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Fully graphic degree sequences and P-stable degree sequences ☆

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A B S T R A C T

The notion of P-stability of an infinite set of degree sequences plays influential role in approximating the permanents, rapidly sampling the realizations of graphic degree sequences, or even studying and improving network privacy. While there exist several known sufficient conditions for P-stability, we don't know any useful necessary condition for it. We also do not have good insight of possible structure of P-stable degree sequence families.

At first we will show that every known infinite P-stable degree sequence set, described by inequalities of the parameters n, c_1, c_2, Σ (the sequence length, the maximum and minimum degrees and the sum of the degrees) is "fully graphic" meaning that every degree sequence from the region with an even degree sum, is graphic. Furthermore, if Σ does not occur in the determining inequality, then the notions of P-stability and full graphicality will be proved equivalent. In turn, this equality provides a strengthening of the well-known theorem of Jerrum, McKay and Sinclair about P-stability, describing the maximal P-stable sequence set by n, c_1, c_2. Furthermore

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we conjecture that similar equivalences occur in cases if Σ also part of the defining inequality.

1. Introduction

In this paper we explore the connection between P-stability and the "fully graphic" property of specific regions of degree sequences, establishing a close relationship between these concepts.

We start the section with recalling and introducing some notions and notations. If $n > c_1 \geq c_2$ and Σ are natural numbers with $n \cdot c_1 \geq \Sigma \geq n \cdot c_2$, let

$$\mathbb{D}(n, c_1, c_2) = \{(d_1, \ldots, d_n) \in \mathbb{N}^n : c_1 \geq d_1 \geq \cdots \geq d_n \geq c_2, \textstyle\sum_{i=1}^n d_i \text{ is even}\},$$
$$\mathbb{D}(n, \Sigma, c_1, c_2) = \{(d_1, \ldots, d_n) \in \mathbb{D}(n, c_1, c_2) : \textstyle\sum_{i=1}^n d_i = \Sigma\}.$$

We will refer the elements of $\mathbb{D}(n, c_1, c_2)$ as **degree sequences**. As usual, an element of $\mathbb{D}(n, c_1, c_2)$ is **graphic** if there is a simple graph with these degrees. Otherwise the element is **non-graphic**. We say that a collection $\mathbb{D}$ of degree sequences is a **simple degree sequence region** if and only if there exists a property $\varphi(n, \Sigma, c_1, c_2)$ such that

$$\mathbb{D} = \mathbb{D}[\varphi] := \bigcup \{\mathbb{D}(n, \Sigma, c_1, c_2) \ : \ n > c_1 \geq c_2 \text{ and } \varphi(n, \Sigma, c_1, c_2) \text{ holds}\}. \tag{SR}$$

Similarly, we say that a collection $\mathbb{D}$ of degree sequences is a **very simple degree sequence region** if and only if there exists a property $\psi(n, c_1, c_2)$ such that

$$\mathbb{D} = \mathbb{D}[\psi] := \bigcup \{\mathbb{D}(n, c_1, c_2) \ : \ n > c_1 \geq c_2 \text{ and } \psi(n, c_1, c_2) \text{ holds}\}. \tag{VSR}$$

We will use the expression **(very) simple region** for short.

A (very) simple region $\mathbb{D}$ will be called **fully graphic** if and only if every degree sequence from the region is graphic. $\mathbb{D}$ is **almost fully graphic** if and only if $\mathbb{D} \setminus \mathcal{D}$ is fully graphic for some finite $\mathcal{D}$.

Given a graphic degree sequence D of length $|D| = n$, denote $\mathcal{G}(D)$ the set of all realizations of graphic sequence D and let

$$\partial(D) = \sum_{1 \leq i < j \leq n} |\mathcal{G}(D + 1^{-i}_{-j})| / |\mathcal{G}(D)|, \tag{1}$$

where the vector 1^{-i}_{-j} is comprised of all zeros, except at the ith and jth coordinates, where the values are -1. The operation $D \mapsto D + 1^{-i}_{-j}$ is called a **perturbation operation** on the degree sequences. We define two more similar operations: $D + 1^{-i}_{+j}$ and $D + 1^{+i}_{+j}$

analogously. Let us emphasize that we do not assume that i and j are different, so for example the operation $D + 1_{+j}^{+j}$ is also defined, and it adds 2 to the j^{th} coordinate.

We say that a family $\mathbb{D}$ of degree sequences is **P-stable** if there is a polynomial $p(n)$ such that $\partial(D) \leq p(|D|)$ for each graphic element D of $\mathbb{D}$. Let us emphasize that we do not require every element of a P-stable family to be graphic. The P-stability has alternative, but equivalent definitions using different perturbation operations. In the Appendix of this paper, we describe a short history of these definitions, and prove their equivalence. (As far as we are aware, this is the first explicit discussion of this topic in writing.)

The P-stability of an infinite family of degree sequences is an important property. It would be enough to mention one result: the **switch Markov Chain** process rapidly mixes on P-stable families (see [5, Theorems 8.3]). Furthermore, that P-stability plays an influential role in approximating the permanent, and even studying and improving network privacy.

Determining whether a particular family is P-stable is typically challenging. There exist only a few results establishing the P-stability of families of degree sequences for simple graphs. First, we examine three of them which implicitly considered simple and very simple regions.

(P1) (Jerrum, McKay, Sinclair [11]) The very simple degree sequence region $\mathbb{D}[\varphi_{JMS}]$ is P-stable, where $\varphi_{JMS} \equiv (c_1 - c_2 + 1)^2 \leq 4c_2(n - c_1 - 1)$.

(P2) (Greenhill, Sfragara [9]) The simple degree sequence region $\mathbb{D}[\varphi_{GS}]$ is P-stable, where $\varphi_{GS} \equiv (2 \leq c_2$ and $3 \leq c_1 \leq \sqrt{\Sigma/9})$. (This result was not announced explicitly, but [9, Lemma 2.5] clearly proved this fact.)

(P3) (Jerrum, McKay, Sinclair [11]) The simple degree sequence region $\mathbb{D}[\varphi^*_{JMS}]$ is P-stable, where $\varphi^*_{JMS} \equiv (\Sigma - nc_2)(nc_1 - \Sigma) \leq (c_1 - c_2)\big\{(\Sigma - nc_2)(n - c_1 - 1) + (nc_1 - \Sigma)c_2\big\}$.

In contrast with P-stability, the classical theorem of Paul Erdős and Gallai ([3]) makes it easy to check if a certain sequence is graphic or not. As an extension of this result, in Section 2 we show that every simple region $\mathbb{D}(n, \Sigma, c_1, c_2)$ contains exactly one **primitive** element, (i.e. a sequence in the form $(c_1, \ldots, c_1, a, c_2, \ldots c_2)$), and the region $\mathbb{D}(n, \Sigma, c_1, c_2)$ is fully graphic if and only if its primitive element is graphic (see Theorem 2.5).

In this way we have established a machinery to decide whether certain (very) simple regions are fully graphic or not. Using that machinery, in Theorems 3.1, 3.4 and 3.6 we show the following

(P1*) The very simple region $\mathbb{D}[\varphi_{JMS}]$ is fully graphic.
(P2*) The simple region $\mathbb{D}[\varphi_{GS}]$ is fully graphic.
(P3*) The simple region $\mathbb{D}[\varphi^*_{JMS}]$ is fully graphic.

When comparing statements (P1) and (P2) with their counterparts (P1*) and (P2*), a fundamental question arises, serving as the central focus of this paper: *What is the connection between P-stability and the fully graphic property of specific (very) simple regions?*

Concerning the "*fully graphic* $\longrightarrow$ *P-stable*" direction, if we consider very simple regions, then we can prove the following strengthening of (P1) which is actually the strongest possible result (Theorem 4.2):

Theorem. *The largest fully graphic very simple region*

$$\mathbb{D}_{\max} := \bigcup \{\mathbb{D}(n, c_1, c_2) : \ \mathbb{D}(n, c_1, c_2) \textit{ is fully graphic}\}$$

is P-stable, and so the switch Markov chain is rapidly mixing on $\mathbb{D}_{\max}$.

We do not know whether a similar statement holds for simple regions.

Problem 1.1. Is it true that the largest fully graphic simple region

$$\mathbb{D}' = \bigcup \{\mathbb{D}(n, \Sigma, c_1, c_2) : \ \mathbb{D}(n, \Sigma, c_1, c_2) \text{ is fully graphic}\}$$

is P-stable?

Concerning the "*P-stable* $\longrightarrow$ *fully graphic*" direction, we have a partial result claiming that a P-stable very simple region should contain large fully graphic very simple regions provided the region satisfies some natural restrictions. To formulate our result precisely, let us say that a very simple region $\mathbb{D}$ is a **cone region** if and only if for some functions $f, g \in \mathbb{N}^{\mathbb{N}}$ we have

$$\mathbb{D} = \mathbb{D}(f, g) := \bigcup \{\mathbb{D}(n, g(n), f(n)) : n \in \mathbb{N}\}.$$

In Section 5 we prove the following results (Theorem 5.10 and Corollary 5.11)

Theorem. *Assume that* $f, g, h \in \mathbb{N}^{\mathbb{N}}$ *are increasing functions. If the cone region* $\mathbb{D}(f, g)$ *is P-stable, then* $\mathbb{D}(f + h, g - h)$ *is almost fully graphic provided*

(1) $f(n + k) \le f(n) + k$ *for each* $n, k \in \mathbb{N}$,
(2) $lim_{n\to\infty} h(n)/\ln(n) = \infty$.

Corollary. *Assume that* $0 \le \varepsilon_2 < \varepsilon_2' < \varepsilon_1' < \varepsilon_1 \le 1$. *If the very simple region* $\mathbb{D} := \bigcup_{n\in\mathbb{N}} \mathbb{D}(n, \lfloor \varepsilon_1 \cdot n \rfloor, \lceil \varepsilon_2 \cdot n \rceil)$ *is P-stable, then the region* $\mathbb{D}' = \bigcup_{n\in\mathbb{N}} \mathbb{D}(n, \lfloor \varepsilon_1' \cdot n \rfloor, \lceil \varepsilon_2' \cdot n \rceil)$ *is almost fully graphic.*

This is a partial answer for the problem raised in the Abstract: the structure of each very simple P-stable region is essentially fully determined. The proof of the previous theorem is based on some observations concerning split graphs and Tyshkevich product.

In Section 4, besides proving that the largest graphic very simple region is P-stable, we also construct new P-stable simple regions. To do so we improved a method which was developed in [11] to prove (P1). In [11] to obtain (P1) Jerrum, McKay and Sinclair actually proved that $\partial(D) \le n^{10}$ for each $D \in \mathbb{D}(n, c_1, c_2)$ provided $\varphi_{JMS}(n, c_1, c_2)$ holds. Using a finer analysis we can estimate $\partial(D)$ for a single degree sequence under milder assumptions. Namely, we will prove the following statement (Theorem 4.4):

Theorem. *If a graphic degree sequence $D = (d_1, d_2, \dots, d_n)$ satisfies*

$$\forall k \in [1, n] \quad \sum_{i=1}^{k} d_i \le k \cdot (k-1) + d_n \cdot (n-k) + 1, \tag{46}$$

then $\partial(D) \le 3 \cdot n^9$.

In [11] the authors proved that $\mathbb{D}[\varphi_{JMS}] \subset \mathbb{D}[\varphi^*_{JMS}]$, i.e. (P3) is a stronger statement than (P1). However, the following result (Theorem 4.8) shows that there exist simple P-stable regions which are completely disjoint from (P3):

Theorem. *The simple region*

$$\mathbb{D}_0 = \bigcup \{\mathbb{D}(2m, 4m, m, 1) : m \ge 4\} \tag{2}$$

is fully graphic and P-stable, but $\mathbb{D}_0 \cap \mathbb{D}[\varphi^*_{JMS}] = \emptyset$.

2. Characterization of fully graphic regions $\mathbb{D}(n, \Sigma, c_1, c_2)$

In this section we study fully graphic simple degree sequence regions.

Definition 2.1. A non-increasing sequence $(d_1, \dots, d_n)$ has the *Erdős-Gallai property* if and only if for each $1 \le k \le n$,

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k). \tag{EG$_k$}$$

The next statement is almost trivial, but will be very convenient later on.

Proposition 2.2. *For all $D \in \mathbb{D}(n, \Sigma, c_1, c_2)$, property (EG$_k$) holds for each $k \le c_2$ and $k > c_1$.*

Proof. When $k \le c_2$ then

$$\sum_{i=1}^{k} d_i \le kc_1 \le k(n-1) = k(k-1) + (n-k)k.$$

If $k > c_1$ then $kc_1 \le k(k-1)$. □

We denote by $(a)_m$ the constant a sequence of length m.

Definition 2.3. If $n > c_1 \ge c_2$ are natural numbers, we say that a sequence $D = (d_1, \ldots, d_n) \in \mathbb{D}(n, c_1, c_2)$ is **primitive** if and only if D has the form

$$D = \underbrace{c_1, \ldots, c_1}_{k}, a, \underbrace{c_2, \ldots, c_2}_{(n-1)-k}, \tag{3}$$

for some $1 \le k \le n$ and $c_2 \le a \le c_1$.

Definition 2.4. If $n > c_1 \ge c_2$ and Σ are natural numbers with $n \cdot c_1 \ge \Sigma \ge n \cdot c_2$, we define the *least Erdős-Gallai sequence* $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ of length n with the given sum Σ and with given maximum and minimum elements c_1 and c_2 as follows:

If $c_1 = c_2$, then let $\mathrm{LEG}(n, \Sigma, c_1, c_2) = (c_1)_n$. If $c_2 < c_1$, then let

$$\mathrm{LEG}(n, \Sigma, c_1, c_2) = \underbrace{c_1, \ldots, c_1}_{\alpha}, a, \underbrace{c_2, \ldots, c_2}_{(n-1)-\alpha}, \tag{4}$$

where

$$\alpha = \left\lfloor \frac{\Sigma - n \cdot c_2}{c_1 - c_2} \right\rfloor \tag{5}$$

and

$$a = \Sigma - \big(\alpha \cdot c_1 + (n-1-\alpha) \cdot c_2\big). \tag{6}$$

Theorem 2.5. *Assume that* $n > c_1 \ge c_2$ *and* Σ *are natural numbers with* $n \cdot c_1 \ge \Sigma \ge n \cdot c_2$, Σ *is even.*

(1) $\mathrm{LEG}(n, \Sigma, c_1, c_2) \in \mathbb{D}(n, \Sigma, c_1, c_2)$ *and it is primitive.*
(2) *If* $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ *is Erdős-Gallai, then every element of* $\mathbb{D}(n, \Sigma, c_1, c_2)$ *is Erdős-Gallai.*

Proof of Theorem 2.5 (1). Clearly, $\sum \mathrm{LEG}(n, \Sigma, c_1, c_2) = \Sigma$, and $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ is primitive by its definition.

Since $\alpha \cdot c_1 + (n-1-\alpha)\cdot c_2 = \alpha\cdot(c_1-c_2)+(n-1)\cdot c_2$, we have

$$a \ge \Sigma - \frac{\Sigma - n\cdot c_2}{(c_1-c_2)}(c_1-c_2)+(n-1)\cdot c_2 = c_2. \tag{7}$$

Moreover, $(\Sigma - n\cdot c_2)/(c_1-c_2)-1 < \alpha$, hence

$$\begin{aligned} a &= \Sigma - (\alpha\cdot(c_1-c_2)+(n-1)\cdot c_2) \le \\ &\le \Sigma - \left(\left(\frac{\Sigma-n\cdot c_2}{c_1-c_2}-1\right)\cdot(c_1-c_2)\right)+(n-1)\cdot c_2 = c_1. \end{aligned} \tag{8}$$

Putting together (7) and (8), we obtain $c_1 \ge a \ge c_2$. So $\mathrm{LEG}(n,\Sigma,c_1,c_2)$ is really an element of $\mathbb{D}(n,c_1,c_2)$. □

Before proving Theorem 2.5 (2), we need some preparation. Denote $\lhd_{\mathrm{lex}}$ the lexicographical order on finite sequences (i.e. $(d_1,\ldots,d_n)\lhd_{\mathrm{lex}}(e_1,\ldots,e_n)$ if and only if $d_j < e_j$, where $j=\min\{i : d_i \ne e_i\}$).

Lemma 2.6. *Assume that $D=(d_1,\ldots,d_n)\in\mathbb{D}(n,\Sigma,c_1,c_2)$, and $1\le \ell < m \le n$ with $d_\ell < c_1$ and $d_m > c_2$. Write $D' = D + 1^{+\ell}_{-m}$.*

(1) *$D'\in\mathbb{D}(n,\Sigma,c_1,c_2)$ and $D \lhd_{\mathrm{lex}} D'$.*
(2) *If D' is Erdős-Gallai, then so is D.*

Proof. (1) It is trivial from the construction.

(2) Assume that (EG_k) fails for D: $\displaystyle\sum_{i=1}^{k} d_i > k(k-1)+\sum_{i=k+1}^{n}\min(d_i,k).$

We will show that (EG_k) fails for $D'=(d'_1,\ldots,d'_n)$. We should distinguish three cases:
Case 1. $k<\ell$.

If $\min(k,d_\ell)<\min(k,d'_\ell)$, then $\min(k,d_\ell)+1=\min(k,d_\ell+1)$, and so $d_\ell<k$. Hence, $\min(d'_m,k)=\min(d_m-1,k)=d_m-1$. Thus, $\min(k,d'_\ell)+\min(k,d'_m)=\min(k,d_\ell)+\min(k,d_m)$. If $\min(k,d_\ell)=\min(k,d'_\ell)$, then $\min(k,d'_\ell)+\min(k,d'_m)\le\min(k,d_\ell)+\min(k,d_m)$.

Thus, $\sum_{i=k+1}^{n}\min(k,d_i)\ge\sum_{i=k+1}^{n}\min(k,d'_i)$, and so

$$\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i > k(k-1)+\sum_{i=k+1}^{n}\min(k,d_i) \ge k(k-1)+\sum_{i=k+1}^{n}\min(k,d'_i). \tag{9}$$

Case 2. $\ell \le k < m$.

Then $d'_i\ge d_i$ for $i\le k$ and $d'_i \le d_i$ for $k+1\le i\le n$, so

$$\sum_{i=1}^{k} d'_i \ge \sum_{i=1}^{k} d_i > k(k-1)+\sum_{i=k+1}^{n}\min(k,d_i) \ge k(k-1)+\sum_{i=k+1}^{n}\min(k,d'_i). \tag{10}$$

Case 3. $m \le k$.

Since $\ell < m$, $d'_\ell = d_\ell + 1$ and $d'_m = d_m - 1$, we have $\sum_{i=1}^k d'_i = \sum_{i=1}^k d_i + 1 - 1 = \sum_{i=1}^k d_i$. Moreover, $d'_i = d_i$ for $k+1 \le i \le n$. Thus

$$\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i > k(k-1) + \sum_{i=k+1} \min(k, d_i) = k(k-1) + \sum_{i=k+1} \min(k, d'_i). \tag{11}$$

Hence, (EG$_k$) really fails for D'. □

Lemma 2.7. $\mathbb{D}(n, \Sigma, c_1, c_2)$ *contains just one primitive element,* $\mathrm{LEG}(n, \Sigma, c_1, c_2)$*, which is the* $\lhd_{\mathrm{lex}}$*-maximal element of* $\mathbb{D}(n, \Sigma, c_1, c_2)$*.*

Proof. We know that $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ is a primitive element of $\mathbb{D}(n, \Sigma, c_1, c_2)$ by Theorem 2.5 (1). Observe now that $\mathbb{D}(n, \Sigma, c_1, c_2)$ can not contain two different primitive elements.

Indeed, assume that $A = (c_1)_k a (c_2)_{n-k-1}$ and $B = (c_1)_\ell b (c_2)_{n-\ell-1}$ are from $\mathbb{D}(n, \Sigma, c_1, c_2)$. If $\Sigma = c_1 \cdot n$, then $\mathbb{D}(n, \Sigma, c_1, c_2)$ contains just one element: the $(c_1)_n$ sequence. So we can assume that $\Sigma < n \cdot c_1$, and so we can assume that $a, b < c_1$.

If $k = \ell$, then $a = \Sigma - (kc_1 + (n-k-1)c_2) = \Sigma - (\ell c_1 + (n-\ell-1)c_2) = b$, so the two sequences are the same. Assume that $k < \ell$. Using that $a < c_1$, we obtain

$$\sum A = kc_1 + a + (n-k-1)c_2 < (k+1)c_1 + (n-k-1)c_2 \le \ell c_1 + b + (n-\ell-1)c_2 = \sum B, \tag{12}$$

contradiction. We proved the observation.

The $\lhd_{\mathrm{lex}}$-maximal element of $\mathbb{D}(n, \Sigma, c_1, c_2)$ is primitive by Lemma 2.6(1), so it must be $\mathrm{LEG}(n, \Sigma, c_1, c_2)$. □

Proof of Theorem 2.5 (2). Assume that the set

$$\mathcal{D} = \{D \in \mathbb{D}(n, \Sigma, c_1, c_2) : D \text{ is not Erdős-Gallai}\}$$

is not empty. Let D^* be the $\lhd_{\mathrm{lex}}$-maximal element of $\mathcal{D}$. If D^* is not primitive then let $\ell = \min\{i : 1 \le i \le n : c_2 < d_i < c_1\}$ and $m = \max\{i : 1 \le i \le n : c_2 < d_i < c_1\}$. Then $\ell < m$ and so $D^* + 1^{+\ell}_{-m} \in \mathcal{D}$ and $D^* \lhd_{\mathrm{lex}} D^* + 1^{+\ell}_{-m}$ by Lemma 2.6. Contradiction, and so the maximal element of $\mathcal{D}$ is primitive. So, by Lemma 2.7, $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ is in $\mathcal{D}$. □

Corollary 2.8. *A simple region* $\mathbb{D}$ *is fully graphic if and only if* $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ *is graphic whenever* $\mathbb{D}(n, \Sigma, c_1, c_2) \neq \emptyset$*, and, consequently,* $\mathbb{D}(n, \Sigma, c_1, c_2) \subset \mathbb{D}$*.*

Proof. If $D \in \mathbb{D}$ is not graphic, then $D \in \mathbb{D}(n, \Sigma, c_1, c_2) \subset \mathbb{D}$ for some parameters n, Σ, c_1, c_2, and $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ is not Erdős-Gallai by Theorem 2.5(2). □

The following, easy to prove statement will simplify some arguments in Section 4.

Lemma 2.9. *Let $\mathbb{D}(n, c_1, c_2)$ be fully graphic. Then, $\varphi_{FG}(n, c_2, c_1)$ holds, where*

$$\varphi_{FG}(n, c_1, c_2) \equiv \forall k \in [1, n] \quad c_1 \cdot k \leq k \cdot (k-1) + c_2 \cdot (n-k) + 1. \tag{13}$$

Indeed, fix k. If $\mathbb{D}(n, c_1, c_2)$ is fully graphic, then either $c_1k+(n-k)c_2$ or $c_1k+(n-k)c_2+1$ is even, so either the sequence $(c_1)_k(c_2)_{n-k}$, or the sequence $(c_1)_k(c_2+1)(c_2)_{n-k-1}$ is in $\mathbb{D}(n, c_2, c_1)$, and so it is graphic. We apply the Erdős-Gallai theorem. In the first case the inequality holds with 1 surplus. In the second case the displayed inequality holds.

3. The known *P*-stable, simple degree sequence regions are fully graphic

In this section we will show that some known P-stable degree sequence regions are also fully graphic.

3.1. Sequences defined by minimum and maximum degrees

Our first result below clearly implies the statement (P1*) from the Introduction. In the proof we will use the machinery we developed in Section 2.

Theorem 3.1. *If $n > c_1 \geq c_2$ are natural numbers such that*

$$(c_1 - c_2 + 1)^2 \leq 4c_2(n - c_1 - 1), \tag{14}$$

then every sequence from $\mathbb{D}(n, c_1, c_2)$ has the Erdős-Gallai property.

Before proving this result we have to point out that this result was already proved by Zverovich and Zverovich ([17]) in 1992. It was somewhat strengthened in [2] by Cairns, Mendan and Nikolayevsky. Actually they used the inequality $4nc_2 \geq (c_1 + c_2 + 1)^2$ and neither paper recognized that this is identical with (14). However, our proof is new, and as construction (58) shows, it also provides a slightly larger always graphic region.

Before we continue our proof we should recall the following theorem. As the authors remarked in the first lines of the proof of the "Theorem" in [14], they actually proved the following statement.

Theorem 3.2 *(Tripathi-Vijay [14]). Assume that $D = (d_1, d_2, \ldots, d_n)$ is a non-increasing sequence of non-negative integers and $n > d_1$. Then the following two statements are equivalent:*

(1) *D has the Erdős-Gallai property,*
(2) *(EG_k) holds for $k \in \{j : 1 \leq j \leq n$ and $d_j > d_{j+1}\}$.*

Proof of Theorem 3.1. Assume on the contrary that $D = (d_1, \ldots, d_n) \in \mathbb{D}(n, c_1, c_2)$ is not Erdős-Gallai. Write $\Sigma = \sum_{i=1}^{n} d_i$. By Theorem 2.5(2), $E = \mathrm{LEG}(n, \Sigma, c_1, c_2) \in$

$\mathbb{D}(n, c_1, c_2)$ is not Erdős-Gallai, as well. E has the form $(c_1)_k a (c_2)_{n-1-k}$, where $c_1 \geq a \geq c_2$. Applying Theorem 3.2 for E we obtain that either (EG$_k$) or (EG$_{k+1}$) fails, and, by Proposition 2.2, we have $c_2 \leq k \leq c_1$.
In the first case:

$$kc_1 > k(k-1) + a + (n-k-1)c_2. \tag{15}$$

Since $a \geq c_2$, we obtain

$$kc_1 > k(k-1) + c_2 + (n-k-1)c_2. \tag{16}$$

Rearranging, we obtain

$$0 > k^2 - (c_2 + c_1 + 1)k + nc_2. \tag{17}$$

If we have two roots, then the discriminant of (17) should be positive:

$$(c_1 + c_2 + 1)^2 - 4nc_4 > 0 \tag{18}$$

Reordering (14) we obtain

$$0 \geq (c_1 - c_2 + 1)^2 - 4c_2(n - c_1 - 1) = c_1^2 + c_2^2 + 1 - 2c_1c_2 + 2c_1 - 2c_2 - 4nc_2 - 4c_1c_2 - 4c_2 = \\ c_1^2 + c_2^2 + 2c_1c_2 + 2c_1 + 2c_2 + 1 - 4nc_2 = (c_1 + c_2 + 1)^2 - 4c_2n,$$

which contradicts (18)
In the second case:

$$kc_1 + a > k(k+1) + (n-k-1)c_2. \tag{19}$$

Since $a \leq c_1$, we obtain

$$(k+1)c_1 > k(k+1) + (n-k-2)c_2. \tag{20}$$

Let $\ell = k+1$. We obtain

$$\ell c_1 > \ell(\ell - 1) + (n - \ell - 1)c_2. \tag{21}$$

Rearranging, we obtain

$$0 > \ell^2 - (c_2 + c_1 + 1)\ell + nc_2. \tag{22}$$

But (22) is just (17), which leads contradiction again. $\square$

3.2. Sequences defined by extremal degrees and degree sums

Concerning (P2) we could prove the following theorem which is stronger than (P2*), where the restriction concerning c_1 was $3 \leq c_1 \leq \frac{1}{3}\sqrt{\Sigma}$.

Definition 3.3. For any real $\epsilon > 0$ define the property φ_ϵ as follows:

$$\varphi_\epsilon(n, \Sigma, c_1, c_2) \quad \equiv \quad 2 \leq c_2 \ \text{ and } \ 3 \leq c_1 \leq \sqrt{(1-\epsilon)\Sigma}. \tag{23}$$

Theorem 3.4. *For each positive ϵ, the simple region $\mathbb{D}[\varphi_\epsilon]$ is almost fully graphic.*

Proof. Assume that $D \in \mathbb{D}[\varphi_\epsilon]$ is not graphic. Then $D \in \mathbb{D}(n, \Sigma, c_1, c_2) \subset \mathbb{D}$ for some parameters n, Σ, c_1, c_2 with $\varphi_\epsilon(n, \Sigma, c_1, c_2)$, and by Theorem 2.5(2), $A = \mathrm{LEG}(n, \Sigma, c_1, c_2)$ is not Erdős-Gallai.

Then sequence A has the form $(c_1)_k a (c_2)_{n-1-k}$, where $c_1 \geq a \geq c_2$. Applying Theorem 3.2 for A we obtain that either (EG_k) or (EG_{k+1}) fails.
In the first case, when (EG_k) fails, we obtain

$$kc_1 > k(k-1) + a + (n-k-1)c_2. \tag{24}$$

Since $a + (n-k-1)c_2 = \Sigma - kc_1$, we have

$$kc_1 > k(k-1) + \Sigma - kc_1. \tag{25}$$

Rearranging, we obtain

$$0 > k^2 - (2c_1 + 1)k + \Sigma, \tag{26}$$

and so, using $c_1 \leq \sqrt{(1-\epsilon)\Sigma}$, we have

$$0 > k^2 - (2\sqrt{(1-\epsilon)\Sigma} + 1)k + \Sigma. \tag{27}$$

Thus, the discriminant of (27) should be positive:

$$\left(2\sqrt{(1-\epsilon)\Sigma} + 1\right)^2 - 4\Sigma > 0. \tag{28}$$

In the second case, when (EG_{k+1}) fails, we obtain

$$kc_1 + a > k(k+1) + (n-k-1) \tag{29}$$

Using $\Sigma - (kc_1 + a) = (n-k-1)c_2$, we obtain

$$kc_1 + a > k(k+1) + (\Sigma - (kc_1 + a)) \tag{30}$$

Rearranging, we obtain

$$0 > k^2 - (2c_1 - 1)k - 2a + \Sigma \geq k^2 - (2c_1 + 1)k + \Sigma, \tag{31}$$

and so

$$0 > k^2 - (2\sqrt{(1-\epsilon)\Sigma} + 1)k + \Sigma. \tag{32}$$

Thus, the discriminant of (32) should be positive:

$$\left(2\sqrt{(1-\epsilon)\Sigma} + 1\right)^2 - 4\Sigma > 0. \tag{33}$$

So far we established the following statement:

$$\text{If } D \in \mathbb{D}(n, \Sigma, c_1, c_2) \subset \mathbb{D}[\varphi_\epsilon] \text{ is not graphic, then } \left(2\sqrt{(1-\epsilon)\Sigma} + 1\right)^2 - 4\Sigma > 0. \tag{34}$$

But $\left(2\sqrt{(1-\epsilon)\Sigma} + 1\right)^2 - 4\Sigma > 0$ if and only if $\Sigma < \frac{1}{4(1-\sqrt{1-\epsilon})^2}$. Since $\Sigma \geq 2n$, we obtain

$$\text{If } D \in \mathbb{D}(n, \Sigma, c_1, c_2) \subset \mathbb{D}[\varphi_\epsilon] \text{ is not graphic, then } n < \frac{1}{8(1-\sqrt{1-\epsilon})^2}. \tag{35}$$

So we proved there is a natural number n_ϵ such that every element of $\mathbb{D}$ with length at least n_ϵ is graphic. This completes the proof. □

Remark 3.5. Using (35) one could estimate the value n_ϵ for concrete real numbers. For example, in the case of (P2*) we have $1 - \epsilon = 1/9$ and then $n_{8/9}$ is 1.

Next we prove the statement of (P3*).

Theorem 3.6. *The simple degree sequence region $\mathbb{D}[\varphi^*_{JMS}]$ is fully graphic.*

Proof. Assume that $D \in \mathbb{D}[\varphi^*_{JMS}]$, where

$$\varphi^*_{JMS} \equiv (\Sigma - nc_2)(nc_1 - \Sigma) \leq (c_1 - c_2)\left\{(\Sigma - nc_2)(n - c_1 - 1) + (nc_1 - \Sigma)c_2\right\}.$$

We need to prove that D is Erdős-Gallai. Fix parameters n, Σ, c_1, c_2 with $D \in \mathbb{D}(n, \Sigma, c_1, c_2) \subset \mathbb{D}[\varphi^*_{JMS}]$. By Theorem 2.5(2), it is enough to prove that the sequence $A = \mathrm{LEG}(n, \Sigma, c_1, c_2)$ is Erdős-Gallai.

The sequence A has the form $(c_1)_k a (c_2)_{n-1-k}$, where $c_1 \geq a \geq c_2$. Applying Theorem 3.2 for A we obtain that A is Erdős-Gallai if and only if (EG_k) and (EG_{k+1}) holds. Since we can assume $c_2 \leq k \leq c_1$ by Proposition 2.2, (EG_k) and (EG_{k+1}) have the following form:

$$kc_1 \leq k(k-1) + (n-k)c_2 + (\min\{a,k\} - c_2), \tag{EG_k}$$

and

$$kc_1 + a \leq (k+1)k + (n-k-1)c_2, \tag{EG_{k+1}}$$

which can be rearranged as

$$kc_1 \leq k(k-1) + (n-k)c_k + (2k - c_2 - a). \tag{EG^*_{k+1}}$$

Consider the φ^*_{JMS} inequality. Using the notation $b = a - c_2$, we have

$$\Sigma = kc_1 + a + (n-k-1)c_2 = nc_2 + k(c_1 - c_2) + b.$$

So, taking $x = \frac{b}{c_1 - c_2}$, the LHS of φ^*_{JMS} can be written as follows:

$$\begin{aligned} LHS = \big[k(c_1 - c_2) + b\big]\big[(n-k)(c_1 - c_2) - b\big] = \\ (nk - k^2)(c_1 - c_2)^2 + (n-2k)b(c_1 - c_2) - b^2 = \\ (c_1 - c_2)^2\big[nk - k^2 + (n-2k)x - x^2\big]. \end{aligned} \tag{36}$$

Now consider the RHS of φ^*_{JMS}:

$$\begin{aligned} RHS = (c_1 - c_2)\Big\{\big[(k(c_1 - c_2) + b\big](n - c_1 - 1) + \big[(n-k)(c_1 - c_2) - b\big]c_2\Big\} = \\ (c_1 - c_2)^2\big[(k+x)(n - c_1 - 1) + (n-k-x)c_2\big]. \end{aligned} \tag{37}$$

Since $c_1 - c_2 > 0$, putting together (36) and (37) we obtain

$$nk - k^2 + (n-2k)x - x^2 \leq (k+x)(n - c_1 - 1) + (n-k-x)c_2,$$

which can be rearranged as:

$$kc_1 \leq k(k-1) + (n-k)c_2 + x(2k + x - c_1 - 1 - c_2). \tag{38}$$

To derive (EG_k) we can assume that $c_2 \leq k \leq c_1$ or (EG_k) holds as we proved it in Proposition 2.2. Moreover, it is enough to show that the RHS of (38) is less than, or equal to RHS of (EG_k), that is,

$$x(2k + x - c_1 - 1 - c_2) \leq \min\{a,k\} - c_2. \tag{39}$$

Since $0 \leq x < 1$, we have

$$x\big(2k + x - (c_1 + c_2) - 1\big) \leq 2k - (c_1 + c_2) = (k - c_1) + (k - c_2) \leq k - c_2. \tag{40}$$

Since $x = \frac{b}{c_1-c_2} \le 1$ and $k \le c_1$, we have

$$x(2k + x - c_1 - 1 - c_2) = b\frac{2k + x - c_1 - 1 - c_2}{c_1 - c_2} = \\ b\frac{(c_1 - c_2) + (2k - 2c_1) + (x - 1)}{c_1 - c_2} \le b\frac{c_1 - c_2}{c_1 - c_2} = b. \quad (41)$$

Since $\min(a, k) - c_2 = \min(b, k - c_2)$, putting together (41) and (40) we obtain (39), which implies that (EG_k) holds.

To derive (EG_{k+1}) we can assume that $c_2 \le k + 1 \le c_1$ by Proposition 2.2, and it is enough to show that RHS of (EG^*_{k+1}) is greater than, or equal to RHS of (38):

$$x(2k + x - c_1 - 1 - c_2) \le 2k - c_2 - a. \quad (42)$$

But $0 \le x \le 1$, so

$$x(2k + x - c_1 - 1 - c_2) = x(2k - c_1 - c_2 + (x - 1)) \le (2k - c_1 - c_2) \quad (43)$$

so (43) holds which implies (EG^*_{k+1}), and so (EG_{k+1}) as well. This finishes the proof that the region $\mathbb{D}[\varphi^*_{JMS}]$ is fully graphic. □

Gao and Greenhill proved in [8] that for any given parameter $\gamma > 2$ the infinite set of scale free degree sequences with the given parameter is P-stable. However this set is clearly not a degree sequence region. However we believe that this set can be embedded into a P-stable simple degree region.

4. *P*-stable degree sequences

In this section we are considering fully graphic, very simple degree sequence regions, and want to prove that they are also P-stable regions. Along this process we will strengthen Jerrum, McKay and Sinclair's theorem (P1).

4.1. Every fully graphic very simple degree sequence region is P-stable

In Section 1 the statement (P1) quoted the seminal result of Jerrum, McKay and Sinclair from 1992:

Theorem 4.1 *([11, Theorem 8.1]). The very simple region*

$$\mathbb{D} = \mathbb{D}[\varphi_{JMS}] := \bigcup \left\{ \mathbb{D}(n, c_1, c_2) : (c_1 - c_2 + 1)^2 \le 4c_2(n - c_1 - 1) \right\} \quad (44)$$

is P-stable.

Since in Theorem 3.1 we proved that the very simple region $\mathbb{D}[\varphi_{JMS}]$ is fully graphic, therefore the next statement is a powerful strengthening of Theorem 4.1:

Theorem 4.2. *The largest fully graphic very simple region*

$$\mathbb{D}_{\max} := \bigcup\{\mathbb{D}(n, c_2, c_1) : \ \mathbb{D}(n, c_2, c_1) \text{ is fully graphic}\}$$

is P-stable, and so the switch Markov chain is rapidly mixing on $\mathbb{D}_{\max}$.

Careful study of the proof of [11, Theorem 8.1] reveals, that Jerrum, McKay and Sinclair actually proved the following, slightly stronger result.

Theorem 4.3. *The very simple region* $\mathbb{D}[\varphi^\circ_{JMS}]$ *is P-stable, where the property* φ°_{JMS} *is defined as follows:*

$$\varphi^\circ_{JMS} \ \equiv \ \forall k \in [1, n] \quad c_1 \cdot k \le k \cdot (k-1) + c_2 \cdot (n-k). \tag{45}$$

Unfortunately, Theorem 4.3 does not yield Theorem 4.2 because the assumption that "*every element of* $\mathbb{D}(n, c_2, c_1)$ *is graphic*" does not imply (45). Fortunately, as we already proved (see Lemma 2.9), that the following, slightly weaker inequality holds for a fully graphic $\mathbb{D}(n, c_2, c_1)$:

$$\varphi_{FG}(n, c_2, c_1) \ \equiv \ \forall k \in [1, n] \quad c_1 \cdot k \le k \cdot (k-1) + c_2 \cdot (n-k) + 1.$$

Using this observation, the following result, which is a direct strengthening of Theorem 4.3, already yields Theorem 4.2.

Theorem 4.4. *If a graphic degree sequence* $D = (d_1, d_2, \dots, d_n)$ *satisfies*

$$\forall k \in [1, n] \quad \sum_{i=1}^{k} d_i \le k \cdot (k-1) + d_n \cdot (n-k) + 1, \tag{46}$$

then $\partial(D) \le 3 \cdot n^9$.

Proof. Given a graph $G = \langle V, E\rangle$, an *alternating trail of length* ℓ is a sequence of vertices $v_0, \dots, v_\ell$ such that $\{v_i v_{i+1}\}$ is edge if and only if i is even, and the pairs $\{\{v_i, v_{i+1}\} : i < \ell\}$ are pairwise distinct. An alternating trail is an *alternating path* if the vertices $x_0, \dots, x_n$ are pairwise different apart from the pair $\{x_0, x_1\}$.

The proof of Theorem 4.4 is based on the following Lemma.

Lemma 4.5. *Let* $D^* = D + 1^{+p}_{+q}$ *for some* $1 \le p, q \le n$, *where* D *satisfies inequality* (46). *If* G *is a graph with vertex set* $V = \{v_1, \dots v_n\}$ *and with degree sequence* D^*, *moreover* $\Gamma(v_p) = \Gamma(v_q)$ *(that is their neighborhoods coincide), then there exists an alternating* *trail of odd length 1, 3, 5 or 7 between v_p and v_q, which contains one more edges than non-edges.*

Proof of Lemma 4.5. Suppose for the contrary, that there is no such alternating trail. We will describe the structure of G. First, observe that either $v_p = v_q$ or the edge v_pv_q is missing (otherwise there is an alternating trail of length 1). Let

$$S = \{v_p, v_q\}, \quad X = \Gamma(v_p), \quad Y = \{y \in V : |X \setminus \Gamma(y)| \geq 2\}, \quad Z = \Gamma(Y) \setminus X. \tag{47}$$

Observe the following facts:

(i) The set X is a clique (otherwise there is an alternating trail of length 3, namely $v_p \to X \to X \to v_q$).

(ii) The set Y is an independent set.
Indeed, if $\{y_0, y_1\} \in [Y]^2 \cap E$, then $|X \setminus \Gamma(y_i)| \geq 2$ for $i < 2$ implies that there is $\{x_0, x_1\} \in [X]^2$ such that $\{x_i, y_i\} \notin E$ for $i < 2$. Thus $v_p, x_0, y_0, y_1, x_1, v_q$ is an alternating trail of length 5.

(iii) The set Z is a clique.
Indeed, if $\{z_0, z_1\} \in [X]^2 \setminus E$, then there are $y_0, y_1 \in Y$ such that $\{z_i, y_i\} \in E$ for $i < 2$, but we can not guarantee that $y_0 \neq y_1$. Since $|X \setminus \Gamma(y_i)| \geq 2$ for $i < 2$, there is $\{x_0, x_1\} \in [X]^2$ such that $\{x_i, y_i\} \notin E$ for $i < 2$. Thus $v_p, x_0, y_0, z_0, z_1, y_1, x_1, v_q$ is an alternating trail of length 7, but not necessarily a path.

(iv) The induced bipartite graph $G[X, Z]$ is complete.
Indeed, if $x \in X$ and $z \in Z$ with $\{x, z\} \notin E$, then pick first $y \in Y$ with $\{y, z\} \in E$. Since $|X \setminus \Gamma(y)| \geq 2$, we can pick $x' \in X$ such that $x' \neq x$ and $\{x', y\} \notin E$. Then x_p, x, z, y, x', v_q is an alternating trail of length 5.

(v) The sets $\{v_p, v_q\}, X, Y, Z$ are pairwise disjoint.

Let

$$K = X \cup Z \quad \text{and} \quad R = V \setminus (K \cup Y \cup S).$$

We have $|K| + |Y| + |R| + |S| = |V| = n$. Putting together (i), (iii) and (iv), we obtain

(vi) K is a clique.

Write $k = |K|$. We will estimate the sum of the degrees of the vertices in K. To start with, write

$$\sum_{i=1}^{k} d_i \geq \sum_{v \in K} \deg(v) =$$

$$\sum_{v\in K}|\Gamma(v)\cap K|+\sum_{v\in S}|\Gamma(v)\cap K|+\sum_{v\in R}|\Gamma(v)\cap K|+\sum_{y\in Y}|\Gamma(y)\cap K| \quad (48)$$

Since K is a clique,

$$\sum_{v\in K}|\Gamma(v)\cap K| = k\cdot(k-1). \quad (49)$$

Since $\Gamma(v)=X\subset K$ for $v\in S$, we have

$$\sum_{v\in S}|\Gamma(v)\cap K| = |S||X|. \quad (50)$$

Since $|X\setminus\Gamma(v)|\le 1$ for $v\in R$, we have

$$\sum_{v\in R}|\Gamma(v)\cap K| \ge |R|(|X|-1). \quad (51)$$

By the construction, $\Gamma(Y)\subset K$, so

$$\sum_{y\in Y}|\Gamma(y)\cap K| \ge |Y|\cdot d_n. \quad (52)$$

Putting together, we have

$$\sum_{i=1}^{k} d_i \ge k\cdot(k-1)+|S|\cdot|X|+|R|\cdot(|X|-1)+|Y|\cdot d_n. \quad (53)$$

Since $|X|=\deg(v_p)\ge d_n+(3-|S|)$, we obtain

$$\sum_{i=1}^{k} d_i \ge k\cdot(k-1)+|S|\cdot(d_n+3-|S|)+|R|(d_n+2-|S|)+|Y|d_n. \quad (54)$$

Observe that $|R|+|Y|+|S|=n-k$. Clearly $|S|=1$ or $|S|=2$.
If $|S|=1$, then

$$|S|\cdot(d_n+3-|S|)+|R|(d_n+2-|S|)+|Y|d_n = d_n+2+(|R|+|Y|)d_n+|R| = (|R|+|Y|+|S|)d_n+2+|R| = (n-k)d_n+2. \quad (55)$$

If $|S|=2$, then

$$|S|\cdot(d_n+3-|S|)+|R|(d_n+2-|S|)+|Y|d_n = 2(d_n+1)+(|R|+|Y|)d_n = (|R|+|Y|+|S|)d_n+2 = (n-k)d_n+2. \quad (56)$$

So in both case, from (54) we obtain

$$\sum_{i=1}^{k} d_i \geq k \cdot (k-1) + (n-k)d_n + 2, \tag{57}$$

which contradicts (46). So we proved Lemma 4.5. □

The proof of Theorem 4.4 from Lemma 4.5 is similar to the proof of [11, Theorem 8.1] from [11, Lemma 1].
Assume that G' is a graph such that the degree sequence of G' is $D' = D + 1^{+i}_{+j}$ for some $1 \leq i < j \leq n$.

If $\Gamma_{G'}(v_i) = \Gamma_{G'}(v_j)$, then we can apply Lemma 4.5 for $G = G'$, $p = i$ and $q = j$ to obtain an alternating trail P of odd length 1, 3, 5 or 7 between v_i and v_j, which contains one more edges than non-edges. Flipping edges and non-edges along the trail P we obtain a graph $G^{\dagger}$ which is a realization of D.

If $\Gamma_{G'}(v_i) \neq \Gamma_{G'}(v_i)$, then there is an alternating trail Q of length 2 between v_i and v_j. Assume that $Q = v_i v_m v_j$, where $v_i v_m$ is an edge, and $v_m v_j$ is a non-edge. Flipping edges along trail Q we obtain a graph G^* with degree sequence

$$D^* = D' + 1^{+j}_{-i} = D + 1^{+i}_{+j} + 1^{-i}_{+j} = D + 1^{+j}_{+j}.$$

Now we can apply Lemma 4.5 for $G = G^*$ with $p = q = j$ to obtain an alternating trail P of odd length 1, 3, 5 or 7 from v_j to v_j, which contains one more edges than non-edges. Flipping edges and non-edges along the trail P we obtain a graph $G^{\dagger}$ which is a realization of D.

How much information should we use to obtain back G' from $G^{\dagger}$? We need to know if we were in case $\Gamma_{G''}(v_i) = \Gamma_{G''}(v_j)$ or in case $\Gamma_{G''}(v_i) \neq \Gamma_{G''}(v_j)$.

If $\Gamma_{G''}(v_i) = \Gamma_{G''}(v_j)$, we should know P. The trail P contains at most 8 vertices, so this is at most n^8 possibilities.

If $\Gamma_{G''}(v_i) \neq \Gamma_{G''}(v_j)$, we should know P and Q. Since the first and the last element of P are the same, we have at most n^7 possibilities for P. Knowing P we can compute G^*, and we also know v_i or v_j. We should know Q. We know one vertex (v_i or j) from Q. So knowing P we have n^2 possibilities for Q. Knowing Q we can compute G''. We should know which endpoint of Q is v_i and which is v_j. In this case we have at most $2 \cdot n^2 \cdot n^7 = 2 \cdot n^9$ possibilities.

Putting together, for a given G^* we have at most $n^8 + 2 \cdot n^9 \leq 3 \cdot n^9$ possibilities for $G^{\dagger}$. □

The next theorem gives us a method to prove that a simple region $\mathbb{D}[\varphi]$ is P-stable. Namely, it is enough to prove that $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ satisfies (46) from Theorem 4.4 whenever $\varphi(n, \Sigma, c_1, c_2)$ holds.

Theorem 4.6. *If $n > c_1 \geq c_2$ and $nc_1 \geq \Sigma \geq nc_2$ are natural numbers, Σ is even, then the following are equivalent:*

(1) $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ *satisfies* (46) *from Theorem 4.4,*
(2) every $D \in \mathbb{D}(n, \Sigma, c_1, c_2)$ *satisfies* (46) *from Theorem 4.4.*

Proof. To show that (1) implies (2), let $D = (d_1, \ldots, d_n)$ be an arbitrary element of $\mathbb{D}(n, \Sigma, c_1, c_2)$, and fix $1 \leq k \leq n$. Write $LEG(n, \Sigma, c_1, c_2) = (e_1, \ldots, e_n)$. Then, $\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} e_i$, and $\sum_{i=1}^{k} e_i \leq k(k-1) + (n-k)e_n + 1$ because $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ satisfies (46). Putting together these two inequalities we obtain

$$\sum_{i=1}^{k} d_i \leq k(k-1) + (n-k)e_n + 1 = k(k-1) + (n-k)c_2 \leq k(k-1) + (n-k)d_n + 1,$$

which implies that (46) holds for D and k. □

By $(P2)$, the simple region $\mathbb{D}[\varphi_{GS}] = \mathbb{D}[\varepsilon_{8/9}]$ is P-stable. We also proved that $\mathbb{D}[\varphi_\varepsilon]$ is fully graphic for $\varepsilon > 0$. The next question is very natural.

Problem 4.7. Is the simple region $\mathbb{D}[\varphi_\epsilon]$ P-stable for $\varepsilon > 0$?

4.2. Construction of P-stable families with special properties

We demonstrate the existence of a fully graphic simple region, whose P-stability can be derived from Theorem 4.4, whereas application of [11, Theorem 8.3] does not yield its P-stability.

Theorem 4.8. *The simple region*

$$\mathbb{D}_0 = \bigcup \{\mathbb{D}(2m, 4m, m, 1) : m \geq 4\} \tag{58}$$

is fully graphic and P-stable, although $\mathbb{D}_0 \cap \mathbb{D}[\varphi^*_{JMS}] = \emptyset$.

Proof. First, observe that

$$D_m := (m)_2(3)_1(1)_{2m-3} = \mathrm{LEG}(2m, 4m, m, 1). \tag{59}$$

Lemma 4.9. *For $m \geq 4$, D_m does not satisfy the φ^*_{JMS} inequality.*

Proof. Indeed, $n = 2m$, $c_1 = m$, $c_2 = 1$, $\Sigma = 4m$, so

$LHS_{\varphi^*_{JMS}} - RHS_{\varphi^*_{JMS}} =$

$$(\Sigma - c_2 n)(c_1 n - \Sigma) - (c_1 - c_2)[(\Sigma - c_2 n)(n - c_1 - 1) + (c_1 n - \Sigma)c_2] = \\ 4(m^2 - 2m)m - [2((m-1)m + m^2 - 2m)(m-1)] = 2m^2 - 6m,$$

which is positive for $m \geq 4$, so we proved the Lemma. □

Lemma 4.10. D_m *is graphic.*

Proof. By the Tripathi-Vijay Theorem 3.2, we should check only EG_2 and EG_3 for D_m. But

$$\sum_{i=1}^{2} d_i = 2m < 2 + (2 + (2m-3)\cdot 1) = 2(1-2) + \sum_{i=3}^{2m} \min(d_i, 2), \tag{EG_2}$$

and

$$\sum_{i=1}^{3} d_i = 2m + 3 = 6 + (2m-3)\cdot 1 = 3(3-1) + \sum_{i=4}^{2m} \min(d_i, 3). \quad \square \tag{EG_3}$$

Lemma 4.11. D_m *satisfies* (46) *from Theorem 4.4.*

Proof. If $k = 1, 2, 3$, inequality (46) is the following:

$$\begin{aligned} d_1 = m &\leq 1(1-0) + (2m-1)1 + 1 && (k=1)\\ d_1 + d_2 = 2m &< 2(2-1) + (2m-2)1 + 1, && (k=2)\\ d_1 + d_2 + d_3 = 2m + 3 &\leq 3(3-1) + (2m-3)1 + 1. && (k=3) \end{aligned} \tag{60}$$

If $k \geq 3$, then EG_k implies EG_{k+1}, because, the LHS is increased by 1, and the RHS is increased by $2k - 1$. □

The lemmas together prove the theorem. □

5. Large fully graphic regions in very simple P-stable regions

In the first two subsections we review the necessary facts about split graphs and Tyshkevich product.

5.1. Split graphs

A $G = (V, E)$ graph is a **split graph** if its vertices can be partitioned into a clique and an independent set. Split graphs were introduced by Földes and Hammer ([7]).

Split graphs are recognizable from their degree sequences:

Theorem 5.1 *(Hammer and Simeone, 1981 [10], Tyshkevich, Melnikov and Kotov [15]). Assume that G is a graph with degree sequence $D = (d_1, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n$. Let m be the largest value of i, such that $d_i \geq i - 1$. Then G is a split graph if and only if*

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i.$$

Remark 5.2. Consequently, if one realization of a degree sequence D is a split graph, then all realizations of D are split graphs as well. Such a degree sequence is referred as *split degree sequence.*

We will write $G = ((U, W), E)$ to mean that G is a split graph with vertex set $U \cup W$, U is a clique and W is an independent set. Let us remark that U and W are not necessarily unique.

Theorem 5.3. *If $\mathbb{D}(n, c_1, c_2)$ is not fully graphic, then $\mathbb{D}(n, c_1, c_2)$ contains a split degree sequence.*

Proof. Fix a non-graphic $D \in \mathbb{D}(n, c_1, c_2)$. Write $\Sigma = \sum D$. By Corollary 2.8 the sequence $\mathrm{LEG}(n, \Sigma, c_1, c_2)$ is not graphic.

Lemma 5.4. *There is $c_2 \leq \ell \leq c_1$ such that*

$$\ell c_1 > \ell(\ell - 1) + (n - \ell)c_2. \tag{61}$$

Proof of the Lemma. Write $\mathrm{LEG}(n, \Sigma, c_1, c_2) = (d_1, \ldots, d_n) = ((c_1)_k, a, (c_2)_{n-2})$, where $c_2 \leq a \leq c_1$.

By the Tripathi-Vijay Theorem 3.2, either EG_k or EG_{k+1} fails. By Proposition 2.2 property (EG_ℓ) holds for each $\ell \leq c_2$ or $\ell > c_1$. So we can assume that $c_2 \leq k < c_1$.
Case 1: If (EG_k) fails, then

$$\sum_{i=1}^{k} d_i > k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k), \tag{62}$$

and so

$$kc_1 > k(k-1) + \min(a, k) + (n - k - 1)c_2, \tag{63}$$

therefore

$$kc_1 > k(k-1) + (n-k)c_2. \tag{64}$$

Case 2: If (EG_{k+1}) fails, then

$$\sum_{i=1}^{k+1} d_i > (k+1)k + \sum_{i=k+2}^{n} \min(d_i, k), \tag{65}$$

and so

$$kc_1 + a > (k+1)k + (n-k-2)c_2, \tag{66}$$

therefore

$$(k+1)c_1 > (k+1)k + (n-k-1)c_2. \tag{67}$$

So either $\ell = k$ or $\ell = k+1$ has the following property: $c_2 \le \ell \le c_1$ and

$$\ell c_1 > \ell(\ell-1) + (n-\ell)c_2. \tag{68}$$

So we proved the Lemma. □

Let

$$\sigma = (n-\ell)c_2, \qquad c = \lfloor \sigma/\ell \rfloor, \qquad \alpha = \sigma - \ell c. \tag{69}$$

Then

$$(n-\ell)c_2 = \sigma = \alpha(c+1) + (\ell-\alpha)c.$$

Consider the following degree sequence

$$D = (\ell+c)_\alpha (\ell+c-1)_{\ell-\alpha} (c_2)_{n-\ell} \tag{70}$$

Lemma 5.5. *The previous degree sequence is $D \in \mathbb{D}(n, c_1, c_2)$, and it is a graphic split sequence.*

Proof. First observe that D is graphic. Really, it has a realization $G = \langle V, E \rangle$ on the vertex set $V = \{v_j : j < \ell\} \cup \{w_k : k < n-\ell\}$ with

$$E = \big\{(v_i, v_j) : i < j < \ell\big\} \cup \big\{(v_{i \bmod \ell}, w_{i \bmod (n-\ell)}) : i < \sigma\big\}. \tag{71}$$

Next observe that ℓ is the largest j such that $d_j \ge j-1$. Moreover,

$$\sum_{i=1}^{\ell} d_i = (\ell+c)\alpha + (\ell+c-1)(\ell-\alpha) = \sigma + \ell(\ell-1)$$

$$= \ell(\ell-1) + c_2(n-\ell) = \ell(\ell-1) + \sum_{i=\ell+1}^{n} d_i.$$

Thus the degree sequence D is a split degree sequence by Theorem 5.1. □

This completes the proof of Theorem 5.3. □

5.2. Tyshkevich product

Definition 5.6 *(Tyshkevich [16]).* Let $G = (\langle U, W\rangle; E)$ be a split graph and $H = (V, F)$ be an arbitrary graph. We define the *composition* graph $K = G \circ H$ as follows: K consists of a copy of G, and a copy of H and of all the possible new edges (u, v) where $u \in U, v \in V$. More formally,

$$V(K) = U \cup W \cup V \text{ and } E(K) = E \cup F \cup \{(u, v) : u \in U, v \in V\}.$$

Observe that the first operand in this operation is always a split graph.
Barrus [6, Theorem 3.5] proved the following (see also [1, Theorem 6]):

Theorem 5.7 *(Barrus). Assume that $G = (\langle U, W\rangle; E)$ is a split graph and $H = (V, F)$ is an arbitrary graph. Let $K = G \circ H$. Then*

$$|\mathcal{G}(\mathbf{d}(K))| = |\mathcal{G}(\mathbf{d}(G))| \cdot |\mathcal{G}(\mathbf{d}(H))|. \tag{72}$$

5.3. How to obtain "not P-stable" from "not almost fully graphic"?

Theorem 5.8. *Assume that*

$$\mathbb{D} = \bigcup_{k\in\mathbb{N}} \mathbb{D}(n_k, c_{k,1}, c_{k,2}) \text{ and } \mathbb{D}' = \bigcup_{k\in\mathbb{N}} \mathbb{D}(n'_k, c'_{k,1}, c'_{k,2})$$

are very simple degree sequence regions. If $\mathbb{D}$ is not almost fully graphic, then $\mathbb{D}'$ is not P-stable provided:

(1) $\lim_{k\to\infty}(n'_k - n_k)/\ln(n'_k) = +\infty$,
(2) $c'_{k,2} \le c_{k,2}$,
(3) $c'_{k,1} \ge c_{k,1} + (n'_k - n_k)$.

Proof. We will use a construction, which is similar to the one Jerrum, McKay and Sinclair derived in [11, Lemma 8.1], and based on the result [4, Corollary 6.2].

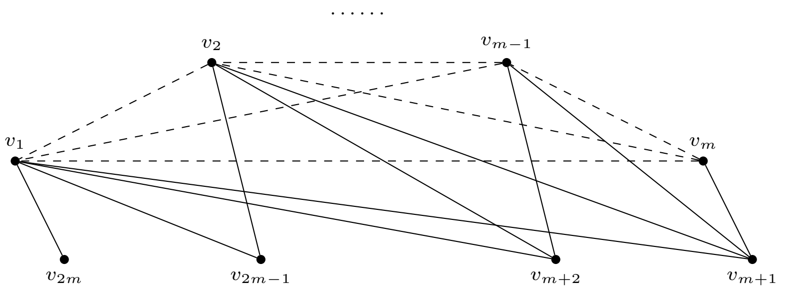

Fig. 1. The unique realization of $\mathbf{h}_m$.

Lemma 5.9. *For each natural number $m \geq 1$ the following sequence*

$$\mathbf{h}_m = (2m-1, 2m-2, \ldots, m+1, m, m, m-1, \ldots, 2, 1), \tag{73}$$

has exactly one realization $H_m = (V, E)$ on the vertex set $V = \{v_1, \ldots, v_{2m}\}$, namely

$$E = \{(i,j) : m+1 \leq i < j \leq 2m\} \cup \{(i,j) : 1 \leq i \leq m < j \leq 2m, i+j \leq 2m+1\}.$$

However the sequence

$$\mathbf{h}'_m = \mathbf{h}_m + 1^{+2m}_{+m} = (2m-1, 2m-2, \ldots, m+1, \mathbf{m+1}, m-1, m-2, \ldots, 2, \mathbf{2}) \tag{74}$$

has at least $\Theta(e^{\delta m})$ realizations for some $\delta > 0$. (See Fig. 1.)

Proof. It is easy to see that the degree sequence $\mathbf{h}_m$ has exactly one realization (or see, for example, [11, Lemma 8.1]).

The calculation concerning the number of realizations of $\mathbf{h}'_m$ follows from [4, Corollary 6.2], which claims that the bipartite degree sequence $((m, m-1, \ldots, 2, 2), ((m, m-1, \ldots, 2, 2))$ has

$$\Theta\left(\left(\frac{3+\sqrt{5}}{2}\right)^m\right)$$

realizations. □

Let $I = \{k \in \mathbb{N} : \mathbb{D}(n_k, c_{k,1}, c_{k,2})$ contain a non-Erdős-Gallai sequence$\}$. By the assumption of Theorem, the set I is infinite. Replacing $\mathbb{D}$ with $\bigcup\{\mathbb{D}(n_k, c_{k,1}, c_{k,2}) : n \in I\}$, we can assume that every $\mathbb{D}(n_k, c_{k,1}, c_{k,2})$ contains a non-Erdős-Gallai sequence. Therefore, by Theorem 5.3, every $\mathbb{D}(n_k, c_{k,1}, c_{k,2})$ contains a split degree sequence D_k. Let G_k be the unique realization of D_k. Furthermore, for each k let $H^*_k = H_{n'_k - n_k}$ be the unique realization of the graphic degree sequence $\mathbf{h}_{n'_k - n_k}$.

Let $\mathbf{e}_k$ be the degree sequence of the Tyshkevich product $G_k \circ H_k$. Clearly $\mathbf{e}_k \in \mathbb{D}(m'_k, c'_{k,2}, c'_{k1})$ by the construction. Then, by Theorem 5.7, $\mathbf{e}_k$ has exactly one realization because both $\mathbf{d}_k$ and $\mathbf{h}_{n'_k - n_k}$ have exactly one realization.

However, for some i and j, the sequence $\mathbf{e}_k + 1^{+i}_{+j}$ has at least

$$C \cdot e^{\delta(n'_k - n_k)}$$

realizations by Theorem 5.7 and by the second part of the Lemma 5.9.

Let p be an arbitrary polynomial. Then

$$\lim_{k\to\infty} \ln\left(\frac{C \cdot e^{\delta(n'_k - n_k)}}{p(n'_k)}\right) \geq \lim_{k\to\infty} \left(C' \cdot (n'_k - n_k) - C'' \cdot \ln(n'_k)\right) = \infty,$$

by assumption (1) of this Theorem. Thus, the ratio of the number of realizations of $\mathbf{e}_k$ and $p(k)$ tends to infinity. So $\mathbb{D}'$ is not P-stable. □

Theorem 5.10. *Assume that $f, g, h \in \mathbb{N}^{\mathbb{N}}$ are increasing functions. If the cone region $\mathbb{D}(f, g)$ is P-stable, then $\mathbb{D}(f + h, g - h)$ is almost fully graphic provided*

(1) $f(n + k) \leq f(n) + k$ *for each* $n, k \in \mathbb{N}$,
(2) $lim_{n\to\infty} h(n)/\ln(n) = \infty$.

Proof of Theorem 5.10. Assume on the contrary that $\mathbb{D}(f + h, g - h)$ is not almost fully graphic. Let

(i) $n_k = k$, $c_{k,1} = g(n_k) - h(n_k)$, $c_{k,2} = f(n_k) + h(n_k)$,
(ii) $n'_k = n_k + h(n_k)$, $c'_{k,1} = g(n'_k)$ $c'_{k,2} = f(n'_k)$.

The assumption Theorem 5.10 (2) implies that 5.8(1) holds. The assumption Theorem 5.10 (1) implies that 5.8(2) holds. Finally Theorem 5.10 (1) also implies that $g(n_{k,1}) \geq c_{k,1} + h(n_k)$, and $g(n'_{k,1}) \geq g(n_{k,1})$ because g is monotone. So 5.8(3) holds.

Hence, we can apply Theorem 5.8 to obtain that $\mathbb{D}(f, g)$ is not P-stable. □

Corollary 5.11. *Assume that $0 \leq \varepsilon_2 < \varepsilon'_2 < \varepsilon'_1 < \varepsilon_1 \leq 1$. If the very simple region $\mathbb{D} := \bigcup_{n\in\mathbb{N}} \mathbb{D}(n, \lfloor\varepsilon_1 \cdot n\rfloor, \lceil\varepsilon_2 \cdot n\rceil)$ is P-stable, then the region $\mathbb{D}' = \bigcup_{n\in\mathbb{N}} \mathbb{D}(n, \lfloor\varepsilon'_1 \cdot n\rfloor, \lceil\varepsilon'_2 \cdot n\rceil)$ is almost fully graphic.*

6. A common bound of the growing rate in P-stable regions

In the literature, various P-stable families of degree sequences are described. It is easy to check that in these cases the polynomial $p_0(n) = n^{10}$ has the following property: if $\mathbb{D}$ is P-stable, then $\partial(D) \leq p_0(|D|)$ for all but finitely many $D \in \mathbb{D}$. This observation is notable and leads to the following bold conjecture.

Conjecture 6.1. *There is a polynomial $p^*(n)$ such that for each P-stable family $\mathbb{D}$ of degree sequences (or, just for each P-stable simple region $\mathbb{D}$)*

$$\partial(D) \le p^*(|D|)$$

for all but finitely many graphic $D \in \mathbb{D}$.

Appendix A. The different definitions of P-stability are really equivalent

Given a degree sequence D of length n, define the following families of degree sequences:

$$\mathcal{D}^{-+} = \left\{D + 1^{-i}_{+j} : 1 \le i \ne j \le n\right\},$$
$$\mathcal{D}^{++} = \left\{D + 1^{+i}_{+j} : 1 \le i \ne j \le n\right\},$$
$$\mathcal{D}^{+2} = \left\{D + 1^{+i}_{+i} : 1 \le i \le n\right\}.$$

The families $\mathcal{D}^{--}$ and $\mathcal{D}^{-2}$ are defined analogously. We will use the notation

$$\mathbb{G}(\mathcal{D}^{-+}) = \bigcup_{1 \le i \ne j \le n} \mathcal{G}(D^{-i}_{+j})$$

and the analogous notations for the other cases.

In 1989 Jerrum and Sinclair introduced the so-called *Jerrum-Sinclair Markov Chain* (JSMC) in their seminal paper [12] on the approximation of the zero-one permanents. Later they used the same Markov chain to sample certain graph realization classes on labelled vertices in [13]. They introduced there the notion of P-stability: a family $\mathcal{D}$ of degree sequences is **P-stable** if and only if there is a polynomial $p(n)$ such that for each graphic sequence $D \in \mathcal{D}$ with length n we have

$$|\mathbb{G}(\mathcal{D}^{--}) \cup \mathbb{G}(\mathcal{D}^{-2})| \le p(n) \cdot |\mathcal{G}(D)|. \tag{JS}$$

In 1992 Jerrum, McKay and Sinclair gave more results about P-stable degree sequences ([11, Subsection 8.2]). However they used there a different definition. They say that a family $\mathcal{D}$ of degree sequences is *P-stable* if and only if there is a polynomial $p_1(n)$ such that for each graphic sequence $D \in \mathcal{D}$ with length n we have

$$|\mathbb{G}(\mathcal{D}^{-+})| \le p_1(n) \cdot |\mathcal{G}(D)|. \tag{JMS}$$

The authors made the remark (without proof) that, while the two definitions formally are different, they are equivalent.

There has been studied another Markov chain based approach to sample graph realizations for at least three decades, the chain is called *switch Markov Chain*. In 2022, the rapid mixing of the switch Markov chain was proven on P-stable degree sequences, encompassing all simple, bipartite, and directed degree sequences ([5]), providing the currently available strongest result. However, that paper presented a third definition for P-stability [5, Definition 1.2]. They say that a family $\mathcal{D}$ of degree sequences is *P-stable* if and only if there is a polynomial $p_2(n)$ such that for each graphic sequence $D \in \mathcal{D}$ with length n we have

$$|\mathbb{G}(\mathcal{D}^{++})| \leq p_2(n) \cdot |\mathcal{G}(D)|. \tag{EGMMSS}$$

The paper states (again, without proof) that this definition is equivalent to the former ones.

The following theorem yields immediately that the three definitions of P-stability are indeed equivalent.

Theorem 6.2. *Assume that $D = (d_1, \ldots, d_n)$ is a graphic degree sequence.*

(a) $\max\left(|\mathbb{G}(\mathcal{D}^{++})|, |\mathbb{G}(\mathcal{D}^{--})|\right) \leq n^2 \cdot \left(|\mathbb{G}(\mathcal{D}^{+-})| + |\mathcal{G}(D)|\right)$,
(b) $\max\left(|\mathbb{G}(\mathcal{D}^{+2})|, |\mathbb{G}(\mathcal{D}^{-2})|\right) \leq n^2 \cdot |\mathbb{G}(\mathcal{D}^{+-})|$,
(c) $|\mathbb{G}(\mathcal{D}^{+-})| \leq (n^4 + n^2) \cdot \min\left(|\mathbb{G}(\mathcal{D}^{++})|, |\mathbb{G}(\mathcal{D}^{--})|\right)$.

Remark. The proofs for (a) and (b) are straightforward. However, proving (c) presents a greater challenge.

Proof. We assume that the vertex set of the realizations is $\{v_1, \ldots, v_n\}$.
(a) Assume that $G \in \mathcal{G}(D + 1^{+i}_{+j}) \subset \mathbb{G}(\mathcal{D}^{++})$ for some $1 \leq i \neq j \leq n$. We will define a graph $G' \in \mathbb{G}(\mathcal{D}^{+-}) \cup \mathcal{G}(D)$ such that the symmetric difference of $E(G)$ and $E(G')$ is one edge.

If $(v_i, v_j) \in E(G)$, then let $G' = G - (v_i, v_j)$. Then $\mathbf{d}(G') = \mathbf{d}(G) + 1^{-i}_{-j} = D$, so G' satisfies the requirements.

Assume that $(v_i, v_j) \notin E(G)$. Since $\deg_G(v_i) = d_i + 1 \geq 1$, there is a k such that $v_i v_k$ is an edge in G. Since (v_i, v_j) is a non-edge, $k \neq j$. Let $G' = G - v_i v_k$. Then $\mathbf{d}(G') = \mathbf{d}(G) + 1^{-i}_{-k} = D + 1^{+i}_{+j} + 1^{-i}_{-k} = D^{+j}_{-k}$, so G' satisfies the requirements.

From G you can get back G' provided you know the symmetric difference of $E(G)$ and $E(G')$, which is just one pair of vertices. Since there are less than n^2 many pairs, for any $H \in \mathbb{G}(\mathcal{D}^{+-}) \cup \mathcal{G}(D)$ there are less than n^2 many $G \in \mathbb{G}(\mathcal{D}^{++})$ such that $G' = H$. So we proved $|\mathbb{G}(\mathcal{D}^{++})| \leq n^2 \cdot \left(|\mathbb{G}(\mathcal{D}^{+-})| + |\mathcal{G}(D)|\right)$.

The inequality $|\mathbb{G}(\mathcal{D}^{--})|\big) \leq n^2 \cdot \left(|\mathbb{G}(\mathcal{D}^{+-})| + |\mathcal{G}(D)|\right)$ can be proved analogously
(b) Assume that $G \in \mathcal{G}(D + 1^{+i}_{+i}) \subset \mathbb{G}(\mathcal{D}^{+2})$ for some $1 \leq i \leq n$.

We will define a graph $G' \in \mathbb{G}(\mathcal{D}^{+-})$ such that the symmetric difference of $E(G)$ and $E(G')$ is one edge. Since $\deg_G(v_i) = d_i + 2 > 0$, there is j such that $(v_i, v_j) \in E(G)$. Then $G' = G - (v_i, v_j)$ meets the requirements because $\mathbf{d}(G') = \mathbf{d}(G) + 1^{-i}_{-j} = D + 1^{+i}_{+i} + 1^{-i}_{-j} = D + 1^{+i}_{-j}$.

From G you can get back G' provided you know the symmetric difference of $E(G)$ and $E(G')$, which is just one pair of vertices. Since there are less than n^2 many pairs, for any $H \in \mathbb{G}(\mathcal{D}^{+-})$ there are less than n^2 many $G \in \mathbb{G}(\mathcal{D}^{+2})$ such that $G' = H$. So we proved $|\mathbb{G}(\mathcal{D}^{+2})| \leq n^2 \cdot |\mathbb{G}(\mathcal{D}^{+-})|$.

The inequality $|\mathbb{G}(\mathcal{D}^{-2})| \leq n^2 \cdot |\mathbb{G}(\mathcal{D}^{+-})|$ can be proved analogously.

(c) For each $G \in \mathbb{G}(\mathcal{D}^{+-})$ we will find a $G' \in \mathbb{G}(\mathcal{D}^{++})$ such that the symmetric difference of $E(G)$ and $E(G')$ is either an edge, or a path of length 3. Fix i, j such that $G \in \mathcal{G}(D + 1_{-i}^{+j})$. If there is $k \neq i, j$ such that $v_i v_k$ is not an edge, then let

$$G' = G + (v_i, v_j).$$

Then $\mathbf{d}(G') = 1_{+k}^{+j}$, so G' satisfies the requirements.

So we can assume that $v_i v_k \in E(G)$ for $k \neq i, j$. Since the $\deg_G(v_i) = d_i - 1 \leq n - 2$, we obtain that $(v_i, v_j) \notin E(G)$. Let $X = \Gamma_G(v_j)$ and write $d = |X|$. Since $\deg_G(v_j) = d_j + 1 > 0$, we have $d \geq 1$.

Claim. *There is $v_k \in X$ such that $(v_k, v_\ell) \notin E(G)$ for some $\ell \neq k$.*

Proof of the claim. Assume on the contrary that $(v_k, v_\ell) \in E(G)$ for each $v_k \in X$ and $\ell \neq k$. Thus $\deg_G(v_k) = n - 1$ for each $v_k \in X$. Since $v_i, v_j \notin X$, it follows that $d_k = \deg_G(v_k) = n - 1$ for each $v_k \in X$.

Let H be a realization of the graphic sequence D on the vertex set $\{v_1, \ldots, v_n\}$. Since $\deg_H(v_k) = d_k = n - 1$ for each $v_k \in X$, and $v_j \notin X$, it follows that $d_j = \deg_H(v_j) \geq |X| = d$. But $|X| = \deg_G(v_j) = d_j + 1$, so $d_j < |X|$. Contradiction, we proved the Claim. □

By the Claim, we can fix $v_k \in X$ such that $(v_k, v_\ell) \notin E(G)$ for some $\ell \neq k$. Then $\ell \neq i, j$ since (v_i, v_k) and (v_j, v_k) are edges in G. Let

$$G' = G + (v_i, v_j) - (v_j, v_k) + (v_k, v_\ell).$$

Then $\mathbf{d}(G') = D + 1_{+\ell}^{+j}$. So G' satisfies the requirements. Thus we could always define G'.

Since there are less than n^2 edges and less than n^4 many paths of length 3, for any $H \in \mathbb{G}(\mathcal{D}^{++})$ there are less than $n^2 + n^4$ many $G \in \mathbb{G}(\mathcal{D}^{+-})$ such that $G'H$. So we proved $|\mathbb{G}(\mathcal{D}^{+-})| \leq (n^4 + n^2) \cdot |\mathbb{G}(\mathcal{D}^{++})|$.

The inequality $|\mathbb{G}(\mathcal{D}^{+-})| \leq (n^4 + n^2) \cdot |\mathbb{G}(\mathcal{D}^{--})|$ can be proved analogously. □

Remark. If D is the constant 0 sequence of length n, then $|\mathbb{G}(\mathcal{D}^{++})| > 0$, but $|\mathbb{G}(\mathcal{D}^{+-})| = 0$ because $\mathcal{D}^{+-} = \emptyset$, so in 6.2.(a), we can not omit $|\mathcal{G}(D)|$ from the RHS of the inequality.

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