remaining cases both q and γ are nonzero, and each term in the preperiod summation has length

$$\begin{aligned}&(q - i)(b + 1) + i(b - 1) + (r + 2i) + 2(\gamma - i) + 1 + 2(k - (\gamma - i)) + 1 \\ &\quad = q(b + 1) - 2i + r + 2i + 2k + 2 = c + b + 2.\end{aligned}$$

Examine the last $b + 1$ values of the last term in the summation, $2^{2(\gamma-\min(q,\gamma)+1)}(01)^{k-(\gamma-\min(q,\gamma)+1)}1$. If $\gamma \le q$, this is $1^2(01)^{k-1}1$, which is equal to the last $b + 1$ values of the period structure. If $q \le \gamma$, this is equal to $2^{2(\gamma-q+1)}(01)^{k-\gamma+q-1}1 = 2^{2(\gamma-q+1)}(01)^{r/2+q}$, which is also equal to the last $b + 1$ values of the period structure. This implies that the preperiod transitions into the period $b + 1$ steps earlier than depicted in Eq. 17, and has length $\min(q, \gamma)(b + c + 2) - b - 1$.

In cases *(vii)* and *(viii)*, Per (A) can be computed simply by counting the length of the strings in Eq. 17. The period for $\gamma \le q$ has length

$$(b + 1)(q - \gamma) + (b + 1 - 2)(\gamma + 1) = (b + 1)q - 2\gamma + b - 2 + 1 = (b + 1)q + r + 1 = c + 1.$$

The period for $q \le \gamma$ has length

$$q(a + 1 - 2) + (r + 2q) + (2\gamma - 2q + 1) + (r + 2q) + 1 = q(a + 1) + 2r + 2\gamma + 2 = b + a.$$

In case *(ix)*, where $\gamma = q$, recall that $r + 2\gamma = 2k - 2$. This allows us to simplify $r/2 + q = r/2 + \gamma = k - 1$, so the two period structures are equivalent and can be expressed as $((01)^{k-1}1)^\infty$, which has period $b - 1$.

We might also notice that Eq. 17 also holds if $r = b$ (with no preperiod) and agrees with case *(iv)*. In this case we would have $\gamma = -1$ so $\gamma \le q$ and $\{w^A\} = ((01)^k 1)^\infty$. □

One result of this is that preperiod lengths can be arbitrarily longer than periods in the $\{1, b, c\}$ case. This can be seen in Fig. 2, where the preperiod length appears quadratic in c.

Example 6 Let $b = 2k$ and $c = k(b + 1)$. Then $A = \{1, 2k, 2k^2 + k\}$. This means $q = k$, $r = 0$, and $\gamma = k - 1$. Noting that $\gamma < q$, Eq. 17 yields

$$\{w^A\} = \sum_{i=0}^{k-2} \Big(((01)^k 1)^{k-i} ((01)^{k-1} 1)^i (01)^i 1^{2(k-i)-1} (01)^{i+1} 1 \Big) \Big((01)^k 1 ((01)^{k-1} 1)^k \Big)^\infty \tag{18}$$

and Theorem 20 (case vii) gives $\mathrm{Per}\,(A) = 2k^2 + k + 1$, and $\mathrm{PrePer}\,(A) = (k-1)(2k + 2k^2 + k + 2) - 2k - 1 = 2k^3 + k^2 - 3k - 3$.

If we instead choose $b = 2k$ and $c = (k-1)(b+1)$, we would have $q = \gamma$ (case ix) so $\mathrm{Per}\,(A) = 2k - 1$ and $\mathrm{PrePer}\,(A) = 2k^3 - 3k^2 + 2k - 4$.

For general 3-sets, (Althöfer and Bültermann 1995, problem (vi)) provided the example $A = \{2s, 4s+1, 22s+2\}$ with $\mathrm{Per}\,(A) = 26s + 3$, though they err in giving $\mathrm{PrePer}\,(A) = 24s^2 - 4s + 1$ for $s \in [2, 20]$ while actually $\mathrm{PrePer}\,(A) = 24s^2 - 4s - 1$ for all $s \in [2, 200]$. Another example follows from (Cairns and Ho 2010, thm 2). If $A = \{k, k+2, 2k+3\}$, then $\mathrm{PrePer}\,(A) = \frac{1}{2}(3k^2 - 5)$ and $\mathrm{Per}\,(A) = 2$. Thus for general 3-sets, $\mathrm{PrePer}\,(A)$ is not bounded by any function of $\mathrm{Per}\,(A)$, whereas for $A = \{1, b, c\}$, Theorem 20 shows that $\mathrm{PrePer}\,(A) = \mathcal{O}(\mathrm{Per}\,(A)^3)$.

In Sect. 3, we used the $A = \{1, b\}$ case to characterize all possible 2-sets using a permutation of w^A. A similar strategy may be possible if we could characterize all periodicities of $\{1, b, c\}$, though this would be more complicated. In particular, note that the length of the periods are highly dependent on seeds, unlike the $\{a, b\}$ case. An example of this is proven in Sect. 6 and visualized in Fig. 7.

Suppose $A = \{a, b, c\}$ and we are given any string Y with $|Y| = p$. If $\gcd(a, p) = 1$, then we let a^{-1} be the multiplicative inverse of a in $\mathbb{Z}_p^\times$ and $b' :\equiv a^{-1}b \pmod p$ and $c' :\equiv a^{-1}c \pmod p$. We see $X = \sigma_{a,p}(Y)$ is a permutation of Y, so we could show in a manner similar to Theorem 18 that $Y^\infty \in \mathcal{W}^{\{a,b,c\}}$ if and only if $X^\infty \in \mathcal{W}^{\{1,b',c'\}}$. Thus as long as p and a are coprime, we have $\mathcal{W}^{a,b,c}(p) = \sigma_{a^{-1},p}\Big[\mathcal{W}^{1,b',c'}(p)\Big]$. This observation carries little information without further understanding of $\mathcal{W}^{1,b,c}$.

As an example we apply Lemma 22. Choose any $n \in \mathbb{N}$ and $d \geq 1$. Let $b = 4n + 2d + 1$ and let $p = 2(n+1)b + 1$. Lemma 22 will imply that $p \in \mathcal{P}^{\{1,b,b+1\}}$. 2 is coprime to p, so calculate that $2^{-1} = (n+1)b + 1$ and $2^{-1}b = (2n + d) + 2^{-1} = (n+1)b + 2n + d + 1$, and finally $2^{-1}(b+1) = 2n + d + 1$. Therefore there is some seed S such that $\mathrm{Per}\,(\{2, 2n+d+1, (n+1)b + 2n + d + 1\}, S) = 2(n+1)b + 1$.

4.1 Building the preperiod sequence for $\{1, b, c\}$

The statements in Theorems 20, 21, and Lemma 22 are somewhat tedious, so we provide three proofs of distinct flavors. In this section we provide a visual verification of Eq. 17 to complete the proof of Theorem 20.

Proof We claim that if $b = 2k$, $q = \lfloor c/(b+1) \rfloor$, $r = c - qb$ is even $c > b + 1$, and $\gamma = \frac{b-r-2}{2}$, then Eq. 17 holds.

To verify the construction of the preperiod, we will confirm that for each term $i \in \{0, \ldots, \min(q-1, \gamma)\}$, the recurrence relation holds. If $i = 0$, note that the first c entries proceed as $w^{1,b}$, so $\{w^A\} = \big((01)^k 1)^q (01)^{r/2} \underset{c}{\ldots}$, which agrees with the $i = 0$ term of Eq. 17. Thus when considering prefixes we may assume $i \geq 1$. Suppose the i^{th} term starts at entry m, and assume that the previous $i - 1$ terms follow Eq. 17.

The “alignment diagram" on the left hand side in Fig. 3 re-writes the structure presented in Eq. 17 on two lines such that $w^A(n)$ on the first line is horizontally justified with $w^A(n - c)$ on the second line. To interpret this diagram, we simply confirm that for all n, if $w^A(n) = 0$, then then look directly below to check that $w^A(n - c) = 1$. Further, if $w^A(n) = 1$, then either $w^A(n - 1) = 0$ on the left, or $w^A(n - c) = 0$ directly below (shown in bold), or neither is true and we must have $w^A(n - b) = 0$ (underlined).

Consider the additional case where $q \leq \gamma$ and $i = q$. The diagram nearly holds until the last $b + 1$ entries. In particular, we note that $((01)^k 1)^{q-i} = \emptyset$, so the $w^A(n - c)$ sequence should conclude with $(01)^{k-1}1(01)0$ instead of $(01)^k 10$. Therefore the q^{th} term can be modified to $((01)^{k-1}1)^q (01)^{r/2+q} 1^{2(\gamma-q)+1} (01)^{k-(\gamma-q)-1} 10101$.

The alignment diagram on the right hand side in Fig. 3 similarly re-writes the structure such that $w^A(n)$ is horizontally justified with $w^A(n - b)$. We must confirm that if $w^A(n) = 0$, then directly below, $w^A(n - b) = 1$. We also confirm that the underlined entry has $w^A(n - b) = 0$.

Therefore for all $i \in [0, \ldots, \min(q - 1, \gamma)]$, the i^{th} term follows the recurrence. Note that the $*$ does not affect the recurrence, but represents that the value is unknown. This value is 1 if $i < \gamma$ but 0 if $i = \gamma$. In either case, the recurrence holds. Now we consider the $\gamma \leq q$ case. As justified above, after the $\gamma - 1^{\text{th}}$ term, the γ^{th} term begins at index m_1 with

$$\{w^A(n)\} = \underset{m}{\ldots}\, \overbrace{0(10)^{k-1}1\mathbf{1}((01)^{k-1}01\mathbf{1})^{q-i-1}}^{((01)^k1)^{q-i}} \overbrace{((01)^{k-2}01\mathbf{1})^{i-1}(01)^{k-1}}^{((01)^{k-1}1)^i} \quad \mathbf{1}\, \overbrace{(01)^{r/2+i-1}01(1\underline{\mathbf{1}})^{\gamma-i}\mathbf{1}}^{(01)^{r/2+i}1^{2(\gamma-i)+1}} \overbrace{(01)^{k-\gamma+i-1}01\mathbf{1}}^{(01)^{k-(\gamma-i)}1}\ldots$$

$$\{w^A(n-c)\} = \underset{m-c}{\ldots}1(01)^{k-1}1\underbrace{\mathbf{0}((10)^{k-1}11}_{((01)^k1)^{q-i-1}}\; \underbrace{\mathbf{0})^{q-i-1}((10)^{k-2}11\mathbf{0}}_{((01)^{k-1}1)^{i-1}} \underbrace{)^{i-1}(10)^{r/2+i-2}1^{2(\gamma-i)+4}}_{(01)^{r/2+i-1}1^{2(\gamma-i+1)+1}}\; \underbrace{\mathbf{0}\,(10)^{r/2+i-1}11}_{(01)^{k-(\gamma-i+1)}1}\; \underbrace{(\mathbf{0}\underline{1})^{\gamma-i}\mathbf{0}(10)^{k-\gamma+i-1}11}_{(01)^k1}\mathbf{0}\ldots$$

$$\{w^A(n)\} = \underset{m}{\ldots}\, \overbrace{(01)^{\gamma-i}(01)^{k-(\gamma-i)-1}01\,1((01)^{k-1}011)^{q-i-1}}^{((01)^k1)^{q-i}} \overbrace{((01)^{k-1}1)\;\; ((01)^{k-1}1)^{i-1}}^{((01)^{k-1}1)^i} \overbrace{(01)^{r/2+i}(1\underline{\mathbf{1}})^{\gamma-i} \qquad 1}^{(01)^{r/2+i}1^{2(\gamma-i)+1}} \overbrace{(01)^{k-(\gamma-i)-1}011}^{(01)^{k-(\gamma-i)}1}\ldots$$

$$\{w^A(n-b)\} = \underset{m-b}{\ldots}(11)^{\gamma-i}(1\,\underbrace{0)^{k-(\gamma-i)-1}11}_{(01)^{k-(\gamma-(i-1))}1}\, \underbrace{0((10)^{k-1}110)^{q-i-1}\;\;(10)^{k-1}\;\;1\;\;((1}_{((01)^k1)^{q-i}} \underbrace{0)^{k-1}1)^{i-1}((10)^{r/2+i}(1\mathbf{0})^{k-r/2-1-i}1)(1}_{((01)^{k-1}1)^i} \underbrace{0)^{r/2+i} \qquad 11}_{(01)^{r/2+i}} *\ldots$$

Fig. 3 Two alignments diagrams re-writing the structure presented in Eq. 17. See text for details

$$\begin{aligned}\{w^A\} &= \underset{m_1}{\ldots} \left((01)^k 1\right)^{q-\gamma} \left((01)^{k-1}1\right)^{\gamma} (01)^{r/2+\gamma} 1^{2(\gamma-\gamma)+1} (01)^{k-(\gamma-\gamma)} 1 \ldots \\ &= \underset{m_1}{\ldots} \left((01)^k 1\right)^{q-\gamma} \left((01)^{k-1}1\right)^{\gamma} (01)^{k-1} 1 (01)^k 1 \ldots \\ &= \underset{m_1}{\ldots} \left((01)^k 1\right)^{q-\gamma} \left((01)^{k-1}1\right)^{\gamma+1} (01)^k 1 \ldots\end{aligned}$$

We now observe that for $Y = \left((01)^k 1\right)^{q-\gamma} \left((01)^{k-1}1\right)^{\gamma+1}$, the sequence Y^∞ satisfies the recurrence. Because Y has length $c + 1$, it suffices to check that $Y(n) = 1$ if and only if $Y(n-1) = 0$ or $Y(n+1) = 0$ or $Y(n-2k) = 0$. Y satisfies this criterion by inspection.

Now consider the $q \leq \gamma$ case. As justified above, after the $q-1^{\text{th}}$ term, the q^{th} term begins at index m_1 with

$$\begin{aligned}\{w^A\} &= \underset{m_1}{\ldots}((01)^k 1)^{q-q}((01)^{k-1}1)^q (01)^{r/2+q} 1^{2(\gamma-q)+1} (01)^{k-(\gamma-q)-1} 10101 \ldots \\ &= \underset{m_1}{\ldots}((01)^{k-1}1)^q (01)^{r/2+q} 1^{2(\gamma-q)+1} (01)^{r/2+q} 1 \quad 0101 \ldots\end{aligned}$$

Now, we show that for $Y = ((01)^{k-1}1)^q (01)^{r/2+q} 1^{2(\gamma-q)+1} (01)^{r/2+q} 1$, the sequence Y^∞ satisfies the recurrence using alignment diagrams. Note that $|Y| = b + c$, and we start the diagram at $Y(-1)$ to simplify the diagram.

$$Y^\infty(n) = \underset{-1}{\ldots}\ 1\overbrace{\left((01)^{k-2}01\mathbf{1}\right)^{q-1}(01)^{k-1} \qquad \mathbf{1}}^{\left((01)^{k-1}1)\right)^q}\overbrace{(01)^{r/2+q-1}01\,(1\underline{\mathbf{1}})^{\gamma-q}\mathbf{1}}^{(01)^{r/2+q}1^{2(\gamma-1)+1}}\overbrace{(01)^{r/2+q-1}\ 01\mathbf{1}}^{(01)^{r/2+q}1}\ldots$$

$$Y^\infty(n-c) = \underset{-1-c}{\ldots}\ \underbrace{\mathbf{0}\left((10)^{k-2}11\mathbf{0}\right)^{i-1}}_{\left((01)^{k-2}1\right)^{q-1}}\underbrace{(10)^{r/2+q-1}1^{2(\gamma-q)+2}}_{(01)^{r/2+q}1^{2(\gamma-q)+1}}\underbrace{\mathbf{0}\,(10)^{r/2+q-1}11}_{(01)^{r/2+q}1}\underbrace{(\mathbf{0}\underline{1})^{\gamma-q}\mathbf{0}(10)^{k-\gamma+q-2}11\mathbf{0}}_{(01)^{k-1}1}\ldots$$

Similarly checking the recurrence for b:

$$Y^\infty(n) = \underset{0}{\ldots}\ \overbrace{\left((01)^{k-1} \qquad 1\right)\left((01)^{k-1}1\right)^{q-1}}^{\left((01)^{k-1}1)\right)^q}\overbrace{(01)^{r/2+q}(1\underline{\mathbf{1}})^{\gamma-i} \qquad 1}^{(01)^{r/2+i}1^{2(\gamma-i)+1}}\overbrace{(01)^{r/2+q}1}^{(01)^{r/2+q}1}\ldots$$

$$Y^\infty(n-b) = \underset{0-b}{\ldots}\ (11)^{\gamma-q}(1\underbrace{0)^{k-\gamma+q-1}1\ \ (}_{(01)^{r/2+q}1}\underbrace{(10)^{k-1}1)^{q-1}\left((10)^{r/2+q}(1\mathbf{0})^{k-r/2-1-i}1\right)(1}_{\left((01)^{k-1}1\right)^q}\underbrace{0)^{r/2+q}1}_{(01)^{r/2+q}}\ldots$$

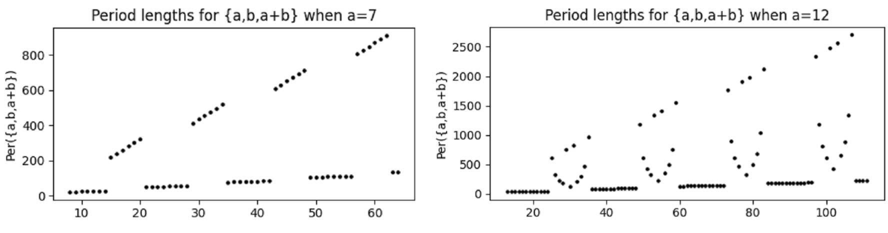

Fig. 4 Two plots of $\mathrm{Per}(\{a, b, a+b\})$ with fixed a. Note the phase transition from linear period lengths to quadratic, as well as the dips when $\gcd(a, b) \neq 1$. See Fig. 6 for a more extreme example

These diagrams verify that Y^∞ satisfies the recurrence, and this completes the proof that Eq. 17 holds. ☐

5 The $\{a, b, a+b\}$ case

Note that we also do not consider seeds in this section. In Berlekamp et al. (2004, p. 531), Berlekamp et al. state without proof that the period lengths of the $\{a, b, a+b\}$ game are quadratic and give a formula for the period length of the Grundy sequence $\mathcal{G}(n)$. In Althöfer and Bültermann (1995), Althöfer and Bülterman prove a particular example of this case (Example 7). In this section we provide a proof for the general $\{a, b, a+b\}$ game by finding $\{w^A\}$ explicitly, which was presented as open by Ho in (Ho 2015, table 4) (Fig. 4).

Theorem 21 *Suppose* $A = \{a, b, a+b\}$ *with* $a < b$. *Define* $k = \left\lfloor \frac{b-1}{2a} \right\rfloor$, $\sigma_i :\equiv ib \pmod{a}$, *and* $\delta_i := \mathbf{1}(\sigma_i > \sigma_{i-1})$, *with the exception* $\delta_1 = 0$.[4] *Let* $\tilde{a} = a/\gcd(a, b)$. *Then*

$$\{w^A\} = \begin{cases} \left(\sum_{i=1}^{\tilde{a}-1} \left((0^a 1^a)^{k-\delta_i} 0^{\sigma_i} 1^b 0^{a-\sigma_i} 1^a\right) 0^a 1^{a+b} \right)^\infty & 1 \le b < a \pmod{2a} \\ \left((0^a 1^a)^k 0^a 1^{a+b}\right)^\infty & \text{Otherwise} \end{cases} \tag{19}$$

It follows that

$$\operatorname{Per}(A) = \begin{cases} \tilde{a}(2b - 2ka) & 1 \le b < a \pmod{2a} \\ b + 2a(k+1) & \text{Otherwise} \end{cases} \tag{20}$$

Berlekamp et al. (2004, p. 531) inspire an alternate formulation of Eq. 20. If we let $b = 2ha + \rho$ for $\rho \in (-a, a]$, then the following also holds:

$$\operatorname{Per}(A) = \begin{cases} \tilde{a}(2b + \rho) & 1 \le \rho < a \\ 2b + \rho & \rho \le 0 \text{ or } \rho = a \end{cases}$$

Proof We separate this proof into 2 cases, beginning with the simpler linear case.

Case 1: If $b \ge a \pmod{2a}$ or $b \equiv 0 \pmod{2a}$, we can let $b = qa + r$ for odd $q = 2k + 1$ and $r \in [0, a]$.

For $n < b$ we find $w^A(n) = w^{\{a\}}(n)$, and $\{w^{\{a\}}\} = (0^a 1^a)^\infty$. Because q is odd,

$$\{w^A\} = \underbrace{(0^a 1^a)^k 0^a 1^r}_{b} \dots$$

[4] This means $\delta_i = 1$ if $\sigma_i > \sigma_{i-1}$ and $\delta_i = 0$ if not.

Next, for all $n \in [b, b+qa)$, we see that if $w(n-b) = 0$ then both $w(n) = 1$ and $w(n+a) = 1$. This fills in the next portion of the sequence; we use underline and bold characters to highlight the structure of the recurrence.

$$\{w^A\} = \ (\underline{0}^a 1^a)^k\, \underline{0}^a 1^r\, (\underline{1}^a \mathbf{1}^a)^k\, \underline{1}^a \mathbf{1}^a \ = \ (0^a 1^a)^k\, 0^a 1^{a+b} \underset{b+qa+a}{\ldots}$$

We now observe that $\mathbf{v}\big(b + (q+1)a\big) = 1^{a+b} = \mathbf{v}(0)$, which proves that $\mathrm{Per}\,(A) = b + 2a(k+1)$ with no preperiod. We now move to the quadratic case.

Case 2: If $1 \le b < a \pmod{2a}$, then we will have $b = qa + r$, for $r \in [1, a)$ and $q = 2k$. Again for $n < b$, the sequence is identical to $w^{\{a\}}$.

$$\{w^A\} = \ (0^a 1^a)^k\, 0^r \underset{b}{\ldots}$$

Once again for $n \in [b, 2b)$, we notice that if $w^A(n-b) = 0$, then by the recurrence $w^A(n) = w^A(n+a) = 1$. This means that for all $n \in [b, 2b) \cup [(2b-r)+a, 2b+a)$, we have $w(n) = 1$, as illustrated in the equation below. This leaves a gap of length $a - r$. In this gap $n \in [2b, (2b-r)+a)$, we find $w(n-a) = w(n-b) = w(n-a-b) = 1$, so indeed $w(n) = 0$.

$$\{w^A\} = \ \underbrace{(\underline{0}^a 1^a)^k\, \underline{0}^r}_{b}\, \underbrace{(\underline{1}^a \mathbf{1}^a)^k \underline{1}^r}_{b}\, 0^{a-r}\ \mathbf{1}^r \underset{2b+a}{\ldots} \ = \ (0^a 1^a)^k\, 0^r 1^b 0^{a-r}\ 1^r \underset{2b+a}{\ldots}$$

Because we showed $n \in [2b, 2b+a-r)$ has $w(n) = 0$, this implies $w(n+a) = 1$, so we may append $(a-r)$ 1's after $2b + a$.

$$\{w^A\} = \ (0^a 1^a)^k\, 0^r 1^b 0^{a-r}\ 1^a \underset{2b+2a-r}{\ldots}$$

This completes the first term in the summation, where $\sigma_1 = r$ and $\delta_1 = 0$. Using this as a base case for induction, we now show that the $i + 1^{\text{th}}$ term in the summation succeeds the i^{th} term. Suppose

$$\{w^A\} = \ \ldots (0^a 1^a)^{k-\delta_i} 0^{\sigma_i} 1^b 0^{a-\sigma_i} 1^a \ldots$$

We need only consider recent values of $w(n)$ back to $w(n-a-b)$, so define the index m such that

$\{w^A\} = \underset{m}{\ldots} 1^{b-a+\sigma_i} 0^{a-\sigma_i}\ 1^a \ldots$ Now let $m_1 = m + (b - a + \sigma_i)$ and notice that $\mathbf{v}(m_1)$ ends with a long string of ones. In particular we find that for all $n \in [m+a+b, m_1+b)$, we have $w^A(n-a-b) = 1$ and $w^A(n-b) = 1$, so w^A proceeds identically to $w^{\{a\}}$ for a length of precisely $b - 2a + \sigma_i$. We must now consider sub-cases for parity, where $\sigma_i + r < a$ or $\sigma_i + r \ge a$. We show the two cases below

$$\{w^A\} = \ \dots (0^a1^a)^{k-\delta_i} 0^{\sigma_i} 1^b \underbrace{0^{a-\sigma_i}1^a \quad 0^a1^a0^a1^a \dots}_{b} \underset{m_1+b}{\dots}$$

$$= \begin{cases} \underset{m_1}{\dots} 0^{a-\sigma_i}1^a \quad (0^a1^a)^{k-1}0^{\sigma_i+r} \qquad \underset{m_1+b}{\dots} & \sigma_i + r < a \\ \underset{m_1}{\dots} 0^{a-\sigma_i}1^a \quad (0^a1^a)^{k-1}0^a1^{\sigma_i+r-a} \underset{m_1+b}{\dots} & \sigma_i + r \geq a \end{cases}$$

Note that in the two cases we can plug in $\sigma_{i+1} = \sigma_i + r =$ and $\sigma_{i+1} = \sigma_i + r - a$ respectively. In the second case, where $\sigma_i + r \geq a$, then for $n \in \big[(m_1 + b), (m_1 + b) + (a - \sigma_{i+1})\big)$, we find $w^A(n - a) = 0$ so $w^A(n - a) = 1$. For $n \in \big[(m_1 + b) + (a - \sigma_{i+1}), (m_1 + b) + a\big)$, we find $w^A(n - a) = w^A(n - b) = w^A(n - a - b) = 1$, so $w(n) = 0$. Extend this second case below, defining m_2 as the frontier of the sequence:

$$\{w^A\} = \begin{cases} \underset{m_1}{\dots} 0^{a-\sigma_i}1^a \quad (0^a1^a)^{k-1}0^{\sigma_{i+1}} \qquad \underset{m_2}{\dots} & \sigma_i + r < a \\ \underset{m_1}{\dots} 0^{a-\sigma_i}1^a \quad (0^a1^a)^{k-1}0^a\mathbf{1}^a0^{\sigma_{i+1}} \underset{m_2}{\dots} & \sigma_i + r \geq a \end{cases}$$

This adds another copy of 0^a1^a to the sequence in the second case, and observe that $\delta_{i+1} = 1$ in the first case and $\delta_{i+1} = 0$ in the second. We can therefore combine cases. Additionally, we find that for $n \in [m_2, m_2 + b)$, if $w(n - b) = 0$ then $w(n) = w(n + a) = 1$. This leads to another string of 1^b with a gap of length $a - \sigma_{i+1}$ which is filled with zeros. This is shown below, where $m_3 = m_2 + b + a$ (Fig. 5),

$$\{w^A\} = \dots \ (\underline{0}^a1^a)^{k-\delta_{i+1}}\underline{0}^{\sigma_{i+1}} \underbrace{(\underline{1}^a\mathbf{1}^a)^k\underline{1}^{\sigma_{i+1}}}_{b} \underbrace{0^{a-\sigma_{i+1}}\mathbf{1}^{\sigma_{i+1}}}_{a} \underset{m_3}{\dots}$$

For $n \in \big[m_3, m_3 + (a - \sigma_{i+1})\big)$, we see $w(n - a) = 0$ so $w(n) = 1$. This completes the inductive step as we show below:

$$\{w^A\} = \dots (0^a1^a)^{k-\delta_i}0^{\sigma_i}1^b0^{a-\sigma_i}1^a \quad (0^a1^a)^{k-\delta_{i+1}}0^{\sigma_{i+1}}1^b0^{a-\sigma_{i+1}}1^a \dots$$

```
-------------1111111111111---11111111111111111111111111111----------1111111111111
                          ------11111111111111111111111111111-------1111111111111
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```

Fig. 5 Let $a = 13$, $b = 2a + 3$, and $A = \{13, 29, 42\}$. We compute $\mathrm{Per}\,(A) = 793$, and the period structure is shown above, with− used for 0

By induction this pattern will repeat. Let $\tilde{a} = \frac{a}{\gcd(a,b)}$, which is the order of b in $\mathbb{Z}_a^+$. Thus the $\tilde{a}^{\text{th}}$ iteration is the first such that $\sigma_{\tilde{a}} = 0$ and $\delta_{\tilde{a}} = 0$ so the sequence is

$$\{w^A\} = \sum_{i=1}^{\tilde{a}-1} \left((0^a1^a)^{k-\delta_i}0^{\sigma_i}1^b0^{a-\sigma_i}1^a\right)(0^a1^a)^k1^b \underset{m_4}{\dots} \tag{21}$$

We see that $\mathbf{v}(m_4) = 1^{a+b} = \mathbf{v}(0)$, so this is the entire period. We see that the i^{th} term of the summation has length $2ka - 2a\delta_i + b + 2a$, and there are $\tilde{a}$ total terms. We also compute that $\frac{r\cdot\tilde{a}}{a}$ is the number of terms j for which $\sigma_j < \sigma_{j-1}$, so if we include the first term $\delta_1 = 0$ and exclude the last term $\delta_{\tilde{a}} = 0$, there are precisely $r/\gcd(a,b)$ terms in the summation where $\delta_j = 0$, and the rest have $\delta_i = 1$. Therefore $\sum_{i=1}^{\tilde{a}-1}\delta_i = (\tilde{a}-1) - r/\gcd(a,b)$. We conclude that the total period length is

$$\begin{aligned}
\text{Per}(A) &= \sum_{i=1}^{\tilde{a}-1}(2ka - 2a\delta_i + b + 2a) + 2ka + b\\
&= (2ka(\tilde{a}-1) - 2a((\tilde{a}-1) - r/\gcd(a,b)) + b(\tilde{a}-1) + 2a(\tilde{a}-1)) + 2ka + b\\
&= 2ka\tilde{a} + 2ar/\gcd(a,b) + b\tilde{a}\\
&= 3\tilde{a}b - 2ka\tilde{a}
\end{aligned}$$

□ □

The following example is given as a Theorem in Althöfer and Bültermann (1995).

Example 7 (Althöfer and Bültermann 1995, Thm 3.1) Let $A = \{a, 2a+1, 3a+1\}$. Then Eq. (19) gives the following sequence, where $b = 2a+1$, $k = 1$, and $\sigma_i = i$ so $\delta_i = 1$ for all $i \in [1, a-1)$.

$$\begin{aligned}
\{w^A\} &= \left(0^a1^a\sum_{i=1}^{a-1}\left(0^i1^b0^{a-i}1^a\right)0^a1^{a+b}\right)^\infty\\
&= \left(0^a1^a\ \ 0^11^b0^{a-1}1^a\ \ 0^21^b0^{a-2}1^a\ \ 0^31^b0^{a-3}1^a\ \ \dots\ \ 0^a1^{a+b}\right)^\infty
\end{aligned}$$

Additionally $\text{Per}(A) = (6a^2 + 3a) - 2a^2 = 4a^2 + 3a \sim \frac{2}{9}\max(A)^2$.

This class of sets appears to be the only case for $|A| = 3$ which can have superlinear period lengths with no seed. Further generalizations using this result and Theorem 20 might bring the following conjecture within reach (Fig. 6).

Conjecture 1 *For all $a < b < c$ such that $a + b \neq c$,*

$$\text{Per}(\{a,b,c\}) < 2c \tag{22}$$

We have verified Conjecture 1 computationally for all $\{a,b,c\} \in \binom{[200]}{3}$. Ward (2016) conjectured that the period length in this case is a divisor of one of $a+b$, $b+c$ and $a+c$. Further, if the period length divides more than one of these numbers, then

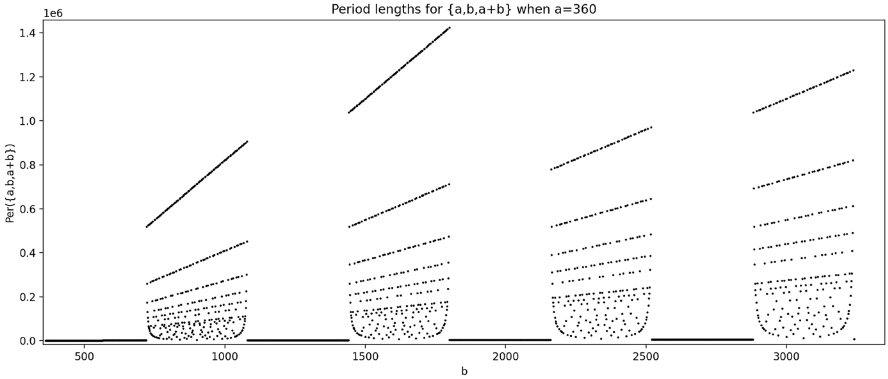

Fig. 6 The period lengths for $\{a, b, a + b\}$ where $a = 360$, a superior highly composite number

it is exactly the greatest common divisor of them. He verified this conjecture for all $\{a, b, c\} \in \binom{[4096]}{3}$. The 2D map of the period lengths for a fixed c and varying $a, b < c$ seems to have a fractal structure, see Larsson (2024); Flammenkamp (1997). If this conjecture is true, it must relate in part to some invariant of the structure caused by the initial seed of 1^α, because it fails entirely for different seeds, even if a, b, and c are coprime. This invariant would therefore be carried through the quadratic length preperiods seen in Sect. 4.

6 Super-polynomial period lengths with initial seeds

In this section we consider a particular family of sets which demonstrate that super-polynomial period lengths exist for 3-sets, given properly chosen initial seeds. This family will relate to an intersection of the cases of Sects. 4 and 5, and now we do consider seeds.

Lemma 22 *For any* $n \in \mathbb{N}$, *choose some odd* $b > 4n + 1$ *and let* $A = \{1, b, b + 1\}$ *and* $S = (01^3)^n$. *Then* $\operatorname{Per}(A, S) = 2(n + 1)b + 1$ *and* $\operatorname{PrePer}(A, S) = 0$.

A proof of this Lemma is found in Sect. 6.1. If we denote $b = 4n + 1 + 2d$, where $d \geq 1$, then the period is obtained by inserting d pairs into structure of a known periodicity of $\{1, 4n + 1, 4n + 2\}$. These inserted pairs are either of the form $(11)^d$ or $(01)^d$, and this is why b must be odd for the construction. The resulting structure is exactly

$$\{w^{A,S}\} = \dots \left(\sum_{i=0}^{n-1} \left((11)^d (1^3 0)^i 11 (01^3)^{n-i} \, (01)^d 0 (1^3 0)^i 11 (01^3)^{n-i-1} 0 \right) (11)^d (1^3 0)^n 1^2 \, (01)^d 0 (1^3 0)^n \right)^\infty,$$

To choose a seed generating this structure it suffices to choose any sub-string of length $a + 1$, so we choose the first $a + 1$ entries, namely $(11)^d 11 (01^3)^n \approx (01^3)^n$.

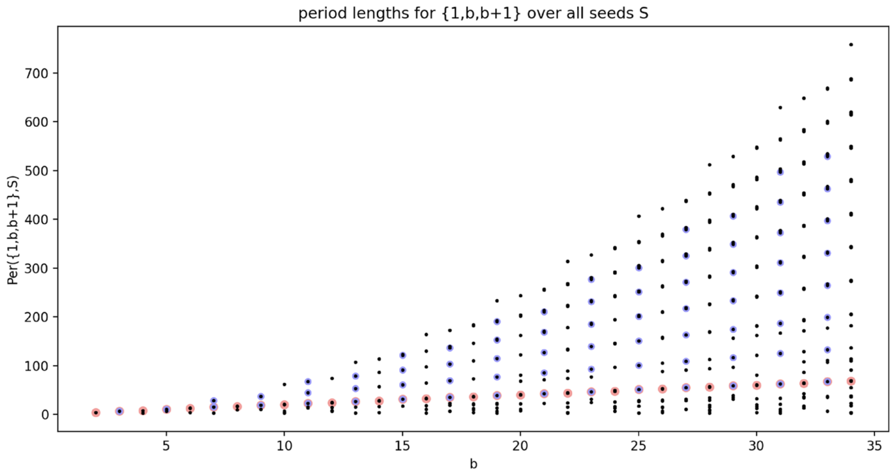

Fig. 7 This figure shows *all* periods of $\{1, b, b+1\}$ for $b+1 \leq 35$. The points highlighted in blue are given by Lemma 22 and the points in red are the default periods. Note that some points close together are overlapping. For example, $\mathcal{P}^{\{1,31,32\}} = \{3, 7, 11, 21, 33, 63, 125, 167, 187, 249, 251, 311, 313, 373, 375, 377, 435, 437, 439, 497, 499, 501, 503, 563, 565, 629\}$, while Lemma 22 gives only $\{63, 125, 187, 249, 311, 373, 435, 497\}$

Lemma 22 contradicts the generalization of Conjecture 1 over all seeds, since it implies quadratic period lengths. Figure 7 plots all period lengths for $\{1, b, b+1\}$, which shows that this Lemma only scratches the surface of the periodicities of 3-sets.

Theorem 23 (Superpolynomial Period Lengths) *For* $n \geq 1$, *define* $b_n = 4n - 1$, $A_n = \{n, nb_n, nb_n + n\}$, *and* $S_{(n)} = \sum_{j=1}^{n-1} 0^j 1^{b_n - j}$. *For the family of pairs* $\{A_n, S_{(n)}\}_{n=1}^{\infty}$, *where* $\alpha_n = \max(A_n)$, *it holds that*

$$\mathrm{Per}\,(A_n, S_{(n)}) = e^{\Omega(\sqrt{\alpha_n})}.$$

Proof Fix n, so we have $b = 4n - 1$ and $A = \{n, bn, bn + n\}$. We now construct the seed S dependent on n. Because A has greatest common divisor n, let $B = \{1, b, b+1\}$ and apply Multiplicative Linearity Proposition 5 to see that the parallel sequences satisfy $(w^{A,S})_i(m) = w^{A,S}(mn + i) = w^{B,S_i}(m)$ where $S_i \in \{0, 1\}^{b+1}$ for $i \in \{0, \ldots, n-1\}$. We now apply Lemma 22 by letting $S_i = (01^3)^i \approx 1^{4(n-i)}(01^3)^i$, so we have $\mathrm{Per}\,(B, S_i) = 2(i+1)b + 1$. By combining all S_i in parallel, this construction yields $S = \sum_{i=0}^{n-1} 1^{n-i} 0^i 1^{3n} = \sum_{j=1}^{n} 0^i 1^{b-i}$. Now, suppose $\{w^{A,S}\}$ is periodic over some $p \in \mathbb{N}$. This implies that for all $m \in \mathbb{N}$ and $i \in \{0, \ldots, n-1\}$, we have

$$w^{B,S_i}(m) = w^{A,S}(mn + i) = w^{A,S}(mn + i + np) = w^{B,S_i}(m + p),$$

so $\{w^{B,S_i}\}$ must also be periodic over p, i.e. $\mathrm{Per}\,(B, S_i) \mid p$. Because this is true for all i, we conclude that $\mathrm{lcm}\,\{2(i+1)b + 1 \mid 0 \leq i < n\} \leq \mathrm{Per}\,(A_n, S)$. A result from (Bateman et al. 2002, Problem 10797) regarding the lcm of arithmetic progressions implies

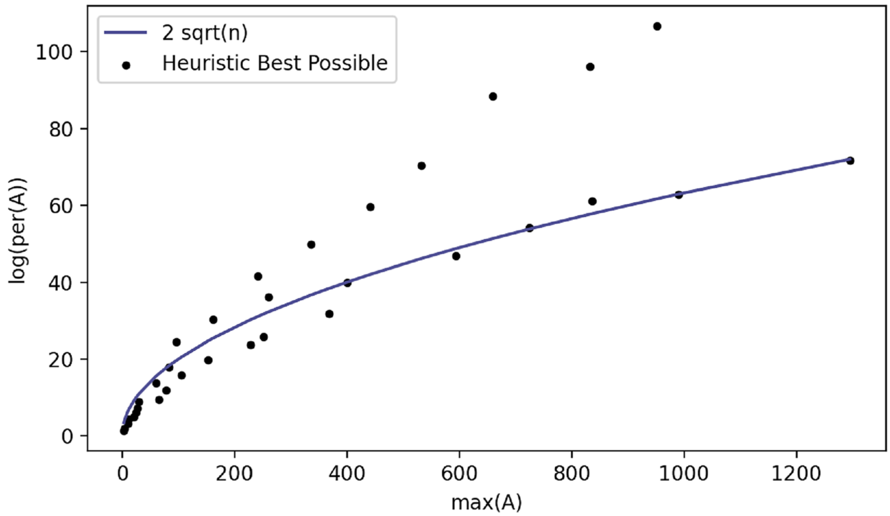

Fig. 8 Using the periods from Fig. 7, we can construct sets A_b with many parallel $\{1, b, b+1\}$ periods, specifically we get $A_b = |\mathcal{P}^{\{1,b,b+1\}}| \cdot \{1, b, b+1\}$ and some seed S comprised of $S_0, \ldots, S_{|\mathcal{P}^{\{1,b,b+1\}}|}$ such that $\text{Per}(A_b, S) \geq \text{lcm}(\mathcal{P}^{\{1,b,b+1\}})$. This gives a heuristic for the longest period possible, which appears to approximate $e^{\mathcal{O}(\sqrt{\max(A_b)})}$

a somewhat loose bound of $\text{lcm}(2b+1, 4b+1, 6b+1, \ldots, 2na+1) \geq e^{n-o(n)}$.[5] Therefore for all n, we note that $\alpha_n \sim 4n^2$, and conclude $\text{Per}(A_n, S_n) = e^{\Omega(n)} = e^{\Omega(\sqrt{\alpha_n})}$. □

Table 2 demonstrates the construction for $n = 2$. This family proceeds as follows

- $n = 1, b = 3, A_1 = \{1, 3, 4\}, S_1 = \emptyset$, and $\text{Per}(A_1, S_1) = \text{lcm}\{7\} = 7$
- $n = 2, b = 7, A_2 = \{2, 14, 16\}, S_2 = 01^6, \text{Per}(A_2, S_2) = 2 \cdot \text{lcm}\{15, 29\} = 870$
- $n = 3$, $b = 11$, $A_3 = \{3, 33, 36\}$, $S_3 = 01^{10}0^21^9$, $\text{Per}(A_3, S_3) = 3 \cdot \text{lcm}\{23, 45, 67\} = 208\,035$
- $n = 4$, $b = 15$, $A_4 = \{4, 60, 64\}$, $S_4 = 01^{14}0^21^{13}0^31^{12}$, $\text{Per}(A_4, S_4) = 4 \cdot \text{lcm}\{31, 61, 91, 121\} = 83\,287\,204$
- $n = 5$, $b = 19$, $A_5 = \{5, 95, 100\}$, $S_5 = 01^{18}0^21^{17}0^31^{16}0^41^{15}$, and $3\,364\,005\,645 \mid \text{Per}(A_5, S_5)$; we have not computed the exact period.

It is natural to predict that we have a multiple of n in the period length, but we have not proven this to be the case. To prove Theorem 23 we used a rough bound on $\text{lcm}\{2b+1, \ldots, 2nb+1\}$ and the few periodicities from Lemma 22, as shown in

[5] In particular it says that $\lim_{n\to\infty} \frac{\log(\text{lcm}(\ldots))}{n} = \frac{1}{\varphi(k)} \sum \frac{2b}{m}$ with the summation taken over units of $\mathbb{Z}_{2b}^{\times}$. This is an average of numbers greater than 1.

Table 2 In the construction for Theorem 23, we have $A_2 = \{2, 14, 16\}$

1		1		1		1		0		1		0		1		0		1		0		1		...
	0		1		1		1		0		1		0		1		1		0		1		1	...

Table 3 **(a)** Setting $n = 1$, we get $B = \{1, 5, 6\}$. **(b)** Setting $n = 2$, we get $B = \{1, 9, 10\}$.

(a)

1	1	0	1	1	1
0	1	1	0		
1	1	1	0	1	1
0	1	1	1	0	

(b)

1	**1**	**0**	**1**	**1**	**1**	**0**	**1**	**1**	**1**
0	*1*	*1*	*0*	*1*	*1*	*1*	*0*		
1	*1*	*1*	*0*	**1**	**1**	**0**	**1**	**1**	**1**
0	**1**	**1**	**1**	**0**	*1*	*1*	*0*		
1	*1*	*1*	*0*	*1*	*1*	*1*	*0*	**1**	**1**
0	**1**	**1**	**1**	**0**	**1**	**1**	**1**	**0**	

(a) We place $2n + 1 = 3$ copies of $X = 1101^30$ into a grid, so odd rows start and end with 0

(b) We place 5 copies of $X = 1101^301^30$ into a grid. Alternating copies of X are bolded to illustrate the pattern

Fig. 7. It is unclear if more periodicities and a stronger bound on their lcm, would yield an asymptotically better result, but this does not appear to be the case. Figure 8 shows that $e^{\Omega(\sqrt{\alpha_n})}$ appears to be best possible.

6.1 Proof of Lemma 22

The final proof uses different approach to the previous ones. Rather than constructing the sequence from scratch, we will exploit a structural pattern to *extend* a simple period. This was the method used to discover Lemma 22, and might be a productive strategy moving forward.

Proof First, we note that if $n = 0$, then Theorems 20 and 21 coincide to give that $\mathrm{Per}(A) = 2b + 1$ with no seed, so we are done. Next, fix $n \geq 1$.

Observe that if $\beta = 4n + 1$ and $B = \{1, \beta, \beta + 1\}$ then the string $X = 110(1^3 0)^n$ is a valid period structure of B with period length $\beta + 2$. This follows from the fact that $X^\infty(n) = 1$ if and only if $X^\infty(n-1) = 0$, $X^\infty(n+1) = 0$, or $X^\infty(n+2) = 0$, meaning each zero is separated by 2 or 3 ones. We use this fact to explicitly build period structures for odd $b > \beta$.

Write $2n + 1$ copies of X in a $(2n+2) \times (\beta + 1)$ grid so that the rows $0, \dots, 2n+1$ are the following: The k^{th} even row indexed from zero reads $(1^3 0)^k 11(01^3)^{n-k}$ and the k^{th} odd row reads $0(1^3 0)^k 11(01^3)^{n-k-1}0$, except for the last row, (the n^{th} odd row), and this reads $0(1^3 0)^n$. Table 3 gives two examples of such a filling. Notice that this filling has the following three properties.

- Reading across all of the rows yields X^{2n+1} in order.
- Even rows are fully filled. These begin and end with strings 1^3 or 11. Their length is $\beta + 1 = |X| - 1$ so they contain all of X except for a 0.
- Odd rows have $\beta - 1$ entries, except for row n which has β entries. These begin and end with 0, and contain all of X except either 1^3 or 11.

Denote $x(i, j)$ as the element in row i and column j, both indexed from 0. We can now re-interpret the recurrence relation on $w^B(n)$ in terms of x. The following identities must hold for all i, j.

$$x(i,j) = \begin{cases} 1 - \min\{x(i,j-1), x(i-1,j), x(i-1,j+1)\} & i \text{ is odd and } j > 0 \\ 1 - \min\{x(i,j-1), x(i-1,j-2), x(i-1,j-1)\} & i > 0 \text{ is even}, \beta > j > 1 \\ \\ 1 - \min\{x(i-1,\beta), x(i-1,0), x(i-1,1)\} & i \text{ is odd and } j = 0 \\ 1 - \min\{x(0,j-1), x(2n+1,j), x(2n+1,j-1)\} & i = 0, \beta > j > 0 \\ 1 - \min\{x(2n+1,\beta-1), x(2n+1,0), x(2n,\beta)\} & i = 0, j = 0 \\ 1 - \min\{x(0,\beta-1), x(0,0), x(2n+1,\beta-1)\} & i = 0, j = \beta \\ 1 - \min\{x(i,\beta-1), x(i,0), x(i-1,\beta-2)\} & i > 0 \text{ is even}, j = \beta \\ 1 - \min\{x(i,0), x(i-1,0), x(i-2,\beta)\} & i > 0 \text{ is even}, j = 1 \\ 1 - \min\{x(i-1,\beta-2), x(i-2,\beta), x(i-2,\beta-1)\} & i > 0 \text{ is even}, j = 0 \end{cases} \tag{23}$$

We focus on the first two cases which are the most general. It might also help to recall that for odd i, $x(i,0) = 0$ and for even i, $x(i,0) = x(i,1) = x(i,\beta-1) = x(i,\beta) = 1$. These follow from the specifications of the grid filling.

Next, we will choose any odd $b = \beta + 2d$ for $d > 0$, and let $A = \{1, b, b+1\}$. We will extend the grid to create a period structure for A with length $2(n+1)b+1$. We add d copies of the first two columns on the left side, as shown below.

We call the elements of this modified grid $y(i, j)$, with i indexed from 0 and j indexed from $-2d$. Thus for $j \geq 0$ we have $y(i,j) = x(i,j)$. If $j < 0$ is even we have $y(i,j) = x(i,0)$ and if $j < 0$ is odd we have $y(i,j) = x(i,1)$. Because of the specifications for filling x, we note that this entails prepending $(11)^d$ to even rows and $(01)^d$ to odd rows (Table 4).

Claim. *Reading across the rows of this table yields Y, a valid period structure for A with length* $2(n+1)b+1$.

First we have added $(2n+2)2d$ entries to the table, and we started with X^{2n+1}, so the total length is

$$(2n+2)2d + (2n+1)(\beta+2) = (2n+2)2d + (2n+2)\beta + 1 = 2b(n+1) + 1$$

Next we show that the recurrence relation is satisfied. To do this, write the recurrence relation $w^A(n) = 1 - \min\{w^A(n-1), w^A(n-\beta-2d), w^A(n-\beta-1-2d)$ in terms of $y(i, j)$. The rows are extended by the same length as the recurrence, so the following rules very are similar to Eq. 23.

$$y(i,j) = \begin{cases} 1-\min\{y(i,j-1), y(i-1,j), y(i-1,j+1)\} & i \text{ is odd and } j > -2d \\ 1-\min\{y(i,j-1), y(i-1,j-2), y(i-1,j-1)\} & i > 0 \text{ is even}, \beta > j > 1-2d \\ & \\ 1-\min\{y(i-1,\beta), y(i-1,0), y(i-1,1)\} & i \text{ is odd and } j = 0 \\ 1-\min\{y(0,j-1), y(2n+1,j), y(2n+1,j-1)\} & i = 0,\ \beta > j > -2d \\ 1-\min\{y(2n+1,\beta-1), y(2n+1,\beta), y(2n+1,-2d)\} & i = 0, j = -2d \\ 1-\min\{y(0,\beta-1), y(0,-2d), y(2n+1,\beta-1)\} & i = 0, j = \beta \\ 1-\min\{y(i,\beta-1), y(i,-2d), y(i-1,\beta-2)\} & i > 0 \text{ is even}, j = \beta \\ 1-\min\{y(i,-2d), y(i-1,-2d), y(i-2,\beta)\} & i > 0 \text{ is even}, j = -2d+1 \\ 1-\min\{y(i-1,\beta-2), y(i-2,\beta), y(i-2,\beta-1)\} & i > 0 \text{ is even}, j = -2d \end{cases} \tag{24}$$

From here it is possible to check that the nine cases in Eq. 24 hold for all $j \geq 0$ and $j < 0$ using Eq. 23, the definition of $y(i, j)$ and the three properties of the filling. We will show only the two main cases; the rest follow suit.

If i is odd and $0 < j < \beta$, then we confirm

$$\begin{aligned} y(i,j) = x(i,j) &= 1 - \min\{x(i,j-1), x(i-1,j), x(i-1,j+1)\} \\ &= 1 - \min\{y(i,j-1), y(i-1,j), y(i-1,j+1)\}. \end{aligned}$$

If i is odd and $-2d < j \leq 0$ is even, then we know $y(i,j) = x(i,0) = 0$, so we confirm

$$y(i,j) = 1 - \min\{y(i,j-1), y(i-1,j), y(i-1,j+1)\} = 1 - \min\{1,1,1\} = 0.$$

If i is odd and $-2d < j < 0$ is odd, then we know $y(i,j) = x(i,1) = 1$, so we confirm

$$y(i,j) = 1 - \min\{y(i,j-1), y(i-1,j), y(i-1,j+1)\} = 1 - \min\{0,1,0\} = 1.$$

If $i > 0$ is even and $1 < j < \beta$, then

$$\begin{aligned} y(i,j) = x(i,j) &= 1 - \min\{x(i,j-1), x(i-1,j-1), x(i-1,j-2)\} \\ &= 1 - \min\{y(i,j-1), y(i-1,j-1), y(i-1,j-2)\}. \end{aligned}$$

If $i > 0$ is even and $-2d+1 < j \le 1$ is odd, then we know $y(i,j) = x(i,1) = 1$, so we confirm

$$y(i,j) = 1 - \min\{y(i,j-1), y(i-1,j-1), y(i-1,j-2)\} = 1 - \min\{1,0,1\} = 1.$$

If $i > 0$ is even and $-2d+1 < j \le 0$ is even, then we know $y(i,j) = x(i,0) = 1$, so we confirm

$$y(i,j) = 1 - \min\{y(i,j-1), y(i-1,j-1), y(i-1,j-2)\} = 1 - \min\{1,1,0\} = 1.$$

Such verification can be completed for the seven edge cases to show that $y(i,j)$ obeys Eq. 24, and therefore $Y^\infty \in \mathcal{W}^A$.

Finally, we check for sub-periods of Y. Note that as long as $d > 0$, there are exactly $(n+1)$ copies of the substring $(01)^{d+1}$ present in Y, each at the beginning of a row. Additionally, the last row uniquely has length b, meaning the instances of $(01)^{d+1}$ are unequally spaced throughout Y. This prevents a sub-period of Y from occurring. □

7 Closing

We close the paper by setting up a few conjectures. First, we recall the observation that the converse of Proposition 8 holds for all (A, S) where $|A| \le 3$, and state it explicitly in a conjecture.

Conjecture 2 *If* $|A| \le 3$, *then for all* $S \in \{0,1\}^\alpha$, *let* $p = \mathrm{Per}(A,S)$. *Then if* $kp + x$ *is an extension of* A *for all* $k \in \mathbb{N}_+$ *and* $x \in A$, *then* $\mathrm{PrePer}(A,S) = 0$.

It suffices to check the case $k = 1$. If $S = \emptyset$ this is precisely the converse of Proposition 8; otherwise the converse is not true.

The following conjecture is based on computer simulations of $A \in \binom{[200]}{3}$. It provides a very limited characterization of the general $\{a,b,c\}$ case but provides an

Table 4 Example of *extending* the grid for $b = \beta + 2d$ with $n = 1$ and $d = 3$

1	1	1	1	1	1	1	1	0	1	1	1		$(11)^{d+1}$	0	1	1	1
0	1	0	1	0	1	0	1	1	0				$(01)^{d+1}$	1	0		
1	1	1	1	1	1	1	1	1	0	1	1	=	$(11)^{d+1}$	1	0	1	1
0	1	0	1	0	1	0	1	1	1	0			$(01)^{d+1}$	1	1	0	

example of its complex behavior. It is also an interesting question how this conjecture could explain the fractal structure observed by Flammenkamp (1997) and Larsson (2024).

Conjecture 3 *Suppose* $A = \{a, b, c\}$ *with* $1 < a < b < c$. *Use the division algorithm to obtain the following variables*:

$$\begin{aligned} b &= q\,a + r \\ c &= q_c(a+b) + r_c \quad r_c = q_a(2a) + r_a \\ c - a &= q'_c(a+b) + r'_c \quad r'_c = q'_a(2a) + r'_a \end{aligned}$$

Then $\mathrm{Per}\,(A) = b + c$ *and* $\mathrm{PrePer}\,(A) = 0$ *if and only if one of the following holds*:

(i) *q is even and $r'_c > 0$, $r'_a \le r$, and $2q'_a \le q$, and if $2q'_a = q$ then $r'_a \le 2r - a$.*
(ii) *q is odd and $r \ne 0$, $r \le r_a \le a$, and if $q_a = 0$ then $r_a < a$.*
(iii) *q is odd and $r = 0$, and $r_a \ne a$*

The following conjecture has been verified for all $A \in \binom{[25]}{3}$, for all seeds S. Together with Conjecture 1 these are a strengthening of a conjecture by Althöfer and Bültermann in (Althöfer and Bültermann 1995, (i)).

Conjecture 4 *If* $A = \{a, b, c\}$ *with* $a < b < c$, *and if* $\gcd(a, b, c) = 1$, *then for all seeds* $S \in \{0, 1\}^\alpha$, *we have* $\mathrm{Per}\,(A, S) < c^2$.

This would imply that the construction in Theorem 23 can only occur if A has a greatest common divisor. It would also provide the general bound that if $g = \gcd(a, b, c)$ and $\tilde{c} = c/g$, then for all seeds S, $\mathrm{Per}\,(A, S) \le \tilde{c}^{2g}$, which is $\mathcal{O}(e^{2c/e})$ in the worst case but ordinarily much less than $(\min(a+b, c) + 1)2^{c-3}$, the bound derived in Theorem 4.

From computer simulations of $A \in \binom{[25]}{3}$ with $\gcd A = 1$, we note that for all seeds S, if the period is super-linear, i.e. $\mathrm{Per}\,(A, S) > 2c$, then $(a + b) \mid c$, with the following eight exceptions. These sets have $\max(\mathcal{P}^A) > 2\alpha$, with example seeds given.

$\mathrm{Per}\,(\{11, 16, 20\}, (01)^2 1^4 01^2)$	$= 61$
$\mathrm{Per}\,(\{3, 11, 21\}, 0^2 101^2 0)$	$= 61$
$\mathrm{Per}\,(\{7, 17, 23\}, (010)^2 01^2)$	$= 73$
$\mathrm{Per}\,(\{10, 21, 23\}, 0^2 101^4)$	$= 78$
$\mathrm{Per}\,(\{5, 11, 24\}, 01^2 0^3 1^2 010)$	$= 65$
$\mathrm{Per}\,(\{11, 16, 25\}, 01^2 010^2 1^4)$	$= 56$
$\mathrm{Per}\,(\{16, 21, 25\}, 0^2 (101^3)^2 01^2)$	$= 61$
$\mathrm{Per}\,(\{13, 23, 25\}, 0^2 1^2 010^2 1^5)$	$= 83$

A characterization of these sets may relate to the solution to Conjecture 1 and/or 4.

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