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Geometric interpretation of efficient weight vectors

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ABSTRACT

Pairwise comparison matrices (PCMs) are frequently used in different multicriteria decision making problems. A weight vector is said to be efficient if no other weight vector is at least as good in estimating the elements of the PCM, and strictly better in at least one position. Understanding the efficient weight vectors is crucial to determine the appropriate weight calculation technique for a given problem. In this paper we study the set of efficient weight vectors for three and four dimensions (alternatives) from a geometric viewpoint, which is a complementary to the algebraic approach used in the literature. Besides providing well-interpretable demonstrations, we also draw attention to the particular role of weight vectors calculated from spanning trees. Weight vectors corresponding to line graphs are vertices of the (polyhedral, but usually nonconvex) set of efficient weight vectors, while weight vectors corresponding to other spanning trees are also on the boundary.

1. Introduction

Pairwise comparisons are popular in many different fields, such as decision making, preference modelling, psychometry, and sports [1–3].

One of the most extensively studied multicriteria decision making methods, the Analytic Hierarchy Process (AHP) [4,5], also applies pairwise comparisons to evaluate the alternatives according to a criterion as well as to determine the importance of the different criteria.

Usually it is easier for the decision maker to determine how many times a given alternative is better compared to another one than to provide direct priorities (weights) of the alternatives (criteria). In the AHP these pairwise comparisons are placed into a matrix, that is a pairwise comparison matrix (PCM).

There have been many weight (priority) calculation techniques proposed in the literature to derive the weights (priorities) from a PCM, e.g., the eigenvector method originally suggested by Saaty [4], the least squares method [6,7], the logarithmic least squares (geometric mean) method [8], or the spanning tree technique [9–13]. For further methods and their comparisons, see, for instance [14].

In the case of a real-world PCM, usually there are some contradictions, inconsistencies among the elements that can be measured several ways (see, for instance, Brunelli [15], Kubler et al. [16], Kułakowski and Talaga [17] and Mazurek [18]). If there is not any contradictions, then we are dealing with a consistent matrix, and in that case we usually expect the weight calculation techniques to provide the same vector [19, Axiom 1]. However, if the matrix is inconsistent, then the different methods can provide different weight vectors [20], and several solutions can be optimal in some sense.

The (Pareto-)efficiency of a w weight vector informally means that there is no weight vector that estimates the elements of the PCM as good as w in every component and even better in at least one element. The efficiency of weight vectors has not been extensively studied in the literature, however, several new findings has been made recently.

Blanquero et al. [21] provided a large set of tools to deal with this problem. They proved that the logarithmic least squares (geometric mean) method provides an efficient weight vector, while the eigenvector method can be inefficient, and developed LP models to test whether a weight vector is efficient.

Conde and de la Paz Rivera Pérez [22] also used linear optimization problems to derive efficient weight vectors. Based on that, they defined a consistency index and a weakly efficient weight vector associated to it.

Bozóki [23] has shown that an arbitrarily small inconsistency of a pairwise comparison matrix can still lead to the inefficiency of the weight vector calculated from it with the eigenvector method.

According to Ábele-Nagy and Bozóki [24] the principal right eigenvector of a PCM is efficient if the matrix is a simple perturbed one, i.e., it only differs from a consistent PCM by one element (and its

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reciprocal). This result has been extended to double perturbed PCMs, and for a special class of triple perturbed matrices, which differ from a consistent PCM by two and three elements (and their reciprocals) by Ábele-Nagy et al. [25] and Fernandes and Furtado [26], respectively. Furtado and Johnson [27] also examined the efficiency of the eigenvector method for so-called block perturbed PCMs.

Bozóki and Fülöp [28] proved that the eigenvector method is weakly efficient, while also developed LP models and provided algorithms to improve an inefficient weight vector.

The efficiency of weight vectors calculated from empirical PCMs of a real-world decision problem has been examined by Duleba and Moslem [29].

In [30] the inconsistent PCMs have been approximated by a consistent one via distance minimization, and an iterative algorithm has been provided that results in a unique, efficient weight vector.

da Cruz et al. [31] and Furtado [32] gave algebraic descriptions of all efficient vectors for simple and double perturbed PCMs, respectively.

Besides proving that the geometric mean of any collection of distinct columns of a PCM is an efficient weight vector, Furtado and Johnson [33] also determined necessary and sufficient conditions for an efficient vector for an $n \times n$ matrix A to be an extension of an efficient vector for an $(n-1)\times(n-1)$ principal submatrix of A. This was used by Furtado and Johnson [34] to provide an algorithm that determines inductively the complete set of efficient weight vectors for a reciprocal PCM.

In this paper, we provide the geometric interpretation of the sets of efficient weight vectors for three and four alternatives—that was so far missing in the literature—as well as analyse which points determine them. Our geometric demonstrations are complementary to the algebraic view [35], and hopefully help understanding better the somehow puzzling set of efficient weight vectors. We also draw attention to the particular role of weight vectors calculated from spanning trees. Weight vectors corresponding to line graphs are vertices of the (polyhedral, but usually nonconvex) set of efficient weight vectors, while weight vectors corresponding to 3-stars are also on the boundary.

The rest of the paper is organized as follows. Section 2 introduces the key definitions and former results used in the paper connected to pairwise comparison matrices and efficient weight vectors. Section 3 provides the main results, the geometric interpretation of efficient weight vectors for three (Section 3.1) and four (Section 3.2) alternatives. Finally, Section 4 concludes and provides further research questions.

2. Pairwise comparison matrices and efficient weight vectors

Let us denote the number of criteria (alternatives, voting powers, etc.) by n.

Definition 1 (*Pairwise Comparison Matrix (PCM)*)**.** The $n \times n$ matrix $A = [a_{ij}]$ is called a pairwise comparison matrix if it is positive ($a_{ij} > 0$ for all i and j) and reciprocal ($1/a_{ij} = a_{ji}$ for all i and j).

The judgements of decision makers usually contain a certain level of inconsistency.

Definition 2 (*Consistent PCM*)**.** A PCM is said to be consistent if $a_{ik} = a_{ij}a_{jk}$ for all i, j, k. If a PCM is not consistent, then it is called inconsistent.

Example 1 (*(In)Consistent PCM*)**.** An illustrative example for an inconsistent PCM for three alternatives can be seen below:

Inconsistent PCM **Consistent PCM**

$$\begin{bmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{bmatrix}$$

One can see in Example 1 that the only triad (i.e., the set of all pairwise comparisons of three given alternatives) of the original matrix is inconsistent, but it can be made consistent by modifying one element.

Bozóki et al. [36] studied PCMs that can be made consistent by modifying at most 3 elements. In other problems [37,38] also used the main idea of comparing two PCMs that differ in only one element.

Definition 3 (*Simple, Double and Triple Perturbed PCMs*)**.** A PCM is said to be

- simple perturbed if it can be made consistent by altering only one element (and its reciprocal);
- double perturbed if it can be made consistent by altering only two elements (and their reciprocals);
- triple perturbed if it can be made consistent by altering only three elements (and their reciprocals).

Example 2 (*Simple, Double and Triple Perturbed PCMs*)**.** An illustrative example for simple, double and triple perturbed PCMs for four alternatives—which will be used later on—can be seen in Box I.

These kinds of matrices are also important for real-world decision problems and quite often appear in empirical matrices, especially among smaller ones [25,36].

In order to use an $n \times n$ PCM A to evaluate alternatives or to compare criteria, the determination of a w component-wise positive weight vector is necessary. The most commonly used weight calculation techniques are the eigenvector method and the logarithmic least squares (geometric mean) method.

Definition 4 (*Logarithmic Least Squares Method (LLSM)*)**.** Let A be an $n \times n$ PCM. The weight vector w of A determined by the LLSM is the optimal solution of the following problem:

$$\min_{w} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln(a_{ij}) - \ln\left(\frac{w_i}{w_j}\right) \right)^2, \tag{1}$$

where w_i is the ith coordinate of w.

Definition 5 (*Eigenvector Method*)**.** Let A be an $n \times n$ PCM. The weight vector w of A determined by the eigenvector method is defined as follows:

$$A \cdot w = \lambda_{\max} \cdot w, \tag{2}$$

where λ_{max} is the principal eigenvalue of matrix A.

The solution vectors of these methods are only unique up to a scalar multiplication, thus, the sum of the components (the weights) are usually normalized to one ($\sum_{i=1}^{n} w_i = 1$).

There can be many cases, when some elements of a PCM are missing, which can happen because of the loss of data, the comparisons are simply not possible (for instance in sports [39]), or the decision maker has no time or willingness to provide them [40]. A PCM with some missing entries is called an incomplete pairwise comparison matrix.

Definition 6 (*Incomplete Pairwise Comparison Matrix (IPCM)*)**.** An $n \times n$ matrix $A = [a_{ij}]$ is an incomplete pairwise comparison matrix (IPCM) if:

- $a_{ij} \in \mathbb{R}_+ \cup \{*\}\ \forall\ 1 \le i, j \le n$ and
  - $a_{ji} = 1/a_{ij}$ if $a_{ij} \in \mathbb{R}_+$,
  - $a_{ji} = *$ if $a_{ij} = *$,

where $*$ denotes the missing elements, and $\mathbb{R}_+$ is the set of positive real numbers.

The analysis of IPCMs can be handled suitably with the help of their graph representation.

$$\begin{array}{cccc} \textbf{Triple} & \textbf{Double} & \textbf{Simple} & \textbf{Consistent} \\ \begin{bmatrix} 1 & 2 & 7 & 5 \\ \frac{1}{2} & 1 & 3 & 8 \\ \frac{1}{7} & \frac{1}{3} & 1 & 4 \\ \frac{1}{5} & \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \Rightarrow & \begin{bmatrix} 1 & 2 & 6 & 5 \\ \frac{1}{2} & 1 & 3 & 8 \\ \frac{1}{6} & \frac{1}{3} & 1 & 4 \\ \frac{1}{5} & \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \Rightarrow & \begin{bmatrix} 1 & 2 & 6 & 5 \\ \frac{1}{2} & 1 & 3 & 12 \\ \frac{1}{6} & \frac{1}{3} & 1 & 4 \\ \frac{1}{5} & \frac{1}{12} & \frac{1}{4} & 1 \end{bmatrix} \Rightarrow & \begin{bmatrix} 1 & 2 & 6 & 24 \\ \frac{1}{2} & 1 & 3 & 12 \\ \frac{1}{6} & \frac{1}{3} & 1 & 4 \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{4} & 1 \end{bmatrix} \end{array}$$

Box I.

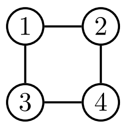

Fig. 1. The representing graph of the IPCM A of Example 3.

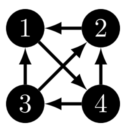

Fig. 2. The BCC directed graph of the PCM A with the weight vector w of Example 4.

Definition 7 (*Representing Graph*)**.** An incomplete pairwise comparison matrix A can be represented by an undirected graph $G = (V, E)$, where:

- the vertices $V = \{1, 2, \dots, n\}$ correspond to the alternatives,
- while the edge set E represents the known elements of A outside the main diagonal:

$$e_{ij} \in E \iff a_{ij} \neq * \text{ and } i \neq j.$$

Example 3 (*Representing Graph*)**.** Let A be the following incomplete pairwise comparison matrix:

$$\begin{bmatrix} 1 & 2 & 6 & * \\ \frac{1}{2} & 1 & * & 12 \\ \frac{1}{6} & * & 1 & 4 \\ * & \frac{1}{12} & \frac{1}{4} & 1 \end{bmatrix}$$

The G representing graph of A can be seen in Fig. 1.

As one can see, the representing graph does not depend on the exact values of the pairwise comparisons, but the fact that whether a given comparison is available or missing. There is an edge between two alternatives (vertices) if and only if the appropriate comparison is known (i.e., there is no comparison between alternatives 1 and 4, and 2 and 3).

There are weight calculation techniques for the incomplete case as well, mainly relying on the completion of the IPCM (for a comparative study see, for instance, Tekile et al. [41]). However, the most often used methods only provide a unique weight vector if the representing graph is connected [42], i.e., there has to be at least an indirect comparison between any two items.

Definition 8 (*Connected Graph*)**.** In an undirected graph, two vertices u and v are called connected if the graph contains a path from u to v. A graph is said to be connected if every pair of vertices in the graph is connected.

Spanning trees have a special role in connection with pairwise comparisons, as they provide the smallest connected system of comparisons, which contains $n - 1$ edges for n vertices.

Definition 9 (*Spanning Tree*)**.** Let $G = (V, E)$ be a connected graph. $G' = (V, E')$ is a spanning tree of G if $E' \subseteq E$ is a minimal set of edges that connects all vertices of G.

Remark 1. An IPCM A with representing graph G can always be complemented to get a consistent complete PCM if G is a spanning tree.

From Remark 1 and Example 1 one can also see that a 3×3 matrix is either a simple perturbed or a consistent one, while Example 2 demonstrates that a 4×4 matrix can also be double or triple perturbed.

Definition 10 (*Efficient Weight Vector*)**.** Weight vector $w = (w_1, w_2, \dots, w_n)^T$ is said to be (Pareto-)efficient, if no positive weight vector $w' = (w'_1, w'_2, \dots, w'_n)^T$ exists such that:

- $\forall i, j : \left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right|$, and
- $\exists k, l : \left| a_{kl} - \frac{w'_k}{w'_l} \right| < \left| a_{kl} - \frac{w_k}{w_l} \right|$.

If a weight vector is not (Pareto-)efficient, then it is called inefficient.

It is worth mentioning that for a consistent PCM (regardless of its size) there is only one possible efficient weight vector, as there exists exactly one component-wise positive and normalized weight vector that estimates the elements of the PCM perfectly in this case. Consequently, a weight vector calculated from an IPCM with a spanning tree as its representing graph is also always efficient because of Remark 1. This is the case, because there are no inconsistencies for an IPCM with a spanning tree representing graph, as there are no cycles in the system of comparisons (there are not enough comparisons for that). However, this way the calculated weight vector will estimate the known elements perfectly, thus, it will be efficient.

We heavily rely on the results of Blanquero et al. [21], who showed that one can determine whether a weight vector is efficient for a PCM using an appropriately defined directed graph.

Definition 11 (*Blanquero–Carrizosa–Conde (BCC) Directed Graph*)**.** Let $A = [a_{ij}]$ be an $n \times n$ PCM, and let $w = (w_1, w_2, \dots, w_n)^T$ be a positive weight vector. The Blanquero–Carrizosa–Conde (BCC) directed graph $G = (V, \vec{E})_{A,w}$ is defined as follows:

- the vertices $V = \{1, 2, \dots, n\}$ correspond to the alternatives,
- $\vec{E} = \{arc(i \to j) \mid w_i / w_j \geq a_{ij}, i \neq j\}$.

Theorem 1 ([21], Corollary 10)**.** *Let $A = [a_{ij}]$ be an $n \times n$ PCM. A weight vector w is efficient if and only if $G = (V, \vec{E})_{A,w}$ is a strongly connected directed graph, that is, there exists a directed path from i to j and from j to i for all pairs of nodes i, j.*

Example 4 (*BCC Directed Graph*)**.** Let A be the following pairwise comparison matrix, and w the corresponding weight vector calculated

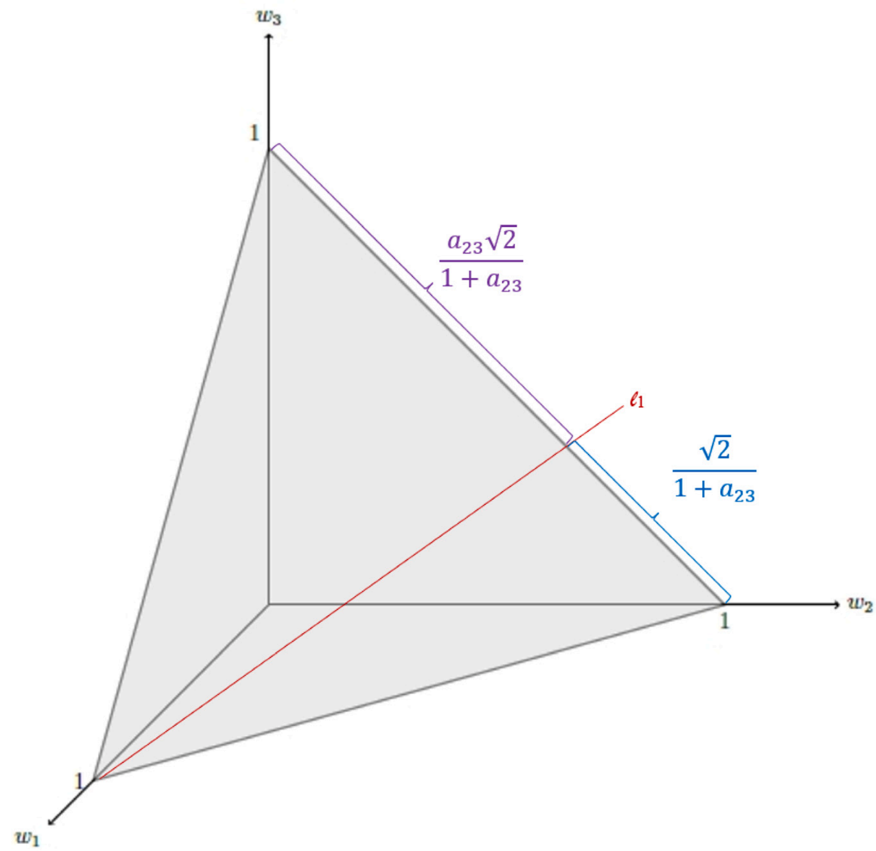

Fig. 3. The unit 3-simplex $\{(w_1, w_2, w_3) \in \mathbb{R}^3_+ \mid w_1 + w_2 + w_3 = 1\}$ for $n = 3$ and line ℓ_1 corresponding to $w_2/w_3 = a_{23}$.

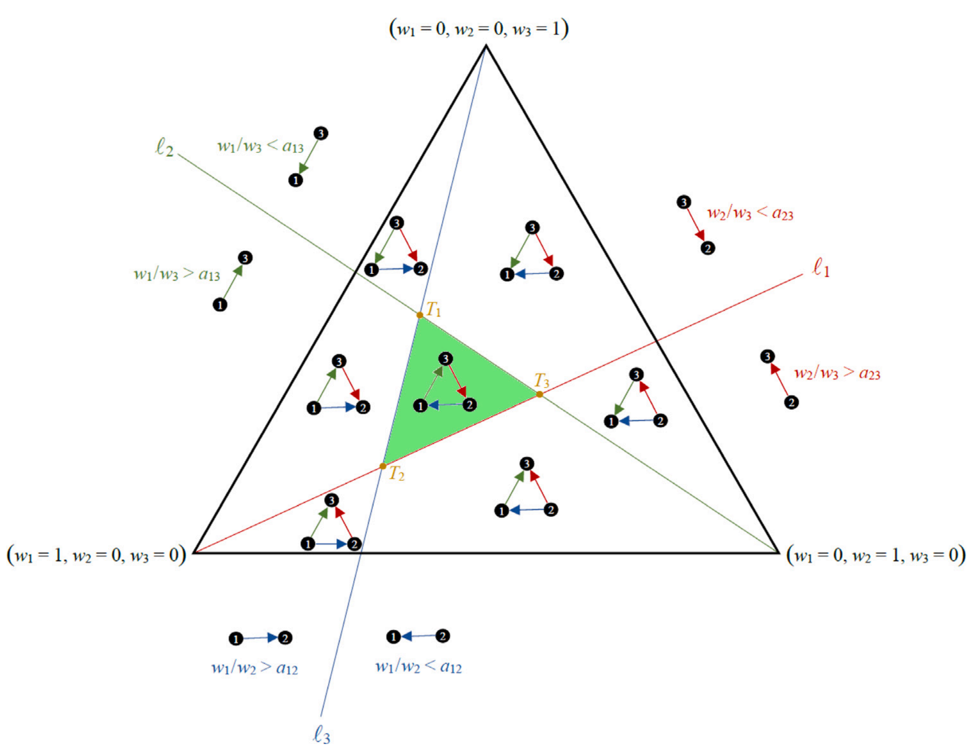

Fig. 4. Efficient weight vectors for $n = 3$ in the case of an inconsistent PCM are highlighted from all the possible weight vectors by green colour.

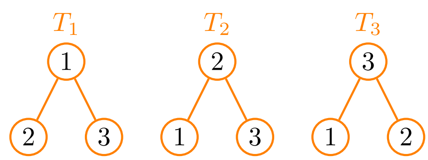

Fig. 5. The spanning trees that determine the efficient weight vectors for $n = 3$ (denoted by orange in Fig. 4).

by the LLSM.

$$\begin{bmatrix} 1 & 2 & 7 & 5 \\ \frac{1}{2} & 1 & 3 & 8 \\ \frac{1}{7} & \frac{1}{3} & 1 & 4 \\ \frac{1}{5} & \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix}$$

$w = (0.5079, 0.3268, 0.1160, 0.0494)^T$

The corresponding G BCC directed graph can be seen in Fig. 2.

It can be seen that G is strongly connected, thus according to Theorem 1, w is an efficient weight vector of PCM A.

Between any pairs of vertices of the BCC digraph, there is an edge in at least one of the two possible orientations. However, there are edges in both possible directions between two vertices only in the case when the comparison between them is estimated perfectly by the given weight vector. We will see that this latter case can only happen at the boundary of the set of efficient weight vectors. However, (Furtado and Johnson [33], Theorem 6) proved that it is a closed set, thus, it is enough to focus on the case when there is an edge between every pair of vertices and it has exactly one orientation, which means that the relevant BCC digraphs are tournaments.

Definition 12 (*Tournament*)**.** A tournament is a directed graph in which every pair of vertices is connected by a directed edge with only one of any one of the two possible orientations.

This also implies that it can be determined whether a weight vector is efficient by focusing on whether its BCC digraph has a Hamiltonian cycle (a cycle that visits each vertex exactly once).

Theorem 2 ([43])**.** *Let G be a tournament. Then, G is strongly connected if and only if G has a Hamiltonian cycle.*

One can see that the BCC directed graph G in Example 4 also has a Hamiltonian cycle along the path of $1 - 4 - 3 - 2 - 1$.

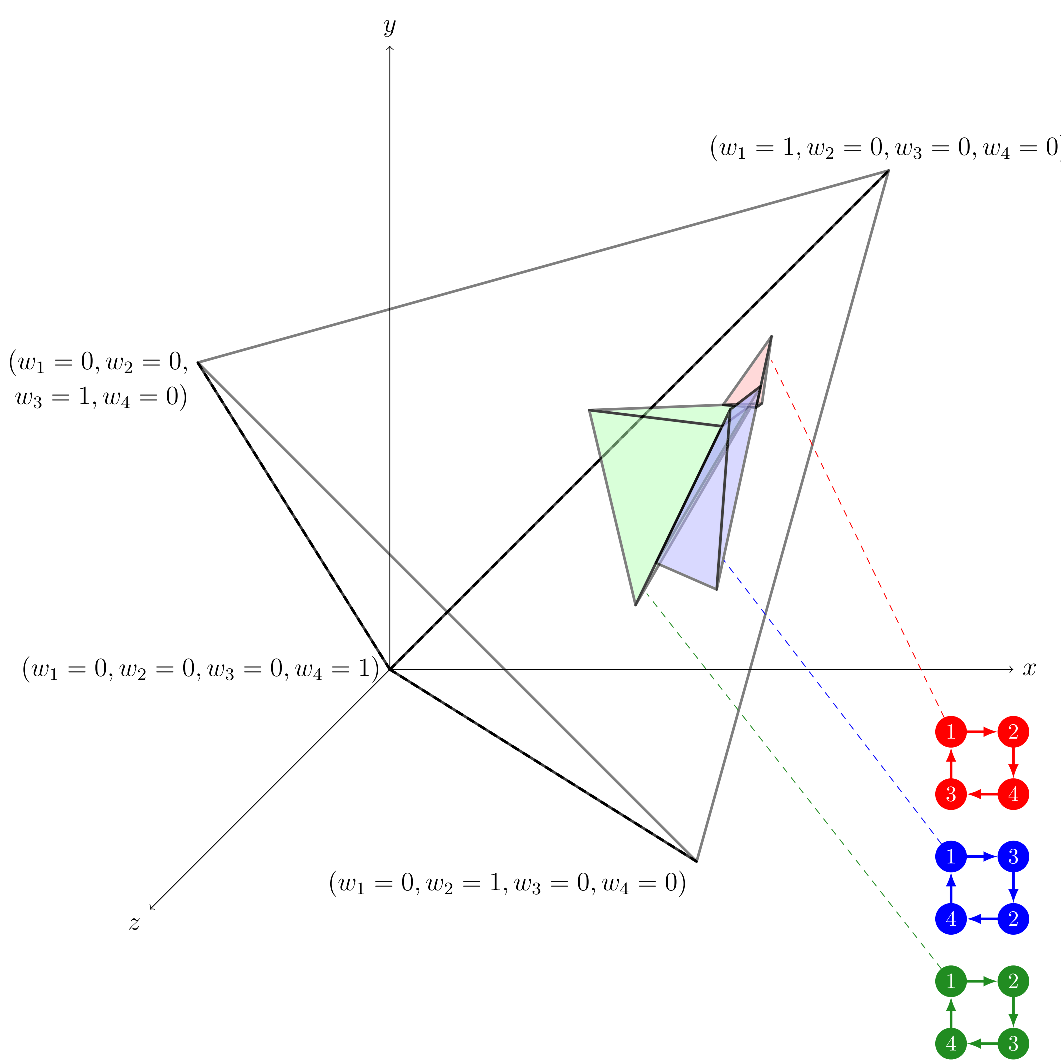

Fig. 6. Efficient weight vectors for the $n = 4$ triple perturbed case.

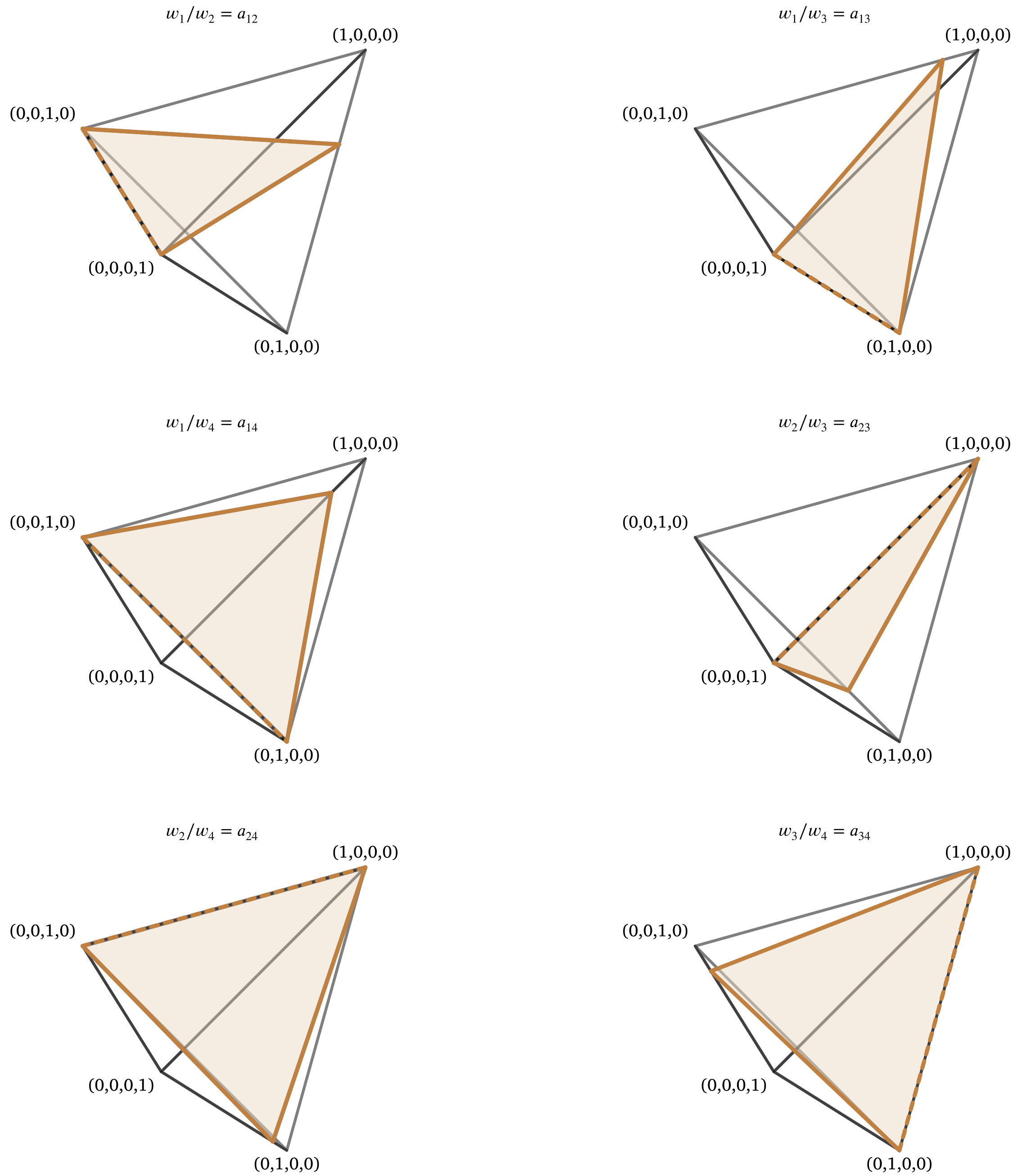

Fig. 7. The planes that determine the polyhedra for the $n = 4$ triple perturbed case. The equation determining the given plane is always placed above the subfigure, e.g., the top left plane is determined by the equation $w_1/w_2 = a_{12}$.

3. Results

3.1. The case of three alternatives

The results of an illustrative example in the case of three alternatives ($n = 3$) can be seen in Fig. 4. As the sum of the components is fixed at 1, the possible weight vectors are in the unit 3-simplex (see Fig. 3) that can be presented in two ($n - 1$) dimensions as a regular triangle (from now on denoted by S_3, see Fig. 4), therefore, we can visualize our findings in a two dimensional figure.

As mentioned above, S_3 (the large triangle in Fig. 4 that is the two dimensional representation of the unit 3-simplex in Fig. 3) contains all the possible weight vectors for three alternatives. The vertices correspond to the cases when two coordinates are zero, while each edge shows the vectors where exactly one component is zero, and the others are positive. Thus, the feasible solutions are inside the triangle, where every element of the vector is positive. Lines ℓ_1, ℓ_2, and ℓ_3 show the set of solutions, in which a given element of the matrix is estimated perfectly, i.e., the given fraction calculated from the gained weight vector is equal to the appropriate element of the PCM:

$$\ell_1 : \quad w_2/w_3 = a_{23},$$

$$\ell_2 : \quad w_1/w_3 = a_{13},$$

$$\ell_3 : \quad w_1/w_2 = a_{12}.$$

These lines determine the polygons of Fig. 4, which correspond to the sets of weight vectors that generate the Blanquero–Carrizosa–Conde directed graphs highlighted by black vertices for each case. The colours of the edges correspond to the colours of the lines, which determine

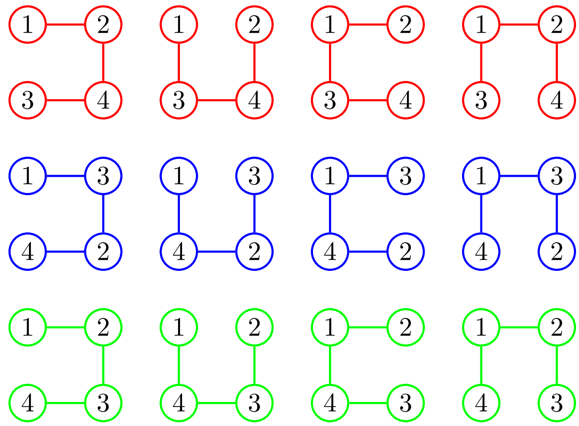

Fig. 8. The spanning trees that determine the efficient weight vectors for the $n = 4$ triple perturbed case.

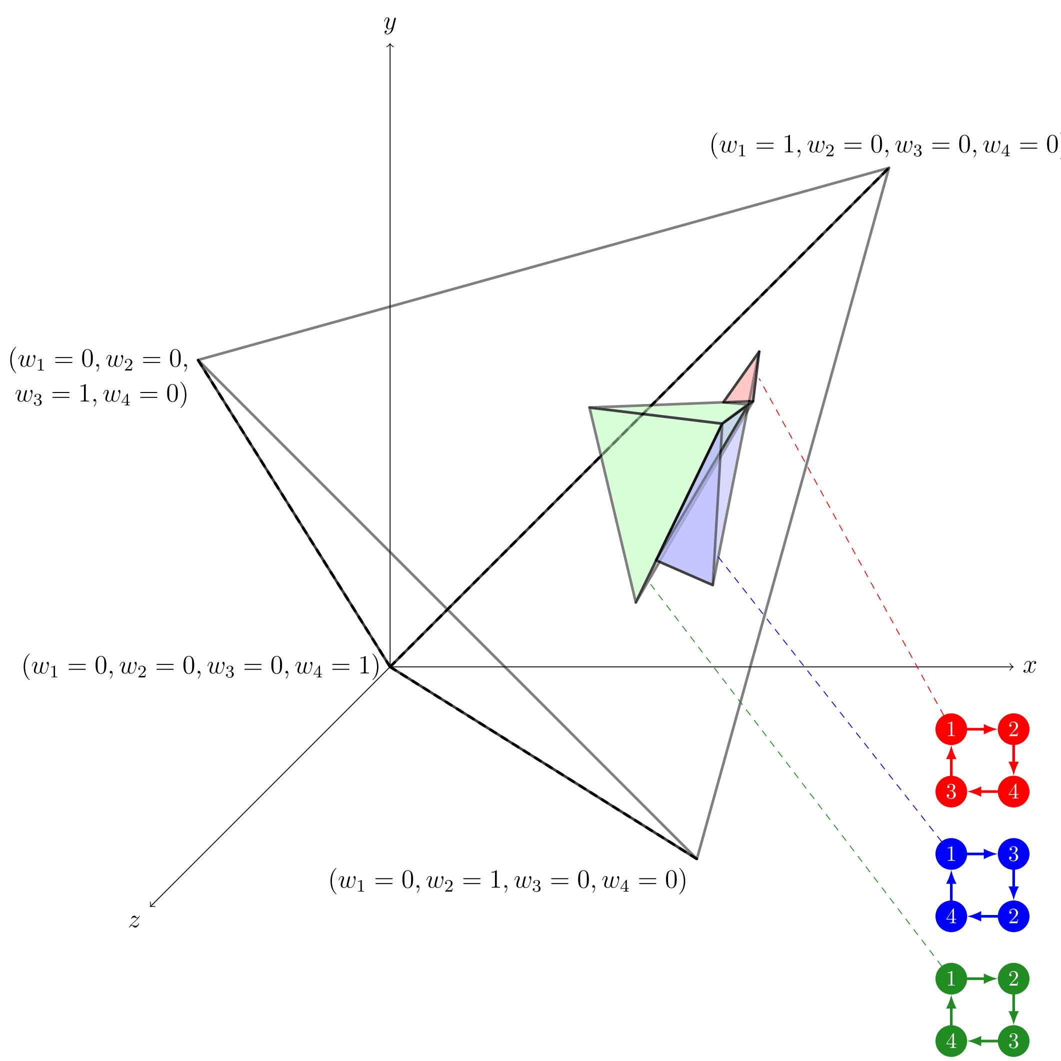

Fig. 9. Efficient weight vectors for the $n = 4$ double perturbed case.

them, e.g., ℓ_1 determines the orientation of the edge between alternatives 2 and 3 (as shown outside of S_3), and they are all highlighted by red.

Based on Theorem 2, the only way to get a strongly connected directed graph for three alternatives is to have a 3-cycle. There is only one such case in our illustrative example, thus, all the efficient weight vectors can be found in the small triangle in the middle highlighted by green background colour.

It is an important fact that the vertices of this triangle (denoted by T_1, T_2, T_3, and highlighted by orange) are the weight vectors calculated from the three possible labelled spanning trees for three alternatives that can be seen in Fig. 5. Thus, the set of efficient weight vectors is

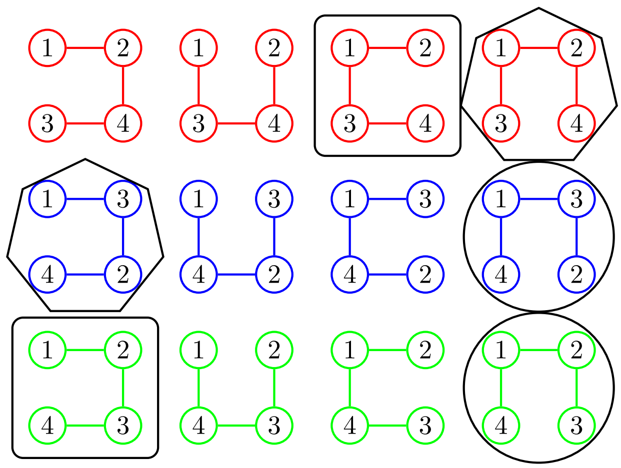

Fig. 10. The structure of the spanning trees in our illustrative example for the $n = 4$ double perturbed case. The spanning trees framed the same way determine the same weight vectors (the same vertices in Fig. 9).

the convex hull of the priority vectors calculated from the spanning trees.

A direct geometric proof of the efficiency of the eigenvector (for three alternatives coinciding with the logarithmic least squares optimal weight vector [44, Section 3.2]) can be given as: the geometric mean of row elements is located in the small green triangle in the middle, the set of efficient weight vectors.

It is also worth mentioning that, although Fig. 4 only shows an illustrative example (where $a_{12} \cdot a_{23} > a_{13}$), the main results are always the same for inconsistent matrices. If we use another PCM, then the lines ℓ_1, ℓ_2, and ℓ_3, and so the exact size and shape of the polygons will be different, but their main properties will be the same. The efficient vectors will be in a similar triangle determined by the spanning trees, except for the case of a consistent PCM (which means $a_{12} \cdot a_{23} = a_{13}$), when there is only one efficient weight vector (lines ℓ_1, ℓ_2, and ℓ_3 are concurrent), i.e., the results calculated from the spanning trees correspond to each other. It can also happen for other PCMs (when $a_{12} \cdot a_{23} < a_{13}$) that the 3-cycle of the Blanquero–Carrizosa–Conde directed graph of the efficient vectors is oriented in the opposite way. It is easy to show that if there is a feasible solution for the inequalities determining one of the two possible directed 3-cycles, then there is no feasible solution for the other case, however, there will be always exactly one strongly connected directed graph among the possible BCC digraphs.

3.2. The case of four alternatives

As mentioned before, in the case of three alternatives, a PCM is either inconsistent (simple perturbed) or consistent. However, for four alternatives it is important to distinguish between triple perturbed, double perturbed, and single perturbed inconsistent matrices as well as the consistent ones (when there is always only one efficient weight vector).

Fig. 6 shows the main results of an illustrative triple-perturbed example (the one that is also presented in Example 2) in the case of four alternatives ($n = 4$). Again, because of the fixed sum of the elements of weight vectors, the possible outcomes are in the unit 4-simplex that can be presented in three ($n - 1$) dimensions as a regular tetrahedron (from now on denoted by S_4).

Thus, S_4 (similarly to the large triangle before) contains all the possible weight vectors. The vertices correspond to the cases, when exactly one coordinate is positive, for the edges there are two positive coordinates, and for the faces there are three positive elements. Thus, the feasible solutions compose the interior of the tetrahedron.

In this case, there are so many possible polyhedra (which has a similar role to the polygons in Fig. 4) that it would be difficult to interpret the chart containing all of them. Thus, we only highlight the set of efficient weight vectors in Fig. 6, and the 6 planes determining these polyhedra (which has a similar role to ℓ_1, ℓ_2, and ℓ_3 in Fig. 4) in Fig. 7, where the equation determining the given plane can be seen above the corresponding subfigure. For instance, the top left plane contains the weight vectors that fulfil the equation $w_1/w_2 = a_{12}$, i.e., the ones that estimate the a_{12} element of the pairwise comparison matrix perfectly. This can also help to understand where the different points and sets are in S_4.

As before, based on Theorem 2, for $n = 4$ the only way to get a strongly connected directed graph is to have a 4-cycle (and the directions of the other edges do not matter). There are six possible ways for this, however, in the half of them, the only difference is the direction of the cycle. For a given PCM, always only one direction is possible (as it was the case for $n = 3$ as well), the inequalities determined by the other direction have no feasible solution.

Thus, we only have to examine three possible cycles. It turns out that each of them determine a small tetrahedron containing efficient weight vectors. These tetrahedra are also touching each other, as the set of efficient vectors is connected as proved by Blanquero et al. [21].

The cycles and the corresponding tetrahedra are highlighted by the same colour in Fig. 6. It is also true that each tetrahedron is determined by four weight vectors (four points) calculated from four spanning trees. The $(3 \cdot 4 =)12$ spanning trees defining the set of efficient vectors are presented in Fig. 8, the ones corresponding to a given tetrahedron are placed in the same row and highlighted by the same colour.

Interestingly, there are 16 possible labelled spanning trees for four vertices. 12 of them are line graphs, which determine our tetrahedra, and the other four are the four possible star graphs. Those also correspond to efficient weight vectors, but they are not placed at the vertices of the three small tetrahedra. They are on one of the edges of a given tetrahedron, at the point where a plane that determines one of the other two tetrahedra would intersect this edge.

It is also important that here the set of efficient weight vectors is not convex, but the union of three convex sets, which are defined by the line graph spanning trees.

When we consider double perturbed matrices for $n = 4$, then the results seem to be quite similar, however, here the set of efficient weight vectors is determined by 9 spanning trees instead of 12, as two vertices of each small tetrahedron correspond to each other. An illustrative example (the one that also can be seen in Example 2) is presented in Fig. 9.

Fig. 10 shows the spanning trees determining the set of efficient weight vectors for this case. The triad of alternatives one, two, and three is consistent in our example. The spanning trees that provide the same weight vectors (framed the same way in Fig. 10) contain two comparisons from this consistent triad, and their third comparison is the same (the comparison between one and four, two and four, or three and four). The consistent triad means that if we know any two of its elements, then those determine the third one as well, thus, in these spanning trees the used information is the same, hence they determine the same weight vectors.

As for the convexity of the set of efficient weight vectors for double perturbed matrices with four alternatives, the conclusion is similar to the previous case. The set of efficient weight vectors is not convex, but the union of three convex sets determined by 9 weight vectors corresponding to 9 spanning trees.

The case of simple perturbed pairwise comparison matrices with four alternatives is also presented via an illustrative example (the one that is used in Example 2) in Fig. 11.

One of the former tetrahedra (denoted by red, and defined by the first (top) cycle) corresponds to only one point in this case, as the comparisons are consistent here. This also means that the only efficient weight vector for the consistent matrix in Example 2 is this point.

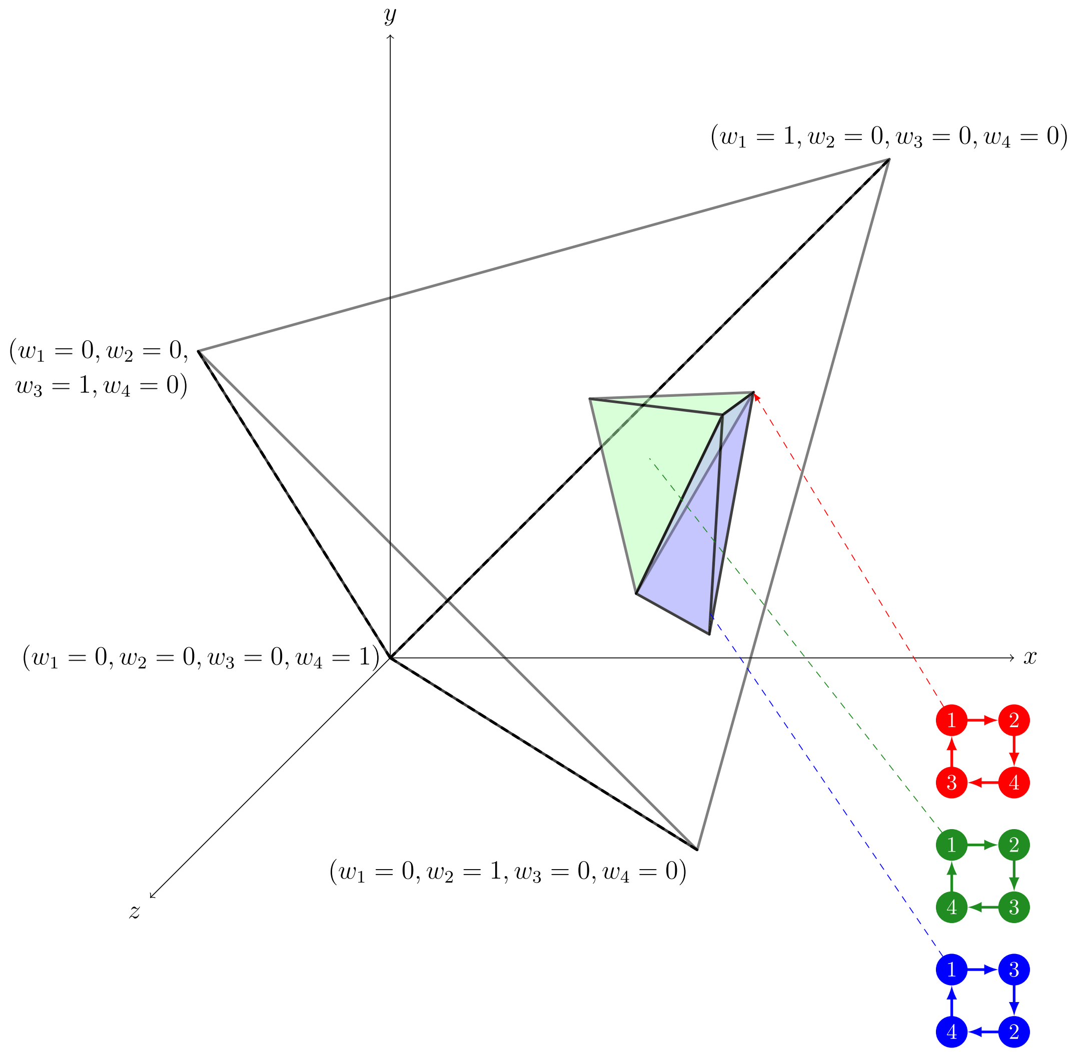

Fig. 11. Efficient weight vectors for the $n = 4$ simple perturbed case.

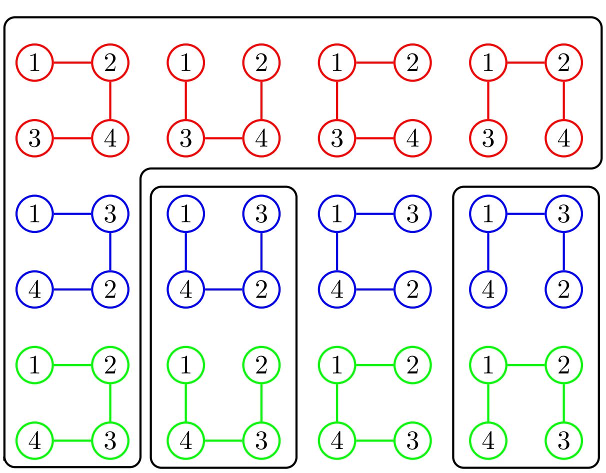

Fig. 12. The structure of the spanning trees in our illustrative example for the $n = 4$ simple perturbed case. The framed spanning trees determine the same weight vectors (same vertices in Fig. 11).

As for the other two tetrahedra, three of their four vertices pairwise correspond to the other tetrahedron's three vertices. Moreover, the point that presents the first tetrahedron is also one of their corresponding vertices. Thus, the set of efficient weight vectors is determined by 5 vectors corresponding to the spanning trees detailed in Fig. 12. This results in a triangular bipyramid (or dipyramid) that is two tetrahedra joined along one face.

As one can see, all the spanning trees that do not contain the comparison between one and four (the only perturbed element), determine the same weight vector, that one point that was mentioned before, as now there are even more consistent elements in the matrix, i.e., the used information coincide with each other. This is why the first (red) row of Fig. 12, thus, the first cycle, determines the same weight vector and so only one point in Fig. 11. On the other hand, the only unique weight vectors for the other two cycles determined by a spanning tree, are the ones, where both the comparison between one and four is included, and the other two comparisons are not possible in the other cycle.

Another interesting fact is that, the faces of S_4 correspond to the triads of the examined matrix. Fig. 13 presents the triads and the faces of S_4 of the simple perturbed example together. This chart is showing the projection of the results to the different triads as well as highlight the connection between $n = 3$ and $n = 4$, as each triad provides a submatrix of the original one with only three elements, and the faces of S_4 are similar to S_3.

It is important to emphasize that none of the points (weight vectors) that can be found on the faces of S_4 are possible weight vectors for us, as we are looking for positive vectors, thus these only help to understand and visualize the different conditions and possible points

Simple perturbed PCM

$$\begin{bmatrix} 1 & 2 & 6 & 5 \\ \frac{1}{2} & 1 & 3 & 12 \\ \frac{1}{6} & \frac{1}{3} & 1 & 4 \\ \frac{1}{5} & \frac{1}{12} & \frac{1}{4} & 1 \end{bmatrix}$$

1-2-3 triad

$$\begin{bmatrix} 1 & 2 & 6 \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{6} & \frac{1}{3} & 1 \end{bmatrix}$$

1-2-4 triad

$$\begin{bmatrix} 1 & 2 & 5 \\ \frac{1}{2} & 1 & 12 \\ \frac{1}{5} & \frac{1}{12} & 1 \end{bmatrix}$$

1-3-4 triad

$$\begin{bmatrix} 1 & 6 & 5 \\ \frac{1}{6} & 1 & 4 \\ \frac{1}{5} & \frac{1}{4} & 1 \end{bmatrix}$$

2-3-4 triad

$$\begin{bmatrix} 1 & 3 & 12 \\ \frac{1}{3} & 1 & 4 \\ \frac{1}{12} & \frac{1}{4} & 1 \end{bmatrix}$$

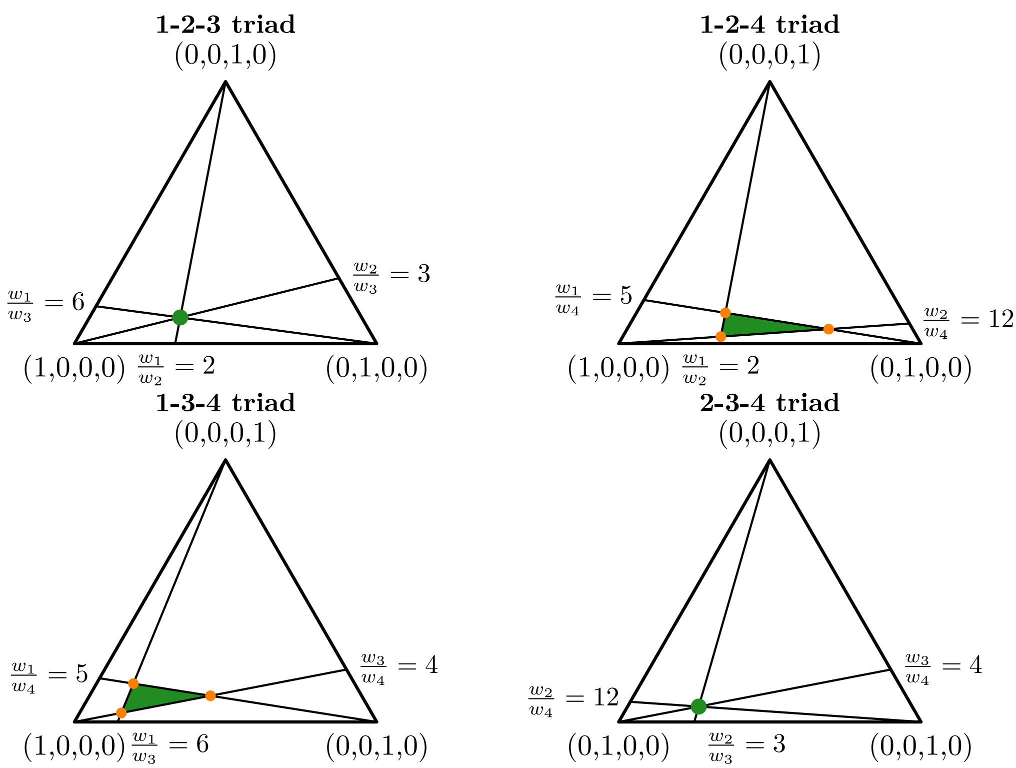

Fig. 13. The faces of S_4 and the corresponding submatrices determined by the given triads for the $n = 4$ simple perturbed case.

in the tetrahedron as well as to see the connection between the PCM and its lower dimensional submatrices.

One can see that in the case of triad 1-2-3 and triad 2-3-4, the equations only determine one point in the given triangle, as the given triads are consistent.

4. Conclusion and further research

In this paper, we filled in the research gap to provide the geometric interpretation of the sets of efficient weight vectors for three and four alternatives. The presented geometric demonstrations are complementary to the algebraic view applied in former studies, and help to better understand these sets. We are first in the literature to draw attention to the special role of spanning trees in this problem.

It turned out that the set of efficient weight vectors is determined by the vectors corresponding to spanning trees. For the simple perturbed case with three alternatives it is the convex hull of the weight vectors of the labelled star (line) graphs that is a triangle. For pairwise comparison matrices with four alternatives, they are determined by the line graphs, and they can be interpreted as the union of three tetrahedra, each of them being the convex hull of four appropriate spanning trees. The difference between triple, double and single perturbed matrices here is that some of the weight vectors determined by the spanning trees coincide, while for consistent PCMs they all provide the same weight vector.

Fig. 4 suggests that the area of the inner triangle (the set of efficient weight vectors) might be related to the level of inconsistency. However, this area depends on the inner triangle's location within S_3, it is larger around the centre (i.e., when the matrix elements are closer to 1), and it is smaller far from it (i.e., when some matrix elements are far from 1). Compare the bottom left triangle's area (0.0304, calculated as $\left((a_{31}a_{12}a_{23}-1)^2\right) / \left((a_{31}a_{12}+a_{12}+1)(a_{12}a_{23}+a_{23}+1)(a_{23}a_{31}+a_{31}+1)\right)$ by Routh's theorem [45, page 82]) in Fig. 13 to that of the triangle of efficient weight vectors corresponding to, e.g., the 3 × 3 PCM $a_{12} =$

$a_{23} = a_{13} = 4$, which is 0.0317. However, the latter matrix seems less inconsistent (with any reasonable inconsistency index) than the 1-3-4 triad. This phenomenon is present also in the case of PCMs of higher order.

It is important to deal with the efficiency of the calculated weight vectors, as it is crucial to determine the weight calculation technique used for a given problem, and it is difficult to argue for an inefficient solution. We believe that our results give significant contribution to better understand the efficient weight vectors, while also raise several further research questions.

The spanning trees determine the set of the efficient weight vectors for larger matrices as well, but almost certainly not all of them do so (as we presented for four alternatives as well). Are the line graph spanning trees retain their property and determine these sets for larger cases as well? Why is their role so unique?

What are the properties of the sets of efficient weight vectors, and how does this relate to the level of perturbation? Where can be the results of known weight calculation techniques in these sets, and why?

CRediT authorship contribution statement

Zsombor Szádoczki: Writing – review & editing, Writing – original draft, Visualization, Methodology, Formal analysis, Conceptualization. **Sándor Bozóki:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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