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ORIGINAL RESEARCH

Convex equilibrium-free stability and performance analysis of discrete-time nonlinear systems

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Abstract

This paper considers the equilibrium-free stability and performance analysis of discrete-time nonlinear systems. Two types of equilibrium-free notions are considered. Namely, the universal shifted concept, which considers stability and performance w.r.t. all equilibrium points of the system, and the incremental concept, which considers stability and performance between trajectories of the system. This paper shows how universal shifted stability and performance of discrete-time systems can be analysed by making use of the time-difference dynamics. Moreover, the existing results are extended for incremental dissipativity for discrete-time systems based on dissipativity analysis of the differential dynamics to more general state-dependent storage functions for less conservative results. Finally, it is shown how both these equilibrium-free notions can be cast as a convex analysis problem by making use of the linear parameter-varying framework, which is also demonstrated by means of an example.

1 | INTRODUCTION

The analysis of nonlinear systems has been an important topic of research over the last decades as systems have become more complex due to the push for higher performance requirements. While a large part of the existing tools for analysing stability and performance of nonlinear systems, such as Lyapunov stability [1] and dissipativity [2, 3], are very powerful, they only analyse these properties w.r.t. a single (equilibrium) point of the state-space representation, often the origin of the state-space. This poses limitations in cases when analysis of stability and performance of a system w.r.t. multiple equilibria or even w.r.t. multiple trajectories, such as in reference tracking and disturbance rejection, is required. Therefore, in recent years, equilibrium-free stability and performance methods have become increasingly popular. These equilibrium-free methods include concepts such as universal shifted and incremental stability and performance. Universal shifted stability and performance, also referred to as equilibrium independent stability and performance, considers these notions w.r.t. all equilibrium points of the system [4–6]. A relating concept to universal shifted stability is parametric stability [7, 8]. For parametric stability, also stability of the system w.r.t. various equilibrium points is investigated, specifically how this stability changes under the influence of uncertain (physical) parameters of the system. Another, stronger, equilibrium-free concept is incremental stability and the connected notion of incremental performance, which consider stability and performance between trajectories of the system [9–13]. This means that in case of incremental stability, trajectories of the system converge towards each other. Often, incremental stability and performance are analysed through so-called contraction analysis [11, 14, 15].

While continuous-time (CT) dynamical systems are often used for analysis and control, the recent surge of developments on data and learning based analysis and control methods heavily relies on discrete-time (DT) formulations. Also, controllers are almost exclusively implemented with digital hardware, hence analysis of the implemented discretized form of the controller with the actuator and sampling dynamics requires DT analysis tools, which are also the first step towards reliable DT synthesis methods. Therefore, analysis of DT nonlinear systems is inevitably important.

For CT nonlinear systems, it has been shown how the velocity form of the system, i.e. the time-differentiated dynamics, imply universal shifted stability and performance [6]. In DT, the time-difference dynamics, analogous to the time-differentiated dynamics in CT, have primarily received attention in the context of (nonlinear) model predictive control methods. In these works, the time-difference dynamics have been mainly used to realize offset free tracking of reference signals [16, 17]. However, to the authors' knowledge, there are no results in literature which connect the time-difference dynamics to stability and performance guarantees w.r.t. equilibrium points in the nonlinear context and on how to cast the corresponding analysis problem as computable tests. Therefore, in this paper, we show how these concepts are connected and how the universal shifted stability and performance analysis problem of DT systems can be solved as a convex optimization problem. Moreover, while parametric stability could be used to analyse universal shifted stability of the system by considering the input to the system as an uncertain parameter, there are, to the authors' knowledge, no tractable computable results for parametric stability for general nonlinear systems. On the other hand, in this paper, we will provide computationally tractable general results to analyse universal shifted stability and performance.

Similarly, w.r.t. incremental stability and performance, most of the literature has focussed on CT systems. For CT systems, it has been shown how the so-called differential form—representing the dynamics of the variations along trajectories—can be used in order to imply incremental stability, dissipativity, and performance properties [9, 11]. There have been some results on incremental and contraction based stability of DT systems, see e.g. [13, 14], or focussing on Lipschitz properties [18–21]. Moreover, for DT nonlinear systems, the link to incremental dissipativity based on the differential form has been made in [22] for quadratic supply functions. However, the work in [22] only considers quadratic storage functions depending on a constant matrix. This is more conservative compared to results available for CT systems, where the links between dissipativity of the differential form and incremental dissipativity has been made for more general storage functions. Therefore, in this paper, we will provide a novel generalization of the CT incremental stability and performance results for DT systems and show how these results can be cast as a convex optimization problem.

To summarize, the main contributions of this paper are as follows

- C1: Show how stability and performance properties of the time-difference dynamics of a DT nonlinear system imply universal shifted stability and performance properties of the system. (Theorems 7 and 9)
- C2: Extend existing CT incremental dissipativity analysis results for DT nonlinear systems based on the differential form, enabling the use of state-dependent storage functions. (Theorem 13)
- C3: Show that both the universal shifted and incremental analysis problems can be cast as an analysis problem of a linear parameter-varying (LPV) representation. This allows these problems to be solved via convex optimization in terms of semi-definite programs (SDPs), providing computable tests for equilibrium-free stability and performance analysis of DT nonlinear systems. (Theorem 14)

The paper is structured as follows, in Section 2, we give an overview of the definitions of universal shifted and incremental stability and performance. Next, in Section 3, we will introduce velocity based analysis for DT nonlinear systems and show how it can be used in order to imply universal shifted stability and performance properties. In Section 4, we will show how dissipativity analysis of the differential form in DT can be used to imply incremental dissipativity. Then, in Section 5, we discuss the connections between the velocity and differential analysis results and the relation between universal shifted and incremental stability and performance. Section 6 shows how the velocity and differential analysis results can be cast as analysis problems of an LPV representation, enabling to solve the equilibrium-free analysis problem via convex optimization in terms of SDPs. In Section 7, the usefulness of the developed analysis tools is demonstrated on an example. Finally, the conclusions are drawn in Section 8.

Notation. $\mathbb{R}$ is the set of real numbers, while $\mathbb{R}_0^+$ is the set of non-negative reals. $\mathbb{Z}_0^+$ is the set of non-negative integers. We denote by $\mathbb{S}^n$ the set of real symmetric matrices of size $n \times n$. Denote the projection operation by π, s.t. for $\mathcal{D} = \mathcal{A} \times \mathcal{B}$, $a \in \pi_a \mathcal{D}$ if and only if $\exists b \in \mathcal{B}$ s.t. $(a, b) \in \mathcal{D}$. For a signal $w : \mathbb{Z}_0^+ \to \mathbb{R}^n$ and a $w_* \in \mathbb{R}^n$, denoted by $w \equiv w_*$ that $w(t) = w_*$ for all $t \in \mathbb{Z}_0^+$. $\mathcal{C}_n$ is the class of n-times continuously differentiable functions. A function $V : \mathbb{R}^n \to \mathbb{R}$ belongs to the class $\mathcal{Q}_{x_*}$, if it is positive definite and decrescent w.r.t. $x_* \in \mathbb{R}^n$ (see [23, Definition 3.3]). A function $V_\mathrm{i} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is in $\mathcal{Q}_\mathrm{i}$, if $V_\mathrm{i}(\cdot, \tilde{x}) \in \mathcal{Q}_{\tilde{x}}$ for all $\tilde{x} \in \mathbb{R}^n$ and $V_\mathrm{i}(x, \cdot) \in \mathcal{Q}_x$ for all $x \in \mathbb{R}^n$. $\|\cdot\|$ is the Euclidean (vector) norm. We use $(\star)$ to denote a symmetric term in a quadratic expression, e.g. $(\star)^\top Q(a-b) = (a-b)^\top Q(a-b)$ for $Q \in \mathbb{S}^n$ and $a, b \in \mathbb{R}^n$. The notation $A \succ 0$ ($A \succeq 0$) indicates that $A \in \mathbb{S}^n$ is positive (semi-)definite, while $A \prec 0$ ($A \preceq 0$) denotes a negative (semi-)definite $A \in \mathbb{S}^n$. Moreover, $\mathrm{col}(x_1, \dots, x_n)$ denotes the column vector $[x_1^\top \ \cdots \ x_n^\top]^\top$.

2 | EQUILIBRIUM-FREE STABILITY AND PERFORMANCE

2.1 | Nonlinear system

We consider DT nonlinear dynamical systems given in the form of

$$x(t+1) = f(x(t), w(t)); \tag{1a}$$

$$z(t) = h(x(t), w(t)); \tag{1b}$$

where $t \in \mathbb{Z}_0^+$ is the discrete-time, $x(t) \in \mathbb{R}^{n_x}$ is the state with initial condition $x(0) = x_0 \in \mathbb{R}^{n_x}$, while $w(t) \in \mathbb{R}^{n_w}$ is the input and $z(t) \in \mathbb{R}^{n_z}$ is the output of the system. Moreover, the functions $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$ and $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_y}$ are considered to be in C_1. We define the set of solutions of Equation (1) as

$$\mathfrak{B} := \{(x, w, z) \in (\mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_z})^{\mathbb{Z}_0^+} \mid (x, w, z) \text{ satisfy Equation (1)}\}, \tag{2}$$

called the behaviour of Equation (1). For a specific input $\bar{w} \in (\mathbb{R}^{n_w})^{\mathbb{Z}_0^+}$,

$$\mathfrak{B}_w(\bar{w}) := \{(x, \bar{w}, z) \in \mathfrak{B}\}, \tag{3}$$

denotes the compatible solution trajectories of Equation (1). We also define the state transition map $\phi_x : \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{R}^{n_x} \times (\mathbb{R}^{n_w})^{\mathbb{Z}_0^+} \to \mathbb{R}^{n_x}$, such that $x(t) = \phi_x(t, 0, x_0, w)$. Moreover, we assume that all solutions are forward complete and unique.

For Equation (1), the equilibrium points satisfy

$$x_* = f(x_*, w_*); \tag{4a}$$

$$z_* = h(x_*, w_*); \tag{4b}$$

where $x_* \in \mathbb{R}^{n_x}$, $w_* \in \mathbb{R}^{n_w}$, and $z_* \in \mathbb{R}^{n_z}$. The set of equilibrium points of the nonlinear system is then defined as

$$\mathscr{E} := \{(x_*, w_*, z_*) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \mid (x_*, w_*, z_*) \text{ satisfy (4)}\}. \tag{5}$$

Define $\mathscr{X} := \pi_{x_*}\mathscr{E}$, $\mathscr{W} := \pi_{w_*}\mathscr{E}$, and $\mathscr{Z} := \pi_{z_*}\mathscr{E}$. For our results concerning universal shifted stability and performance, we take the following assumption:

Assumption 1 (Unique equilibria). For the nonlinear system given by Equation (1) with equilibrium points $(x_*, w_*, z_*) \in \mathscr{E}$, we assume that there exists a bijective map $\kappa : \mathscr{W} \to \mathscr{X}$ such that $x_* = \kappa(w_*)$, for all $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$. This means that, for each $w_* \in \mathscr{W}$, there is a unique corresponding $x_* \in \mathscr{X}$, and vice versa, for each $x_* \in \mathscr{X}$ there is a unique corresponding $w_* \in \mathscr{W}$.

Note that this assumption is only taken in order to simplify the discussion.

2.2 | Equilibrium-free stability

As aforementioned, in this paper, we will consider two forms of equilibrium-free stability. Namely, universal shifted stability and incremental stability. In DT, we consider the same definition for universal shifted stability as has been considered in CT in [6], i.e. a system given by Equation (1) is universally shifted (asymptotically) stable if it is (asymptotically) stable w.r.t. to all its forced equilibrium points. More concretely:

Definition 1 (Universal shifted stability [6]). The nonlinear system given by Equation (1) is *universally shifted stable* (USS), if for each $\epsilon > 0$ and $x_* \in \mathscr{X}$ with corresponding $w_* \in \mathscr{W}$, i.e. $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$, there exists a $\delta(\epsilon) > 0$ s.t. any $x \in \mathfrak{B}_w(w \equiv w_*)$ with $\|x(0) - x_*\| < \delta(\epsilon)$ satisfies $\|x(t) - x_*\| < \epsilon$ for all $t \in \mathbb{Z}_0^+$. The system is *universally shifted asymptotically stable* (USAS) if it is USS and, for each $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$, there exists a $\delta > 0$ such that for $w \equiv w_*$ we have that $\|x(0) - x_*\| < \delta(\epsilon)$ implies $\lim_{t\to\infty} \|\phi_x(t, 0, x(0), w) - x_*\| = 0$.

To the authors' knowledge, a Lyapunov based theorem to imply US(A)S for DT systems does not yet exist in the literature. Therefore, we provide the following novel result:

Theorem 1 (Universal shifted Lyapunov stability). *The nonlinear system given by Equation (1) is USS, if there exists a function $V_s : \mathbb{R}^{n_x} \times \mathscr{W} \to \mathbb{R}_0^+$ with $V_s(\cdot, w_*) \in C_1$ and $V_s(\cdot, w_*) \in \mathcal{Q}_{x_*}$ for every $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$, such that*

$$V_s(x(t+1), w_*) - V_s(x(t), w_*) \leq 0, \tag{6}$$

holds for every $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$ and for all $t \in \mathbb{Z}_0^+$ and $x \in \pi_x\mathfrak{B}_w(w \equiv w_*)$. If Equation (6) holds, but with strict inequality except when $x(t) = x_*$, then the system is USAS.

Proof. See Section A.1 □

Similarly to how standard Lyapunov stability gives rise to invariant sets around stable equilibrium points of a system, we can also extend the notion of invariance for universal shifted Lyapunov stability:

Theorem 2 (Universal shifted invariance). *For a nonlinear system given by Equation (1), for which there exists a function $V_s : \mathbb{R}^{n_x} \times \mathscr{W} \to \mathbb{R}_0^+$ such that it is USS in terms of Theorem 1, any level set:*

$$\mathbb{X}_{w_*,\gamma} = \{x \in \mathbb{R}^{n_x} \mid V_s(x, w_*) \leq \gamma\}, \tag{7}$$

with $\gamma \geq 0$ is invariant, meaning that

$$\phi_x(t, 0, x_0, w \equiv w_*) \in \mathbb{X}_{w_*,\gamma}, \tag{8}$$

for all $t \in \mathbb{Z}_0^+$, $x_0 \in \mathbb{X}_{w_*,\gamma}$.

Proof. See Section A.2 □

Note that this notion of universal shifted invariance can be interpreted as the existence of standard Lyapunov based invariant sets around each (forced) equilibrium point (x_*, w_*) of the system. A visual illustration of universal shifted invariance is depicted in Figure 1.

Incremental stability is a stronger notion than universal shifted stability and considers that all trajectories should be stable w.r.t. each other, meaning that all trajectories converge towards each other. Therefore, incremental stability also implies

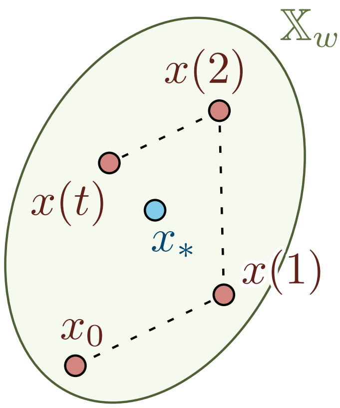

FIGURE 1 The invariant set $\mathbb{X}_{w_*,\gamma}$ for universal shifted invariance.

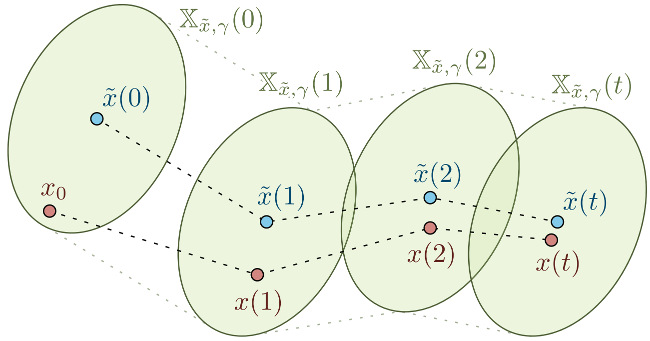

FIGURE 2 The invariant tube $\mathbb{X}_{\tilde{x},\gamma}$ for incremental invariance.

universal shifted stability. For incremental stability, various definitions exist in the literature, such as in [13, 14]. Here, we will consider the following slightly more general definition:

Definition 2 (Incremental stability). The nonlinear system given by Equation (1) is *incrementally stable* (IS), if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ s.t. any $x, \tilde{x} \in \mathfrak{B}_{\mathrm{w}}(w)$ with $\|x(0) - \tilde{x}(0)\| < \delta(\epsilon)$ satisfies $\|x(t) - \tilde{x}(t)\| < \epsilon$ for all $t \in \mathbb{Z}_0^+$. The system is *incrementally asymptotically stable* (IAS) if it is IS and there exists a $\delta > 0$ such that $\|x(0) - \tilde{x}(0)\| < \delta(\epsilon)$ implies that $\lim_{t\to\infty} \|\phi_{\mathrm{x}}(t, 0, x(0), w) - \phi_{\mathrm{x}}(t, 0, \tilde{x}(0), w)\| = 0$.

Similar as for US(A)S, a Lyapunov theorem for I(A)S can also be formulated, which we adopt from [13, 14]:

Theorem 3 Incremental Lyapunov stability [13, 14]:. *The nonlinear system given by Equation (1) is IS, if there exists an incremental Lyapunov function $V_{\mathrm{i}} : \mathbb{R}^{n_{\mathrm{x}}} \times \mathbb{R}^{n_{\mathrm{x}}} \to \mathbb{R}_0^+$ with $V_{\mathrm{i}} \in \mathcal{C}_1$ and $V_{\mathrm{i}} \in \mathcal{Q}_{\mathrm{i}}$, such that*

$$V_{\mathrm{i}}(x(t+1), \tilde{x}(t+1)) - V_{\mathrm{i}}(x(t), \tilde{x}(t)) \le 0, \tag{9}$$

for all $t \in \mathbb{Z}_0^+$ and $x, \tilde{x} \in \pi_{\mathrm{x}}\mathfrak{B}_{\mathrm{w}}(w)$ under all measurable and bounded $w \in (\mathbb{R}^{n_z})^{\mathbb{Z}_0^+}$. Moreover, the nonlinear system is IAS, if Equation (9) holds, but with strict inequality except when $x(t) = \tilde{x}(t)$.

See [13, 14] for the proof. Finally, we can also define a notion of invariance for incremental stability:

Theorem 4 (Incremental invariance). *For a nonlinear system given by Equation (1), for which there exists an incremental Lyapunov function $V_{\mathrm{i}} : \mathbb{R}^{n_{\mathrm{x}}} \times \mathbb{R}^{n_{\mathrm{x}}} \to \mathbb{R}_0^+$ such that it is IS, and for any given trajectory $(\tilde{x}, w) \in \pi_{\mathrm{x,w}} \in \mathfrak{B}_{\mathrm{w}}(w)$ for which w is bounded and measurable, a $\gamma \geq 0$ defines a time-varying invariant set:*

$$\mathbb{X}_{\tilde{x},\gamma}(t) = \left\{x \in \mathbb{R}^{n_{\mathrm{x}}} \mid V_{\mathrm{i}}(x, \tilde{x}(t)) \le \gamma\right\}, \tag{10}$$

i.e. an invariant tube, such that

$$\phi_{\mathrm{x}}(t, 0, x_0, w) \in \mathbb{X}_{\tilde{x},\gamma}(t), \tag{11}$$

for all $t \in \mathbb{Z}_0^+$ and $x_0 \in \mathbb{X}_{\tilde{x},\gamma}(0)$.

Proof. See Section A.3 □

In the case of IAS, there exists a $\gamma : \mathbb{Z}_0^+ \to \mathbb{R}_0^+$, which is a monotonically decreasing function such that $\mathbb{X}_{\tilde{x},\gamma}(t) = \left\{x \in \mathbb{R}^{n_{\mathrm{x}}} \mid V_{\mathrm{i}}(x, \tilde{x}(t)) \le \gamma(t)\right\}$.

A visual illustration of the time-varying invariant set $\mathbb{X}_{\tilde{x},\gamma}$ for incremental invariance is depicted in Figure 2. As it is visible in the figure, incremental invariance can be interpreted the existence of an invariant tube around a given trajectory $\tilde{x} \in \pi_{\mathrm{x}}\mathfrak{B}_{\mathrm{w}}(w)$. By comparing Figure 2 to Figure 1, the difference between universal shifted stability and incremental stability can also be clearly seen. Namely, universal shifted stability is stability w.r.t. (target) equilibrium points while incremental stability is stability w.r.t. (target) trajectories. This makes universal shifted stability useful in analysing stability w.r.t. holding set points, while incremental stability is a stronger notion that is useful for analysing stability w.r.t. general (time-varying) reference tracking problems.

2.3 | Equilibrium-free dissipativity

The concept of dissipativity [2] allows for simultaneous analysis of stability and performance of systems. The concept of "classical" dissipativity can be interpreted as analysing the internal energy of the system over time. However, this analysis of internal energy of the system is only concerned with respect to a single "minimum" point, called the neutral storage, which is often taken as the origin of the state-space associated with the nonlinear representation. Nevertheless, it is often of interest to analyse a set of equilibrium points/trajectories, e.g. in the case of reference tracking or disturbance rejection, which is cumbersome to be performed with the classical dissipativity results for nonlinear systems. Hence, there is a need for equilibrium-free dissipativity notions such as universal shifted dissipativity and incremental dissipativity, as they allow to handle these cases efficiently without the restriction of a single point of neutral storage.

The concept of universal shifted dissipativity (USD) [6] allows for analysing the energy flow between trajectories and equilibrium points of the system. More concretely, similar to the CT USD notion in [6, Definition 2], we formulate the following definition of DT USD:

Definition 3 (Universal shifted dissipativity). The nonlinear system given by Equation (1) is universally shifted dissipative w.r.t. the supply function $s_{\mathrm{s}} : \mathbb{R}^{n_{\mathrm{w}}} \times \mathcal{W} \times \mathbb{R}^{n_{\mathrm{z}}} \times \mathcal{Z} \rightarrow \mathbb{R}$, if there exists a storage function $\mathcal{V}_{\mathrm{s}} : \mathbb{R}^{n_{\mathrm{x}}} \times \mathcal{W} \rightarrow \mathbb{R}_0^+$ with $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_0$ and $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ for every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*}\mathscr{E}$, such that

$$\mathcal{V}_{\mathrm{s}}(x(t_1+1), w_*) - \mathcal{V}_{\mathrm{s}}(x(t_0), w_*) \leq \sum_{t=t_0}^{t_1} s_{\mathrm{s}}(w(t), w_*, z(t), z_*), \tag{12}$$

for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$ and $(x, w, z) \in \mathfrak{B}$.

Incremental dissipativity, see [9], is an even stronger notion of dissipativity which takes into account multiple trajectories of a system and can be thought of as analysing the energy flow between trajectories. Similar to the incremental dissipativity definition for CT systems in [9], we formulate the definition of incremental dissipativity of DT nonlinear systems as follows:

Definition 4 (Incremental dissipativity). The system given by Equation (1) is called *Incrementally Dissipative* (ID) w.r.t. the supply function $s_{\mathrm{i}} : \mathbb{R}^{n_{\mathrm{w}}} \times \mathbb{R}^{n_{\mathrm{w}}} \times \mathbb{R}^{n_{\mathrm{z}}} \times \mathbb{R}^{n_{\mathrm{z}}} \rightarrow \mathbb{R}$, if there exists a storage function $\mathcal{V}_{\mathrm{i}} : \mathbb{R}^{n_{\mathrm{x}}} \times \mathbb{R}^{n_{\mathrm{x}}} \rightarrow \mathbb{R}_0^+$ with $\mathcal{V}_{\mathrm{i}} \in \mathcal{C}_0$ and $\mathcal{V}_{\mathrm{i}} \in \mathcal{Q}_{\mathrm{i}}$, such that, for any two trajectories $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$,

$$\begin{aligned}\mathcal{V}_{\mathrm{i}}\big(x(t_1+1), \tilde{x}(t_1+1)\big) - \mathcal{V}_{\mathrm{i}}\big(x(t_0), \tilde{x}(t_0)\big) \\ \leq \sum_{t=t_0}^{t_1} s_{\mathrm{i}}\big(w(t), \tilde{w}(t), z(t), \tilde{z}(t)\big),\end{aligned} \tag{13}$$

for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$.

For classical dissipativity, supply functions of a quadratic form are often studied as they allow us to link dissipativity of a system to (quadratic) performance notions such as the ℓ_2-gain and passivity. Similarly, for this reason, we will also focus on quadratic supply functions for USD and ID in this paper. More concretely, we will consider quadratic supply functions of the form

$$s_{\mathrm{s}}(w, w_*, z, z_*) = \begin{bmatrix} w - w_* \\ z - z_* \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w - w_* \\ z - z_* \end{bmatrix}, \tag{14}$$

where $Q \in \mathbb{S}^{n_{\mathrm{w}}}$, $R \in \mathbb{S}^{n_{\mathrm{z}}}$, and $S \in \mathbb{R}^{n_{\mathrm{w}} \times n_{\mathrm{z}}}$. In the incremental case, we choose $s_{\mathrm{i}}(w, \tilde{w}, z, \tilde{z})$ to be defined similarly to Equation (14), We will refer to USD and ID w.r.t. supply functions of the form Equation (14) as (Q, S, R)-USD and (Q, S, R)-ID, respectively.

Like in CT, it can easily be shown that (Q, S, R)-USD or (Q, S, R)-ID of a DT nonlinear system given by Equation (1) with $(Q, S, R) = (\gamma^2, 0, -I)$ implies that the system has a universal shifted or incremental ℓ_2-gain bound of γ, respectively (see [6, Definition 3] and [22]). Similarly, a DT nonlinear system is universally shifted or incrementally passive if it is (Q, S, R)-USD or (Q, S, R)-ID with $(Q, S, R) = (0, I, 0)$, respectively.

The definitions of USD and ID give us conditions to analyse universal shifted and incremental stability and performance properties of DT nonlinear systems. However, using these conditions directly to analyse these notions is difficult, as they require finding a storage function that satisfies the corresponding conditions w.r.t. all equilibria or w.r.t. any solution pairs of the system. Therefore, in the next two sections, we will show how other dissipativity notions can be used to simplify the analysis of USD and ID of DT nonlinear systems.

3 | VELOCITY ANALYSIS

3.1 | The DT velocity form and velocity dissipativity

In this section, we will focus on analysing US(A)S and universal shifted performance (USP) properties of DT nonlinear systems using so-called velocity based analysis. In [6], it has been shown how these properties for CT nonlinear systems could be analysed through the time-differentiated dynamics, i.e. velocity form of the system. In DT, the counterpart to time-differentiation is taking difference of the dynamics in time. Due to the different nature of the difference and derivative operators, the resulting velocity form in DT is different from the CT version. This also results in proofs that are of different nature, than their CT counterparts. As a contribution of this paper, we will show in this section how the time-difference dynamics in DT can be used to imply USS and USP of the original DT nonlinear system.

Let us introduce the forward increment signals $x_\Delta(t) := x(t+1) - x(t) \in \mathbb{R}^{n_{\mathrm{x}}}$, $w_\Delta(t) := w(t+1) - w(t) \in \mathbb{R}^{n_{\mathrm{w}}}$, and $z_\Delta(t) := z(t+1) - z(t) \in \mathbb{R}^{n_{\mathrm{z}}}$, which sometimes are also called as DT velocities. Analogously, we can introduce the more commonly used backward increment signals $x_\nabla(t) := x(t) - x(t-1) \in \mathbb{R}^{n_{\mathrm{x}}}, \ldots, z_\nabla(t) := z(t) - z(t-1) \in \mathbb{R}^{n_{\mathrm{z}}}$.

Based on these variables, the forward time-difference dynamics of Equation (1) can be expressed as

$$x_\Delta(t+1) = f(x(t+1), w(t+1)) - f(x(t), w(t)); \tag{15a}$$

$$z_\Delta(t) = h(x(t+1), w(t+1)) - h(x(t), w(t)); \tag{15b}$$

while the backward time-difference dynamics of Equation (1) are

$$x_\nabla(t+1) = f(x(t), w(t)) - f(x(t-1), w(t-1)); \tag{16a}$$

$$z_\nabla(t) = h(x(t), w(t)) - h(x(t-1), w(t-1)). \tag{16b}$$

Let us define the operator Δ for the behaviour $\mathfrak{B}$ of Equation (1) such that

$$\begin{aligned}\Delta\mathfrak{B} = \Big\{ &(x_\Delta, w_\Delta, z_\Delta) \in (\mathbb{R}^{n_{\mathrm{x}}} \times \mathbb{R}^{n_{\mathrm{w}}} \times \mathbb{R}^{n_{\mathrm{z}}})^{\mathbb{Z}_0^+} \mid \\ &x_\Delta(t) = x(t+1) - x(t),\ w_\Delta(t) = w(t+1) - w(t), \\ &z_\Delta(t) := z(t+1) - z(t),\ \forall\, t \in \mathbb{Z}_0^+,\ (x, w, z) \in \mathfrak{B} \Big\},\end{aligned} \tag{17}$$

which defines the solution set of the forward dynamics (15). If q is the forward-time shift operator, meaning that $qx(t) = x(t+1)$, then $\Delta\mathfrak{B} = q\nabla\mathfrak{B}$, where $\nabla\mathfrak{B}$ is the solution set of the backward dynamics (16). As Equation (1) is time-invariant, this concludes that (15) and (16) are equivalent representations. For technical convenience, we will formulate our results w.r.t. the forward dynamics (15), but all derivations can be equivalently formulated for (16) as well. Please also note that both representations (16) and (15) are causal.

By the (second) fundamental theorem of calculus, we can equivalently write Equation (15) in an alternative form, which we will refer to as the DT velocity form:

Definition 5 (Discrete-time velocity form). The velocity form of a nonlinear system, given by Equation (1) with $f, h \in \mathcal{C}_1$, is

$$x_\Delta(t+1) = \bar{A}_\mathrm{v}\big(\xi(t+1), \xi(t)\big)x_\Delta(t) + \bar{B}_\mathrm{v}\big(\xi(t+1), \xi(t)\big)w_\Delta(t); \tag{18a}$$

$$z_\Delta(t) = \bar{C}_\mathrm{v}\big(\xi(t+1), \xi(t)\big)x_\Delta(t) + \bar{D}_\mathrm{v}\big(\xi(t+1), \xi(t)\big)w_\Delta(t); \tag{18b}$$

where $(x, w, z) \in \mathfrak{B}$, $\xi = \mathrm{col}(x, w)$, and

$$\bar{A}_\mathrm{v}(x_+, w_+, x, w) = \int_0^1 \frac{\partial f}{\partial x}(\bar{x}(\lambda), \bar{w}(\lambda))\,\mathrm{d}\lambda, \tag{19a}$$

$$\bar{B}_\mathrm{v}(x_+, w_+, x, w) = \int_0^1 \frac{\partial f}{\partial w}(\bar{x}(\lambda), \bar{w}(\lambda))\,\mathrm{d}\lambda, \tag{19b}$$

$$\bar{C}_\mathrm{v}(x_+, w_+, x, w) = \int_0^1 \frac{\partial h}{\partial x}(\bar{x}(\lambda), \bar{w}(\lambda))\,\mathrm{d}\lambda, \tag{19c}$$

$$\bar{D}_\mathrm{v}(x_+, w_+, x, w) = \int_0^1 \frac{\partial h}{\partial w}(\bar{x}(\lambda), \bar{w}(\lambda))\,\mathrm{d}\lambda, \tag{19d}$$

with $\bar{x}(\lambda) = x + \lambda(x_+ - x)$, $\bar{w}(\lambda) = w + \lambda(w_+ - w)$.

Note that by the fundamental theorem of calculus [24], (15) and (18) are equivalent. To distinguish the velocity form of Equation (1) from the original nonlinear system, we will call Equation (1) to be the primal form.

Based on Equation (17), the solution set of Equation (18) is given by $\mathfrak{B}_\mathrm{v} := \Delta\mathfrak{B}$, and we can also define $\mathfrak{B}_{\mathrm{v,w}}(w) := \Delta\mathfrak{B}_\mathrm{w}(w)$ for a $w \in \mathcal{W}^{\mathbb{Z}_0^+}$. The resulting DT velocity form represents the dynamics of the change between consecutive time-instances of the original dynamics. This is analogous to the CT velocity form introduced in [6], which represents the dynamics of the instantaneous change in time (i.e. time derivative) of the original dynamics. Next, we will show that the DT velocity form has a direct relation to USS and USP. Before presenting this connection, we will first show some analysis results on the DT velocity form.

Definition 6 (Velocity stability). The nonlinear system given by Equation (1) with velocity form Equation (18) is velocity (asymptotically) stable (V(A)S), if the velocity form is (asymptotically) stable in the Lyapunov sense w.r.t. the origin (see also Definition 1), i.e. the velocity state x_Δ is (asymptotically) stable w.r.t. 0.

As V(A)S is nothing more than (asymptotic) stability of velocity form, we can easily formulate the following Lyapunov based theorem in order to verify it:

Theorem 5 (Velocity Lyapunov stability). *The nonlinear system given by Equation (1) is VS, if there exists a function $V_\mathrm{v} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ with $V_\mathrm{v} \in \mathcal{C}_0$ and $V_\mathrm{v} \in \mathcal{Q}_0$, such that*

$$V_\mathrm{v}(x_\Delta(t+1)) - V_\mathrm{v}(x_\Delta(t)) \leq 0, \tag{20}$$

for all $t \in \mathbb{Z}_0^+$ and $x_\Delta \in \pi_{x_\Delta}\mathfrak{B}_{\mathrm{v},\mathcal{W}}$. If Equation (20) holds, but with strict inequality except when $x_\Delta(t) = 0$, then the system is VAS.

The proof of Theorem 5 simply follows from standard Lyapunov stability theory, see e.g. [1, 25]. Next, we formulate a notion of dissipativity regarding the velocity form, which enables the analysis of stability and performance of nonlinear systems in the velocity sense:

Definition 7 (Velocity dissipativity). The nonlinear system given by Equation (1) is velocity dissipative (VD) w.r.t. the supply function $s_\mathrm{v} : \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}} \to \mathbb{R}$, if there exists a storage function $\mathcal{V}_\mathrm{v} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ with $\mathcal{V}_\mathrm{v} \in \mathcal{C}_0$ and $\mathcal{V}_\mathrm{v} \in \mathcal{Q}_0$, such that, for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$,

$$\mathcal{V}_\mathrm{v}(x_\Delta(t_1+1)) - \mathcal{V}_\mathrm{v}(x_\Delta(t_0)) \leq \sum_{t=t_0}^{t_1} s_\mathrm{v}(w_\Delta(t), z_\Delta(t)), \tag{21}$$

for all $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_\mathrm{v}$.

Note that VD can be seen as 'classical' dissipativity of the velocity form of the system Equation (18). Next, let us consider quadratic (Q, S, R) supply functions for VD of the form

$$s_\mathrm{v}(w_\Delta, z_\Delta) = \begin{bmatrix} w_\Delta \\ z_\Delta \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\Delta \\ z_\Delta \end{bmatrix}, \tag{22}$$

where again $Q \in \mathbb{S}^{n_\mathrm{w}}$, $S \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{z}}$, and $R \in \mathbb{S}^{n_\mathrm{z}}$. Moreover, we also consider the storage function $\mathcal{V}_\mathrm{v}$ to be quadratic:

$$\mathcal{V}_\mathrm{v}(x_\Delta) = x_\Delta^\top M x_\Delta, \tag{23}$$

where $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$. Under these considerations, we can derive the following (infinite dimensional) linear matrix inequality (LMI) feasibility condition for VD:

Theorem 6 (DT (Q, S, R)-VD condition). *The system given by Equation (1) is (Q, S, R)-VD on the convex set $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{x} \cdot n_\mathrm{w}}$, where $R \preceq 0$, if there exists an $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$, such that for all*

$(x, w) \in \mathcal{X} \times \mathcal{W}$, *it holds that*

$$(\star)^\top \begin{bmatrix} -M & 0 \\ \star & M \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\mathrm{v}(x, w) & B_\mathrm{v}(x, w) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\mathrm{v}(x, w) & D_\mathrm{v}(x, w) \end{bmatrix} \preceq 0, \quad (24)$$

where $A_\mathrm{v} = \frac{\partial f}{\partial x}$, $B_\mathrm{v} = \frac{\partial f}{\partial w}$, $C_\mathrm{v} = \frac{\partial h}{\partial x}$, $D_\mathrm{v} = \frac{\partial h}{\partial w}$.

Proof. See Section A.4. □

Remark 1. When we talk about a system being stable or dissipative on (a set) $\mathcal{X} \times \mathcal{W}$, we mean the system is stable or dissipative under all trajectories of the system for which holds that $(x(t), w(t)) \in \mathcal{X} \times \mathcal{W}$ for all $t \in \mathbb{Z}_0^+$.

Later, in Section 4, we will also show how, for performance in terms of the ℓ_2-gain and passivity, the results of Theorem 6 can be turned into LMIs. With the result of Theorem 6, we have a novel condition to analyse velocity dissipativity of DT nonlinear systems. This is enabled by the fact that, in the proof, it is shown that the condition for (Q, S, R)-VD in DT can be expressed in terms of the matrix functions $A_\mathrm{v}, \dots, D_\mathrm{v}$ instead of the matrix functions $\bar{A}_\mathrm{v}, \dots, \bar{D}_\mathrm{v}$ of the DT velocity form Equation (18). Expressing the condition for VD in terms of $A_\mathrm{v}, \dots, D_\mathrm{v}$ instead of $\bar{A}_\mathrm{v}, \dots, \bar{D}_\mathrm{v}$ simplifies it. Namely, $A_\mathrm{v}, \dots, D_\mathrm{v}$ only depend on two arguments, which results in the condition needing to be verified at all $(x, w) \in \mathcal{X} \times \mathcal{W}$. On the other hand, a condition using $\bar{A}_\mathrm{v}, \dots, \bar{D}_\mathrm{v}$ takes four arguments and would need to be verified at all $(x, w) \in \mathcal{X} \times \mathcal{W}$ and all $(x_+, w_+) \in \mathcal{X} \times \mathcal{W}$.

Moreover, note that the condition in Theorem 6 corresponds to a feasibility check of an infinite dimensional set of LMIs, as for a fixed $(x, w) \in \mathcal{X} \times \mathcal{W}$, Equation (24) becomes an LMI. Later, in Section 6, we will see how we can reduce this infinite dimensional set of LMIs to a finite dimensional set, which can computationally efficiently be verified. This will then give us efficient tools to analyse (Q, S, R)-VD of a system.

3.2 | Induced universal shifted stability

In the literature, see [6, 26, 27], it has been shown how the velocity form in CT can be used to formulate a condition to imply US(A)S of CT systems. Likewise, we will show that also in DT, we can formulate a condition for US(A)S of a system using the DT velocity form that we have introduced in Definition 5. Before doing so, let us first introduce the behaviour $\mathfrak{B}_{\mathrm{v},\mathscr{W}} := \bigcup_{w_* \in \mathscr{W}} \mathfrak{B}_{\mathrm{v,w}}(w \equiv w_*)$, i.e. the behaviour of the velocity form for which the input is $w(t) = w_* \in \mathscr{W}$, hence $w_\Delta(t) = 0$, for all $t \in \mathbb{Z}_0^+$.

Theorem 7 (Implied universal shifted stability). *The nonlinear system given by Equation (1) is USS, if there exists a function* $V_\mathrm{v} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ *with* $V_\mathrm{v} \in \mathcal{C}_0$ *and* $V_\mathrm{v} \in \mathcal{Q}_0$, *such that Equation (20) holds for all* $t \in \mathbb{Z}_0^+$ *and* $x_\Delta \in \pi_{x_\Delta} \mathfrak{B}_{\mathrm{v},\mathscr{W}}$, *i.e. the system is velocity stable. If Equation (20) holds, but with strict inequality except when* $x_\Delta(t) = 0$, *meaning it is velocity asymptotically stable, then the system is USAS.*

Proof. See Section A.5. □

The proof for Theorem 7 relies on the construction of the universally shifted Lyapunov function based on V_v. In CT, a similar construction is often referred to as the Krasovskii method [1, 6, 27]. However, the novel result and construction that we present in Theorem 7 for DT nonlinear systems are, to the authors' knowledge, not available in literature. Moreover, to the authors' knowledge, this is also the first time that properties of the time-difference dynamics have been connected to US(A)S of the system. Note that condition Equation (20) means that (asymptotic) stability of the velocity form (5) implies US(A)S of system Equation (1). Which implies that by analysing (asymptotic) stability of the velocity form Equation (18), we can infer US(A)S of the primal form.

Using Theorem 7, we can also connect velocity dissipativity to US(A)S of the nonlinear system:

Theorem 8 (USS from VD). *Assume the nonlinear system given by Equation (1) is VD under a storage function* $\mathcal{V}_\mathrm{v} \in \mathcal{C}_1$ *w.r.t. a supply function* s_v *that satisfies*

$$s_\mathrm{v}(0, z_\mathrm{v}) \leq 0, \quad (25)$$

for all $z_\mathrm{v} \in \mathbb{R}^{n_\mathrm{z}}$, *then, the nonlinear system is USS. If the supply function satisfies Equation (25), but with strict inequality when* $x_\Delta \neq 0$, *then the nonlinear system is USAS.*

Proof. See Section A.6. □

Corollary 1 (VD-condition induced universal shifted stability). *For the nonlinear system given by Equation (1), let Equation (24) hold on the convex set* $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{x} \cdot n_\mathrm{w}}$ *w.r.t. a supply function* s_v *that satisfies Equation (25), i.e. the system is VD and V(A)S on* $\mathcal{X} \times \mathcal{W}$. *Then, for any input* $w \equiv w_* \in \mathcal{W}$, *the system is US(A)S and invariant on* $\mathbb{X}_{w_*,\gamma}$ *given by Equation (7) with* $V_\mathrm{s}(x(t), w_*) = \mathcal{V}_\mathrm{v}(f(x(t), w_*) - x(t))$, *if* $\gamma \geq 0$ *satisfies* $\mathbb{X}_{w_*,\gamma} \subseteq \mathcal{X}$.

The proof simply follows from the fact that the system is US(A)S by Lemma 8, which by Theorem 7 implies the system is US(A)S w.r.t. the (universal shifted) Lyapunov function $V_\mathrm{s}(x(t), w_*) = \mathcal{V}_\mathrm{v}(f(x(t), w_*) - x(t))$. By Theorem 2, this then implies universal shifted invariance for any input $w \equiv w_* \in \mathcal{W}$. Note that Corollary 1 means that verification of Equation (24) on a convex set $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{x} \cdot n_\mathrm{w}}$ only implies universal shifted stability on the maximum invariant set $\mathbb{X}_{w_*,\gamma}$, constructed based on the function V_s assembled from $\mathcal{V}_\mathrm{v}$, which is still contained in $\mathcal{X}$. This is due to the fact that we can only give guarantees for $(x(t), w(t)) \in \mathcal{X} \times \mathcal{W}$ for all $t \in \mathbb{Z}_0^+$, as also stated in Remark 1. For initial conditions in $\mathcal{X} \setminus \mathbb{X}_{w_*,\gamma}$, there is no guarantee that the state trajectory will not leave $\mathcal{X}$ momentarily and take values where Equation (24) has not been verified. Increasing the sets $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$ allows one to conclude US(A)S on larger regions of the state space.

With these results, we have shown so far that velocity stability and dissipativity imply universal shifted stability and invariance of nonlinear systems.

3.3 | Induced universal shifted dissipativity

Next, we are interested if (Q, S, R)-VD also implies (Q, S, R)-USD. In [6], this is also investigated for CT nonlinear systems. However, a full proof for the implication that (Q, S, R)-VD implies (Q, S, R)-USD is not presented and to the authors' knowledge does not exist in the literature for either CT or DT. In this section, we will present novel dual DT conditions that link (Q, S, R) velocity dissipativity and (Q, S, R) universal shifted dissipativity.

Instead of considering nonlinear systems that can be represented in the form of Equation (1), in this section, for technical reasons, we will restrict ourselves to nonlinear systems that can be represented as

$$x(t+1) = f(x(t)) + Bw(t); \tag{26a}$$

$$y(t) = Cx(t). \tag{26b}$$

For a system represented by Equation (1), we can transform Equation (1) to the form Equation (26) at the cost of increasing the state dimension, e.g. by using appropriate input and output filters (see e.g. [6, Appendix II]). For Equation (26), we will also assume in this section that $x(t) \in \mathcal{X}$, with $\mathcal{X}$ being convex and compact.

For a nonlinear system given by Equation (26), the equilibrium points $(x_*, w_*, z_*) \in \mathscr{E}$ satisfy

$$x_* = f(x_*) + Bw_*; \tag{27a}$$

$$y_* = Cx_*; \tag{27b}$$

and the velocity form of Equation (26) is given by

$$x_\Delta(t) = \bar{A}_v(x(t+1), x(t))x_\Delta(t) + Bw_\Delta(t); \tag{28a}$$

$$z_\Delta(t) = Cx_\Delta(t); \tag{28b}$$

for which $\bar{A}_v(x_+, x) = \int_0^1 \frac{\partial f}{\partial x}(x + \lambda(x_+ - x))\, d\lambda$.

We will next connect (Q, S, R)-VD for (Q, S, R) tuples with $S = 0$, $Q \succeq 0$, and $R \preceq 0$ to USP notions that can be characterized by a similar (Q, S, R) universal shifted supply function. We take the following assumptions:

Assumption 2. For the nonlinear system given by Equation (26), assume that $CB = 0$.

While Assumption 2 may seem restrictive, it can relatively easily be satisfied by interconnecting low pass filters to the inputs and outputs of the system, e.g. see [6, Appendix II]. Furthermore, we take the following commonly used assumption in literature [28, 29], namely that the (generalized) disturbances are generated by a stable exosystem:

Assumption 3. For a given $(x_*, w_*, z_*) \in \mathscr{E}$ and $\beta \in \mathbb{R}_0^+$, assume that w is generated by the exosystem

$$w(t+1) = A_w(w(t) - w_*) + w_*, \tag{29}$$

where $A_w \in \mathbb{R}^{n_w \times n_w}$ is Schur and $\|A_w - I\| \leq \beta$. The corresponding signal behaviour is

$$\mathfrak{W}_{(w_*, \beta)} := \left\{ w \in \mathcal{W}^{\mathbb{Z}_0^+} \mid w \text{ satisfies Equation (29)} \right\}. \tag{30}$$

Before presenting our results, we first give the following technical proposition:

Proposition 1. *Given a matrix $R \in \mathbb{S}^{n_z}$ with $R \preceq 0$, then there exists an $\alpha \in \mathbb{R}^+$, such that for all $x_* \in \mathfrak{X}$ and $x \in \mathcal{X}$*

$$(\star)^\top RC\left(\bar{A}_v(x, x_*) - I\right)(x - x_*) \leq \alpha^{-1}(\star)^\top RC(x - x_*). \tag{31}$$

In case that $\bar{A}_v$ is bounded, there always exists an α for a given R such that the condition in Proposition 1 holds, as $\mathcal{X}$ is compact.

Under Assumptions 2 and 3, we can show the following result:

Theorem 9 (USP from VD). *If a nonlinear system given by Equation (26) is (Q, S, R)-VD with $S = 0$, $Q \succeq 0$, $R \preceq 0$, and R satisfies the condition in Proposition 1, then under Assumptions 2 and 3, for every $(x_*, w_*, z_*) \in \mathscr{E}$, it holds that*

$$\sum_{t=0}^{T} \beta^2(\star)^\top Q(w(t) - w_*) + \alpha^{-1}(\star)^\top R(z(t) - z_*) \geq 0, \tag{32}$$

for all $T \geq 0$ and $(w, z) \in \pi_{w,z}\mathfrak{B}$ with $w \in \mathfrak{W}_{(w_*, \beta)}$ and[1] $x_\Delta(0) = 0$.

Proof. See Section A.7. □

Applying the result of Theorem 9 to the (Q, S, R) tuple $(Q, S, R) = (\gamma^2 I, 0, -I)$, corresponding to (universal shifted) ℓ_2-gain, we obtain the following corollary:

Corollary 2 (Bounded ℓ_{s2}-gain from velocity dissipativity). *If a nonlinear system given by Equation (26) is velocity (Q, S, R) dissipative for $(Q, S, R) = (\gamma^2 I, 0, -I)$, where $R = -I$ satisfies Proposition 1, then under Assumptions 2 and 3, the system has an ℓ_{s2}-gain bound of $\tilde{\gamma} = \sqrt{\alpha\beta^2\gamma^2}$.*

[1] The results can also be extended to $x_\Delta(0) \neq 0$, which will introduce an additional constant positive term on the left-hand side of Equation (32).

Note that this result follows from Theorem 9 by multiplying Equation (32) by α. The resulting inequality then corresponds to a (Q, S, R) US supply function with $(Q, S, R) = (\alpha\beta^2\gamma^2 I, 0, -I)$, which corresponds to an ℓ_{s2}-gain of $\tilde{\gamma} = \sqrt{\alpha\beta^2\gamma^2}$.

Combining these results with the result of Theorem 6 gives us a condition to analyse universal shifted performance of DT nonlinear systems in terms of an infinite dimensional set of LMIs given by Equation (24) on a chosen convex set $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$. Through Corollary 1, we can clearly characterize for any generalized disturbance $w(t) \in \mathcal{W}$ the region of the state space $\mathcal{X}$ where universal shifted stability and performance are guaranteed. As aforementioned, in Section 6, we will discuss how can we turn the infinite dimensional set of LMIs into a finite dimensional set in order to cast the analysis problem as convex optimization problem.

The results that we have presented in this section on the connection between universal shifted stability and performance and velocity analysis for DT systems can be seen as the dual of the CT results that have been presented in [6]. While the results in DT that we have presented in this paper are analogous to the CT results in [6], the proofs of the underlying results are very much different due to different nature of the time operators and the velocity forms in CT and DT.

Next, we will show how a different, but similar, dissipativity notion can be used to analyse incremental stability and performance of DT nonlinear systems.

4 | DIFFERENTIAL ANALYSIS

4.1 | The differential form

For CT systems, it has been show in [9] how dissipativity of the differential form implies incremental dissipativity. Similarly, in [22], preliminary results have also shown this for DT systems, however, under a restricted form of the storage function. In this section, we provide a novel generalization of these results to show how differential dissipativity implies incremental dissipativity under a state-dependent storage function.

Let us first introduce the following notation: $\Gamma(\varphi, \tilde{\varphi})$ denotes the set of (smooth) paths between points $\varphi, \tilde{\varphi} \in \mathbb{R}^n$, i.e.

$$\Gamma(\varphi, \tilde{\varphi}) := \{\bar{\varphi} \in (\mathbb{R}^n)^{[0,1]} \mid \bar{\varphi} \in \mathcal{C}_1, \bar{\varphi}(0) = \tilde{\varphi}, \bar{\varphi}(1) = \varphi\}. \tag{33}$$

Next, consider two arbitrary trajectories of the system Equation (1): (x, w, z), $(\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$. We parameterize any two trajectories between these in terms of a path connecting their initial conditions: $\bar{x}_0 \in \Gamma(x_0, \tilde{x}_0)$ and a path connecting their input trajectories: $\bar{w}(t) \in \Gamma(w(t), \tilde{w}(t))$, resulting in the state transition map $\bar{x}(t, \lambda) = \phi_x(t, t_0, \bar{x}_0(\lambda), \bar{w}(\lambda)) \in \mathbb{R}^{n_x}$. This gives that for any $\lambda \in [0, 1]$ and all $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$, it holds that

$$\bar{x}(t+1, \lambda) = f(\bar{x}(t, \lambda), \bar{w}(t, \lambda)); \tag{34a}$$

$$\bar{z}(t, \lambda) = h(\bar{x}(t, \lambda), \bar{w}(t, \lambda)); \tag{34b}$$

where $(\bar{x}(\lambda), \bar{w}(\lambda), \bar{z}(\lambda)) \in \mathfrak{B}$. Note that for $\lambda = 0$, we obtain $(\bar{x}(0), \bar{w}(0), \bar{z}(0)) = (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$, while for $\lambda = 1$, we get $(\bar{x}(1), \bar{w}(1), \bar{z}(1)) = (x, w, z) \in \mathfrak{B}$. Differentiating the parameterized dynamics w.r.t. λ, results in the so-called differential form of Equation (1), given by

$$x_\delta(t+1) = A_\delta(\bar{x}(t), \bar{w}(t))x_\delta(t) + B_\delta(\bar{x}(t), \bar{w}(t))w_\delta(t); \tag{35a}$$

$$z_\delta(t) = C_\delta(\bar{x}(t), \bar{w}(t))x_\delta(t) + D_\delta(\bar{x}(t), \bar{w}(t))w_\delta(t); \tag{35b}$$

where we omitted dependency on λ for the sake of readability. In Equation (35), $x_\delta(t, \lambda) = \frac{\partial \bar{x}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_x}$, $w_\delta(t, \lambda) = \frac{\partial \bar{w}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_w}$, $z_\delta(t, \lambda) = \frac{\partial \bar{z}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_z}$, and

$$A_\delta = \frac{\partial f}{\partial x}, \quad B_\delta = \frac{\partial f}{\partial w}, \quad C_\delta = \frac{\partial h}{\partial x}, \quad D_\delta = \frac{\partial h}{\partial w}, \tag{36}$$

where $(\bar{x}(\lambda), \bar{w}(\lambda)) \in \pi_{x,w}\mathfrak{B}$ for all $\lambda \in [0, 1]$. The differential form represents the dynamics of the variations along the trajectories of the system represented by Equation (1).

The differential form allows us to define differential stability:

Definition 8 (Differential stability). The nonlinear system given by Equation (1) with differential form Equation (35) is differentially (asymptotically) stable (D(A)S), if the differential form is (asymptotically) stable in the Lyapunov sense w.r.t. the origin (see also Definition 1), i.e. the differential state x_δ is (asymptotically) stable w.r.t. 0.

Similar to the definition of velocity stability in Definition 6, differential stability considers standard stability of the differential form. Results for this have been discussed in [13, 14], which we will briefly recap:

Theorem 10 Differential Lyapunov stability [13, 14]. *The nonlinear system given by Equation (1) is DS, if there exists a function $V_\delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ with $V_v \in \mathcal{C}_0$ and $V_\delta(\bar{x}, \cdot) \in \mathcal{Q}_0, \forall \bar{x} \in \mathbb{R}^{n_x}$, such that*

$$V_\delta(\bar{x}(t+1), x_\delta(t+1)) - V_\delta(\bar{x}(t), x_\delta(t)) \leq 0, \tag{37}$$

for all $t \in \mathbb{Z}_0^+$ and for all $\bar{x} \in \pi_x\mathfrak{B}_w(w)$ under all measurable and bounded $w \in (\mathbb{R}^{n_w})^{\mathbb{Z}_0^+}$. If Equation (37) holds, but with strict inequality except when $x_\delta(t) = 0$, then the system is DAS.

See also [13, 14] for the proof. Similarly, using the differential form, we formulate the definition of differential dissipativity, which so far has received little attention in literature in the DT setting:

Definition 9 (Differential dissipativity). Consider the system given by Equation (1) and its differential form Equation (35). The system is *Differentially Dissipative* (DD) w.r.t. a supply function $s_\delta : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \to \mathbb{R}$, if there exists a storage function $\mathcal{V}_\delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ with $\mathcal{V}_\delta \in \mathcal{C}_0$ and $\mathcal{V}_\delta(\bar{x}, \cdot) \in$

$\mathcal{Q}_0$, $\forall\,\bar{x} \in \mathbb{R}^{n_x}$, such that

$$\mathcal{V}_\delta\big(\bar{x}(t_1+1), x_\delta(t_1+1)\big) - \mathcal{V}_\delta\big(\bar{x}(t_0), x_\delta(t_0)\big) \leq \sum_{t=t_0}^{t_1} s_\delta\big(w_\delta(t), z_\delta(t)\big), \tag{38}$$

for all $(\bar{x}, \bar{w}) \in \pi_{x,u}\mathfrak{B}$ and for all $t_0, t_1 \in \mathbb{Z}_0^+$, with $t_0 \leq t_1$.

As in Section 3, we consider quadratic supply functions of the form

$$s_\delta(w_\delta,\, z_\delta) = \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}. \tag{39}$$

Differential dissipativity w.r.t. supply functions of this form will be referred to as (Q, S, R)-DD.

Note that checking (Q, S, R)-DD of the (primal form of the) system Equation (1) can be seen as checking "classical (Q, S, R) dissipativity" of the differential form of the system.

For CT systems in [9], it has been show how (Q, S, R)-DD can be analysed through feasibility check of a(n) (infinite dimensional) set of LMIs. In DT, this has also been shown in [22]. However, the work in [22] only considers a quadratic (differential) storage function with a constant matrix. As a contribution of this paper, we will show that how to formulate a similar result using a quadratic storage function with a state-dependent matrix, and importantly how this condition will imply incremental dissipativity of the nonlinear system. Due to the state-dependent nature of the (differential) storage function, showing this implication is more involved. As mentioned, to obtain the results, we consider storage functions of a quadratic form:

$$\mathcal{V}_\delta(\bar{x}, x_\delta) = x_\delta^\top M(\bar{x}) x_\delta, \tag{40}$$

with M satisfying the following condition:

Condition 1. The matrix function $M : \mathbb{R}^{n_x} \to \mathbb{S}^{n_x}$ with $M \in \mathcal{C}_1$ is real, symmetric, bounded and positive definite, i.e. $\exists\, \alpha_1, \alpha_2 \in \mathbb{R}^+$, such that for all $\bar{x} \in \mathbb{R}^{n_x}$, $\alpha_1 I \preceq M(\bar{x}) \preceq \alpha_2 I$.

Moreover, for the system Equation (1), let us also consider the set $\mathcal{D}$, which is the smallest convex set such that $(x(t+1) - x(t)) \in \mathcal{D}$ for all $t \in \mathbb{Z}_0^+$ corresponding to a given set $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$. This allows us to obtain the following result to analyse differential dissipativity in DT:

Theorem 11 ((Q, S, R)-DD condition). *A nonlinear system given by Equation (1) is (Q, S, R)-DD on $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$, if there exists a storage function Equation (40) with M satisfying Condition 1, such that*

$$(\star)^\top \begin{bmatrix} -M(\bar{x}) & 0 \\ 0 & M(\bar{x}+\bar{x}_v) \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\delta(\bar{x},\bar{w}) & B_\delta(\bar{x},\bar{w}) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\delta(\bar{x},\bar{w}) & D_\delta(\bar{x},\bar{w}) \end{bmatrix} \preceq 0, \tag{41}$$

for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\bar{x}_v \in \mathcal{D}$.

Proof. See Section A.8. □

Note that Equation (41) is similar to the check for (Q, S, R)-VD in Equation (24) in Theorem 6. We will discuss the connections and similarities between differential and velocity dissipativity in more detail in Section 5.

Later, in Section 6, we will discuss how we can formulate computable tests for checking feasibility of this infinite dimensional set of LMIs.

4.2 | Induced incremental stability

Before showing how (Q, S, R) differential dissipativity implies (Q, S, R) incremental dissipativity, we will first briefly discuss how it connects to I(A)S.

In literature, the connections between differential dynamics and incremental stability have extensively been discussed, also for DT nonlinear systems [13, 14]. For completeness, we will briefly recap these results:

Lemma 1 (Implied incremental stability). *The nonlinear system given by Equation (1) is incrementally stable, if there exists a quadratic storage Lyapunov function V_δ of the form Equation (40) with M satisfying Condition 1, such that*

$$V_\delta(\bar{x}(t+1), x_\delta(t+1)) - V_\delta(\bar{x}(t), x_\delta(t)) \leq 0 \tag{42}$$

for all $\bar{x} \in \pi_x\mathfrak{B}_w(w)$ under all measurable and bounded $w \in (\mathbb{R}^{n_w})^{\mathbb{Z}_0^+}$. If Equation (42) holds, but with strict inequality except when $x(t) = \tilde{x}(t)$, corresponding to $x_\delta(t) = 0$, then the system is incrementally asymptotically stable.

Similar to the implication of US(A)S from velocity dissipativity, see Lemma 8, we also have that I(A)S is implied form differential dissipativity under restrictions of the supply function, see also [9, Remark 11] for these results in CT.

Theorem 12 (IS from DD). *Assume the nonlinear system given by Equation (1) is DD under a storage function $\mathcal{V}_\delta$ of the form Equation (40) with M satisfying Condition 1 w.r.t. a supply function s_δ that satisfies*

$$s_\delta(0, z_\delta) \leq 0, \tag{43}$$

for all $z_\delta \in \mathbb{R}^{n_z}$, then, the nonlinear system is IS. If the supply function satisfies Equation (43), but with strict inequality when $x(t) \neq \tilde{x}(t)$, then the nonlinear system is IAS.

Proof. See Section A.9. □

Furthermore, we can also connect this result directly to the DD condition:

Corollary 3 (Induced incremental stability and invariance). *For the nonlinear system given by Equation (1), let Equation (41) holds on the convex set $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$ w.r.t. a supply function s_δ that satisfies Equation (43), i.e. the system is DD and D(A)S on $\mathcal{X} \times \mathcal{W}$. Then,* *for any input $w \in (\mathcal{W})^{\mathbb{Z}_0^+}$, there exists a $\gamma \geq 0$, such that the system is I(A)S and invariant in the tube $\mathbb{X}_{x,\gamma}(t) \subseteq \mathcal{X}, \forall t \in \mathbb{Z}_0^+$ given by Equation (10) around the trajectory (x, w) with $x \in \pi_x \mathfrak{B}_w(w)$ and $x(t) \in \mathcal{X}, \forall t \in \mathbb{Z}_0^+$.*

The proof simply follows from the fact that the system is I(A)S by Lemma 12, which by Lemma 1 implies I(A)S under the (incremental) Lyapunov function $V_i(x, \tilde{x}) = \mathcal{V}_i(x, \tilde{x})$ with $\mathcal{V}_i$ given by[2] Equation (A.35). By Theorem 4, this then implies incremental invariance for any $w \in (\mathcal{W})^{\mathbb{Z}_0^+}$ around $x \in \pi_x \mathfrak{B}_w(w)$ with $x(t) \in \mathcal{X}$. The implications are very similar to the velocity case, meaning that verifying Equation (41) on a bounded set $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$ only allows one to conclude I(A)S w.r.t an appropriate set of initial conditions $\mathbb{X}_{x,\gamma}(0)$ in $\mathcal{X}$, ensuring invariance (convergence) of all solutions along the signal (x, w). Increasing the sets $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$, allows one to conclude I(A)S on larger regions of the state space.

4.3 | Induced incremental dissipativity

Next, we will present how we can use (Q, S, R) differential dissipativity in order to analyse (Q, S, R) incremental dissipativity of DT nonlinear systems under mild restrictions of the supply function.

Theorem 13 (Induced incremental dissipativity). *If the nonlinear system given by Equation (1) is (Q, S, R)-DD with $R \preceq 0$ under a storage function $\mathcal{V}_\delta$ of the form Equation (40), then the system is (Q, S, R)-ID for the same tuple (Q, S, R).*

Proof. See Section A.10. □

We can then combine the results of Theorems 11 and 13 and Corollary 3 to arrive at the following corollary:

Corollary 4 (Incremental dissipativity condition). *If for a given (Q, S, R) with $R \preceq 0$ Equation (41) holds for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $x_v \in \mathcal{D}$ with M satisfying Condition 1, then the nonlinear system given by Equation (1) is (Q, S, R)-ID on $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$ in the following sense. For any inputs $w, \tilde{w} \in (\mathcal{W})^{\mathbb{Z}_0^+}$ and the corresponding invariant tubes $\mathbb{X}_{x,\gamma}(t)$, $\mathbb{X}_{\tilde{x},\gamma}(t)$ in terms of Corollary 3, all trajectories $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ with $x(0) \in \mathbb{X}_{x,\gamma}(0)$ and $\tilde{x}(0) \in \mathbb{X}_{\tilde{x},\gamma}(0)$ satisfy Equation (13) with the (Q, S, R) quadratic supply function.*

With these results, we have a powerful tool, through the matrix inequality condition in Theorem 11, to check (Q, S, R)-ID of DT systems. In [22], specifically Theorems 10 and 12, it has been shown how in DT, an infinite dimensional set of LMIs can be formulated in order to analyse the incremental ℓ_2-gain and incremental passivity of a nonlinear system for a quadratic storage function of the form Equation (40) where M is constant. Using the results of Theorems 13 and Corollary 4, we can now also extend those results to the case with a state-dependent matrix M:

Corollary 5 (Incremental ℓ_2-gain analysis). *A nonlinear system given by Equation (1) has a finite incremental ℓ_2-gain of γ on $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$, if there exists a matrix function M satisfying Condition 1 such that for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\bar{x}_v \in \mathcal{D}$*

$$\begin{bmatrix} M(\bar{x} + \bar{x}_v) & A_\delta(\bar{x}, \bar{w}) M(\bar{x}) & B_\delta(\bar{x}, \bar{w}) & 0 \\ \star & M(\bar{x}) & 0 & M(\bar{x}) C_\delta(\bar{x}, \bar{w})^\top \\ \star & \star & \gamma I & D_\delta(\bar{x}, \bar{w})^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{44}$$

Corollary 6 (DT incremental passivity analysis). *A nonlinear system given by Equation (1) is incrementally passive on $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x \cdot n_w}$, if there exists a matrix function M satisfying Condition 1, such that for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\bar{x}_v \in \mathcal{D}$*

$$\begin{bmatrix} M(\bar{x} + \bar{x}_v) & A_\delta(\bar{x}, \bar{w}) M(\bar{x}) & B_\delta(\bar{x}, \bar{w}) \\ \star & M(\bar{x}) & M(\bar{x}) C_\delta(\bar{x}, \bar{w})^\top \\ \star & \star & D_\delta(\bar{x}, \bar{w}) + (\star) \end{bmatrix} \succeq 0. \tag{45}$$

The results of Theorems 5 and 6 give us conditions for a bounded incremental ℓ_2-gain and incremental passivity of nonlinear systems represented by Equation (1). These conditions are both given in terms of feasibility checks of an infinite dimensional set of LMIs. As mentioned, in Section 6, we will discuss how we can formulate computable tests to check these conditions.

5 | THE RELATION BETWEEN VELOCITY AND DIFFERENTIAL DISSIPATIVITY

In Section 3, we have shown how (dissipativity) properties of the velocity form Equation (18) imply universal shifted properties of the primal form Equation (1) of the nonlinear system. Similarly, in Section 4, we have shown how dissipativity properties of the differential form Equation (35) imply incremental properties of the primal form. In both these cases, we use an alternative form of the system, the velocity and differential form, to imply equilibrium-free properties of the corresponding primal form of the nonlinear system.

Based on their definitions, it is clear that incremental properties imply universal shifted properties of the system, as incremental properties are properties between all the trajectories, while universal shifted properties are only between trajectories and equilibrium points. Similarly, we also have that (Q, S, R)-DD implies (Q, S, R)-VD. Namely, when we compare Equation (41) in Theorem 11 to Equation (24) in Theorem 6, it is evident that these conditions become equivalent in case M in Equation (41) is a constant matrix. This means that with condition Equation (24) for (Q, S, R)-VD in Theorem 6, we actually imply the stronger notion of (Q, S, R)-DD. This is because in Theorem 6, we do not explicitly exploit the fact that x_Δ, w_Δ,

[2] This will be derived in the proof of Theorem 13.

and z_Δ represent actual time differences in the state, input, and output, respectively. This leads to conservativeness of the actual result, making it in this case equivalent to checking (Q, S, R)-DD of the system. Quantifying how conservative these results are is however difficult, as this will also highly depend on the dynamical system that is considered in the analysis. For example, for LTI systems, there is no conservativeness whatsoever, as incremental dissipativity is equivalent to universal shifted dissipativity as well as to "classical" dissipativity.

This does not mean that (Q, S, R)-VD in this form does not have any use. Namely, similarly in the CT case in [6], it has been shown how properties of velocity form can still be exploited for controller synthesis in order to achieve closed-loop US stability and performance. Therefore, the velocity based analysis results presented in this paper can serve as a similar stepping stone for developing controller synthesis algorithms for DT nonlinear systems with such stability and performance guarantees.

6 | CONVEX EQUILIBRIUM-FREE ANALYSIS

In the previous sections, we have shown how the DT velocity and differential forms can be used to imply universal shifted and incremental stability and performance properties of the nonlinear system, respectively. In these sections, we have also discussed how through the (infinite dimensional set of) LMIs in Equations (24) and (41), (Q, S, R)-VD and (Q, S, R)-DD can be analysed. In the analysis of LPV systems, similar problems are encountered due to the variation of the scheduling-variable for which various tools have been developed to make these problems computationally tractable. Hence, as for the CT case in [6, 9, 22], we can use the analysis results of the LPV framework to turn the proposed checks for universal shifted and incremental stability and performance analysis to convex finite dimensional optimization problems which can be efficiently solved as SDPs.

As discussed in Section 5, for a constant matrix M, the matrix inequalities in Equations (24) and (41) are equivalent. Therefore, we will only discuss the (Q, S, R)-DD case Equation (41), as the (Q, S, R)-VD case trivially follows from it. Furthermore, we discussed in Section 4 that the (Q, S, R)-DD condition can be interpreted as analysing classical (Q, S, R) dissipativity of the differential form Equation (35). Therefore, to cast the (Q, S, R)-DD analysis problem to an LPV analysis problem, we embed the differential form of the nonlinear system in an LPV representation, which we call a differential parameter-varying (DPV) embedding of the nonlinear system Equation (1):

Definition 10 (DPV embedding). Given a nonlinear system in the form of Equation (1) with differential form given by Equation (35). Then, the LPV representation given by

$$x_\delta(t+1) = A(p(t))x_\delta(t) + B(p(t))w_\delta(t), \tag{46a}$$

$$z_\delta(t) = C(p(t))x_\delta(t) + D(p(t))w_\delta(t), \tag{46b}$$

where $p(t) \in \mathcal{P} \subset \mathbb{R}^{n_\mathrm{p}}$ is the scheduling-variable, is a DPV embedding of Equation (1) on the compact convex region $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}}$, if there exists a function $\eta : \mathcal{X} \times \mathcal{W} \to \mathcal{P}$, called the scheduling-map, such that under a given choice of function class for $A, \ldots, D$, e.g. affine, polynomial etc., $A(\eta(\bar{x}, \bar{w})) = A_\delta(\bar{x}, \bar{w}), \ldots, D(\eta(\bar{x}, \bar{w})) = D_\delta(\bar{x}, \bar{w})$ for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\eta(\mathcal{X}, \mathcal{W}) \subseteq \mathcal{P}$.

Let us denote by $v(t) = p(t+1) - p(t) \in \Pi$. We assume that the set Π is considered such that it includes $(x(t+1) - x(t)) \in \mathcal{D}$. Through the DPV embedding, we can then cast the (Q, S, R)-DD (and (Q, S, R)-VD) check as an LPV analysis problem:

Theorem 14 (LPV based (Q, S, R)-DD condition). *A nonlinear system given by Equation (1) with a corresponding DPV embedding given by Equation (46) on $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}}$ is (Q, S, R)-DD on $\mathcal{X} \times \mathcal{W}$, if there exists a matrix function $M : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$ satisfying Condition 1, such that*

$$(\star)^\top \begin{bmatrix} -M(p) & 0 \\ 0 & M(p+v) \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{47}$$

for all $p \in \mathcal{P}$ and $v \in \Pi$.

Proof. See Section A.11. □

Note that the proposed analysis condition Equation (47) can be seen as a classical (Q, S, R) dissipativity analysis condition of an LPV representation. Therefore, all the techniques from the LPV framework can be used to reduce the infinite dimensional set of LMIs to a finite set. The most common techniques for this are polytopic techniques [30, 31], grid-based techniques [32, 33], and multiplier-based techniques [34, 35], see also [36] for an overview. For the polytopic and multiplier-based techniques, $A, \ldots, D$ are needed to be restricted to an affine or rational function in the embedding Equation (46), respectively. With the results of Theorem 14, combined with Theorem 13, we have a convex analysis condition in order to analyse (Q, S, R)-ID of DT nonlinear systems. Similarly, connecting to Theorem 6, we then also have convex analysis tools for universal shifted stability, through Theorem 7, and performance, through Theorem 9. Note that in these cases, it is important that we also induce an invariant set in $\mathcal{X} \times \mathcal{W}$ through the result of Corollaries 1 and 3. As this invariant set describes the region where the implied stability and performance conditions hold, we can also maximize its volume. For example, for a constant matrix M, the invariant set will correspond to an ellipsoidal region. There exist various result on casting the maximization of the volume of an ellipsoid as a convex problem, e.g. see [37]. However, this is outside the scope of this paper.

Also note that although the same tools from the LPV framework can be used for checking (Q, S, R)-DD, (Q, S, R)-VD and classical (Q, S, R) dissipativity of nonlinear systems, we

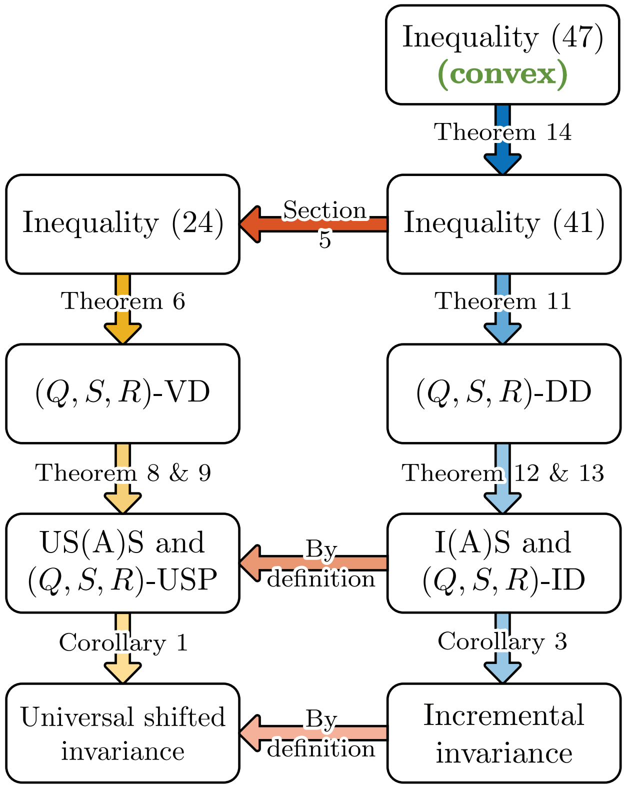

FIGURE 3 Overview of the results and their connections.

would like to emphasize that the underlying dissipativity and stability concepts and the matrix functions on which these sets are applied are very different. Namely, the (Q, S, R)-DD and (Q, S, R)-VD concepts connect to the equilibrium-free concepts of incremental and universal shifted stability and performance. This means that these analysis results are not dependent on a particular trajectory or equilibrium point, respectively. On the other hand, the standard LPV analysis results applied on a direct LPV embedding of a nonlinear system use classical dissipativity and can only provide performance and stability analysis with respect to single equilibrium point, often the origin of the state-space representation of the nonlinear system, which make them equilibrium-dependent.

An overview of all the results and their connections presented in this paper is given in Figure 3.

7 | EXAMPLE

In this section, we apply the results of the previous sections in order to analyse equilibrium-free stability and performance of a discrete-time nonlinear system.

We consider the following CT state-space representation of an actuated Duffing oscillator:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -8x_1 - 10x_1^3 - 4x_2 + w\end{bmatrix}; \tag{48a}$$

$$z = x_1, \tag{48b}$$

where x_1 represents the position of the spring, x_2 its velocity, and w is an input force. We discretize this model using a fourth order Runge–Kutta (RK4) method with a sampling time of 0.01 s, where we assume the input w to be constant in between samples. The resulting model is of the form Equation (1) and is not given due to its complexity. For our analysis, we consider the operating region $x_1(t) \in [-1\ 1]$, $x_2(t) \in [-1\ 1]$, $w(t) \in [-1\ 1]$ for all $t \in \mathbb{Z}_0^+$, i.e. we perform dissipativity analysis on the set $\mathcal{X} \times \mathcal{W}$ with $\mathcal{X} = [-1\ 1] \times [-1\ 1]$ and $\mathcal{W} = [-1\ 1]$. Based on these sets, we compute the corresponding set $\mathcal{D}$, which is given by: $\mathcal{D} = [-0.011\ 0.011] \times [-0.23\ 0.23]$ such that $x(t+1) - x(t) \in \mathcal{D}$. Consequently, we construct a DPV embedding of the nonlinear model on $\mathcal{X} \times \mathcal{W} = [-1\ 1] \times [-1\ 1] \times [-1\ 1]$. If we do a direct DPV embedding, we obtain as scheduling-map $\eta(x, w) = \mathrm{col}(x_1, x_2, w)$, therefore $p(t) = \mathrm{col}(x_1(t), x_2(t), w(t)) \in \mathcal{P}$ where $\mathcal{P} = [-1\ 1] \times [-1\ 1] \times [-1\ 1]$. Moreover, we consider $\Pi = \mathcal{D} \times \mathbb{R}$, such that $v(t) = p(t+1) - p(t) \in \Pi$.

Under these considerations, we minimize γ in Theorem 14 with $(Q, S, R) = (\gamma^2, 0, -I)$, corresponding to an incremental and universal shifted ℓ_2-gain bound of γ. We use a grid-based LPV approach and consider our quadratic storage matrix M to be of the form:[3]

$$M(p) = M_0 + M_1 p_1^2 \tag{49}$$

where $M_i \in \mathbb{R}^{2\times 2}$ for $i = 1, \dots, 2$. Our problem then corresponds to grid-based ℓ_2-gain analysis of an LPV representation, which has been implemented in the LPVcore Toolbox [38]. Using the LPVcore Toolbox, the resulting γ that we obtain is 0.13, which is then our upperbound for the incremental and universal shifted ℓ_2-gain bound of the discretized version of Equation (48) on the region $[-1\ 1] \times [-1\ 1] \times [-1\ 1]$.

Computing γ for a constant quadratic storage matrix M only results in a upperbound for the incremental and universal shifted ℓ_2-gain of 0.42. Therefore, this shows that for this example, the approach using a constant matrix M in the storage function presented in [22] is much more conservative than the approach using a state-dependent storage function presented in this paper for analysing the incremental and universal shifted ℓ_2-gain of DT systems.

We also simulate the system for two different inputs to additionally verify the equilibrium-free properties:

$$w(t) = 0.7e^{-t}\sin(2t) + 0.3\sin(0.2t), \tag{50}$$

$$\tilde{w}(t) = 0.3e^{-t}\cos(t) + 0.3\sin(0.2t), \tag{51}$$

and initial conditions $x_0 = \mathrm{col}(-0.08, 0.22)$ and $\tilde{x}_0 = \mathrm{col}(-0.50, -0.20)$, respectively. Note that these two input trajectories converge as $t \to \infty$. Therefore, as the system is incrementally stable, the state and output trajectories will also converge, as is visible in Figure 4. The cumulative incremental supply plus the initial storage

[3] Note that M only depends on the scheduling-variable p_1 which corresponds to the state x_1, consistent with Equation (41).

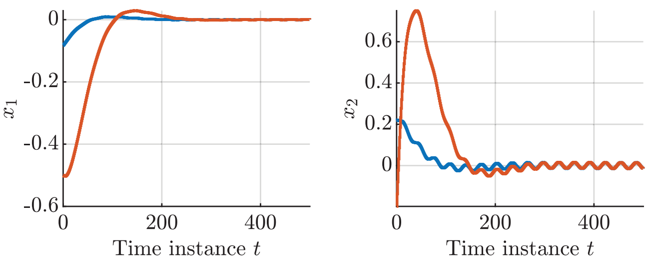

FIGURE 4 State trajectories for input w with initial condition x_0 (—) and for input $\tilde{w}$ with initial condition $\tilde{x}_0$ (—).

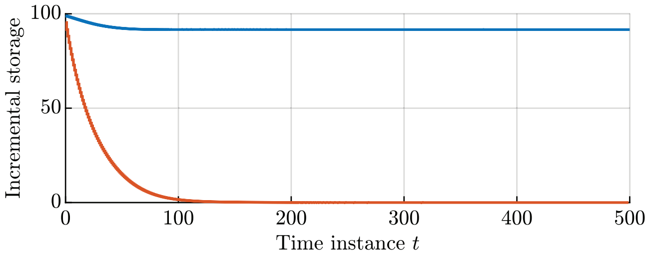

FIGURE 5 The cumulative incremental supply plus the initial storage $\sum_{\tau=0}^{t} s_{\mathrm{i}}(w(\tau), \tilde{w}(\tau), z(\tau), \tilde{z}(\tau)) + \mathcal{V}_{\mathrm{i}}(x(0), \tilde{x}(0))$ (—) and the incremental storage $\mathcal{V}_{\mathrm{i}}(x(t+1), \tilde{x}(t+1))$ (—) for the trajectories generated by the inputs w and $\tilde{w}$.

$\sum_{\tau=0}^{t} s_{\mathrm{i}} \left(w(\tau), \tilde{w}(\tau), z(\tau), \tilde{z}(\tau)\right) + \mathcal{V}_{\mathrm{i}} \left(x(0), \tilde{x}(0)\right)$ and the incremental storage $\mathcal{V}_{\mathrm{i}} \left(x(t+1), \tilde{x}(t+1)\right)$ are also plotted in Figure 5. The incremental storage $\mathcal{V}_{\mathrm{i}}$ is obtained by solving Equation (A.34) and using it in Equation (A.35). To simplify this computation, the integrals are approximated by summations and the smooth path $\chi_{(x,\tilde{x})}$ is approximated as a piecewise linear function. We then solve problem Equation (A.34) using `fmincon` in MATLAB. From Figure 5, it can be seen that the incremental storage is always smaller than or equal to the cumulative incremental supply plus the initial storage. Therefore, this also verifies that the system satisfies the ID inequality, see Equation (13), for these trajectories.

8 | CONCLUSIONS

In this paper, we have developed convex conditions for equilibrium-free analysis of discrete-time nonlinear systems. We have shown how time-difference dynamics can be used in order to analyse universal shifted stability and performance of discrete-time nonlinear systems. Similarly, we have shown how dissipativity of the differential form can be used in order to analyse incremental dissipativity using a state-dependent storage function. Finally, we have shown how both these analysis results can be cast as an analysis problem of an LPV representation. These results give us convex tools for equilibrium-free stability and performance analysis of discrete-time nonlinear systems. For future research, we aim to use these results in order to develop equilibrium-free controller synthesis methods for discrete-time nonlinear systems.

AUTHOR CONTRIBUTIONS

Patrick J. W. Koelewijn: Conceptualization; data curation; formal analysis; investigation; methodology; software; validation; visualization; writing—original draft; writing—review & editing. **Siep Weiland**: Conceptualization; resources; supervision; writing—review & editing. **Roland Tóth**: Conceptualization; methodology; funding acquisition; project administration; resources; supervision; writing—review & editing.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A: PROOFS

A.1 | Proof of Theorem 1

Consider the function $V : x \mapsto V_s(x, w_*)$, which for every $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$ satisfies the conditions for a Lyapunov function for the equilibrium point x_*. Namely, given a $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$, we have that $V = (x \mapsto V_s(x, w_*)) \in \mathcal{Q}_{x_*}$. Therefore, for a given $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$, by Equation (6), it holds that

$$V(x(t+1)) - V(x(t)) \leq 0, \tag{A.1}$$

for all $t \in \mathbb{Z}_0^+$ and $x \in \pi_x\mathfrak{B}_w(w \equiv w_*)$. Due to the properties of V_s and construction of V, Equation (A.1) then also holds for each $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$. Consequently, each equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$ is stable, under $w \equiv w_*$, for the whole state-space by the Lyapunov theory, see e.g. [1, 25]. Therefore, by Definition 1, it is USS. In a similar manner, if Equation (6) holds, but with a strict inequality except when $x(t) = x_*$, this implies that Equation (A.1) holds. This then implies asymptotic stability of each equilibrium point, meaning that the system is USAS.

A.2 | Proof of Theorem 2

Given $w_* \in \mathscr{W}$ and a $\gamma > 0$, define the set Equation (7). We have that Equation (6) holds for every $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$ and for all $t \in \mathbb{Z}_0^+$ and $x \in \pi_x\mathfrak{B}_w(w \equiv w_*)$. Therefore, for $x(0) \in \mathbb{X}_{w_*,\gamma}$, it holds that

$$V_s(x(t), w_*) \leq \cdots \leq V_s(x(1), w_*) \leq V_s(x(0), w_*) \leq \gamma, \tag{A.2}$$

for $t \geq 1$. Consequently, for the nonlinear system given by Equation (1) with initial condition $x(0) = x_0$ and input $w \equiv w_*$, we have by Equation (A.2) that $x(t) = \phi_x(t, 0, x_0, w \equiv w_*) \in \mathbb{X}_{w_*,\gamma}$ for all $t \in \mathbb{Z}_0^+$.

A.3 | Proof of Theorem 4

The proof follows similarly as the proof for Theorem 2. Given a trajectory $(\tilde{x}, w) \in \pi_{x,w} \in \mathfrak{B}_w(w)$ and a $\gamma > 0$ define the time-varying set Equation (10). We have that Equation (9) holds for all $t \in \mathbb{Z}_0^+$ and $x, \tilde{x} \in \pi_x \mathfrak{B}_w(w)$ under all measurable and bounded $w \in (\mathbb{R}^{n_z})^{\mathbb{Z}_0^+}$. Therefore, for $x(0) \in \mathbb{X}_{\tilde{x},\gamma}(0)$, it holds that

$$V_i(x(t), \tilde{x}(t)) \leq \cdots \leq V_i(x(1), \tilde{x}(1)) \leq V_i(x(0), \tilde{x}(0)) \leq \gamma, \tag{A.3}$$

for $t \geq 1$. Consequently, for the nonlinear system given by Equation (1) with initial condition $x(0) = x_0$ and input w, we have by Equation (A.3) that $x(t) = \phi_x(t, 0, x_0, w) \in \mathbb{X}_{\tilde{x},\gamma}(t)$ for all $t \in \mathbb{Z}_0^+$.

A.4 | Proof of Theorem 6

If Equation (24) holds for all $(x, w) \in \mathcal{X} \times \mathcal{W}$, we have by pre- and post multiplication of Equation (24) with $\mathrm{col}(x_\Delta, w_\Delta)^\top$ and $\mathrm{col}(x_\Delta, w_\Delta)$, respectively, that

$$\begin{aligned}
&(\star)^\top M(A_v(x,w)x_\Delta + B_v(x,w)w_\Delta) - x_\Delta^\top M x_\Delta \\
&- w_\Delta^\top Q w_\Delta - 2w_\Delta^\top S\big(C_v(x,w)x_\Delta + D_v(x,w)w_\Delta\big) \\
&- (\star)^\top R\big(C_v(x,w)x_\Delta + D_v(x,w)w_\Delta\big) \leq 0,
\end{aligned} \tag{A.4}$$

for all $x_\Delta \in \mathbb{R}^{n_x}$, $w_\Delta \in \mathbb{R}^{n_w}$, and $(x, w) \in \mathcal{X} \times \mathcal{W}$. As $\mathcal{X}$ and $\mathcal{W}$ are assumed to be convex, we can represent any $\bar{x} \in \mathcal{X}$ and $\bar{w} \in \mathcal{W}$ by a $\lambda \in [0, 1]$, $x_+, x \in \mathcal{X}$, and $w_+, w \in \mathcal{W}$, such that $\bar{x}(\lambda) = x + \lambda(x_+ - x)$ and $\bar{w}(\lambda) = w + \lambda(w_+ - w)$. Consequently, if Equation (A.4) holds, it also holds that

$$\begin{aligned}
&(\star)^\top M(A_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta) \\
&- x_\Delta^\top M x_\Delta - w_\Delta^\top Q w_\Delta \\
&- 2w_\Delta^\top S\big(C_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big) \\
&- (\star)^\top R\big(C_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big) \leq 0,
\end{aligned} \tag{A.5}$$

for any $\lambda \in [0, 1]$, $x_+, x \in \mathcal{X}$, $w_+, w \in \mathcal{W}$, $x_\Delta \in \mathbb{R}^{n_x}$ and $w_\Delta \in \mathbb{R}^{n_w}$. Hence, we also have by integration over λ that

$$\begin{aligned}
\int_0^1 &(\star)^\top M(A_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta) \\
&- x_\Delta^\top M x_\Delta - w_\Delta^\top Q w_\Delta \\
&- 2w_\Delta^\top S\big(C_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big) \\
&- (\star)^\top R\big(C_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big)\, d\lambda \leq 0,
\end{aligned} \tag{A.6}$$

for any $x_+, x \in \mathcal{X}$, $w_+, w \in \mathcal{W}$, $x_\Delta \in \mathbb{R}^{n_x}$ and $w_\Delta \in \mathbb{R}^{n_w}$. By [22, Lemma 16], as $M \succ 0$, we have that

$$\begin{aligned}
&(\star)^\top M\left(\int_0^1 A_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\, d\lambda\right) \\
&\leq \int_0^1 (\star)^\top M(A_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta)\, d\lambda,
\end{aligned} \tag{A.7}$$

and similarly, as $R \preceq 0$, we have that

$$\begin{aligned}
&(\star)^\top (\text{-}R)\left(\int_0^1 C_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\, d\lambda\right) \\
&\leq \int_0^1 (\star)^\top (\text{-}R)\big(C_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big)\, d\lambda.
\end{aligned} \tag{A.8}$$

Note that $A_v = \frac{\partial f}{\partial x}$, $B_v = \frac{\partial f}{\partial w}$, $C_v = \frac{\partial h}{\partial x}$, $D_v = \frac{\partial h}{\partial w}$. Hence, using the definition of $\bar{A}_v, \ldots, \bar{D}_v$ in Equation (19), we have

$$\begin{aligned}
&\int_0^1 A_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\, d\lambda \\
&\quad = \bar{A}_v(x_+, x, w_+, w)x_\Delta + \bar{B}_v(x_+, x, w_+, w)w_\Delta,
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
&\int_0^1 C_v(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_v(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\, d\lambda \\
&\quad = \bar{C}_v(x_+, x, w_+, w)x_\Delta + \bar{D}_v(x_+, x, w_+, w)w_\Delta.
\end{aligned} \tag{A.10}$$

Combining Equations (A.7)–(A.10) with Equation (A.6), we obtain that

$$\begin{aligned}
&(\star)^\top M\big(\bar{A}_v(x_+, x, w_+, w)x_\Delta + \bar{B}_v(x_+, x, w_+, w)w_\Delta\big) \\
&- x_\Delta^\top M x_\Delta - w_\Delta^\top Q w_\Delta \\
&- 2w_\Delta^\top S\big(\bar{C}_v(x_+, x, w_+, w)x_\Delta + \bar{D}_v(x_+, x, w_+, w)w_\Delta\big) \\
&- (\star)^\top R\big(\bar{C}_v(x_+, x, w_+, w)x_\Delta + \bar{D}_v(x_+, x, w_+, w)w_\Delta\big) \leq 0,
\end{aligned} \tag{A.11}$$

for any $x_+, x \in \mathcal{X}$, $w_+, w \in \mathcal{W}$, $x_\Delta \in \mathbb{R}^{n_x}$, and $w_\Delta \in \mathbb{R}^{n_w}$. Substituting $x_+ = x(t+1)$, $x = x(t)$, $x_\Delta = x(t+1) - x(t)$, $w = w(t)$, $w_\Delta = w(t+1) - w(t)$ in Equation (A.11) and summing over time from t_0 to t_1 where $t_0 \leq t_1$, we obtain Equation (21) where $\mathcal{V}_v$ is given by Equation (23) and s_v is given by Equation (22).

A.5 | Proof of Theorem 7

For each equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$, consider

$$V_s(x(t), w_*) := V_v(f(x(t), w_*) - x(t)) = V_v(x_\Delta(t)). \tag{A.12}$$

For each $(x_*, w_*, z_*) \in \mathscr{E}$, this choice implies that $\mathcal{V}_s(\cdot, w_*) \in \mathcal{Q}_{x_*}$, as $\mathcal{V}_v \in \mathcal{Q}_0$. Note that this requires uniqueness of the equilibrium points (see Assumption 1), as otherwise there exists multiple x_* for which $V_s(x_*, w_*) = 0$. By this choice of V_s, we have that for each $(x_*, w_*, z_*) \in \mathscr{E}$,

$$\begin{aligned} &V_s(x(t+1), w_*) - V_s(x(t), w_*) \\ &\quad = V_v(x_\Delta(t+1)) - V_v(x_\Delta(t)) \leq 0, \end{aligned} \tag{A.13}$$

for all $t \in \mathbb{Z}_0^+$ and $x_\Delta \in \pi_{x_\Delta}\mathfrak{B}_{v,w}(w \equiv w_*)$ and correspondingly for all $x \in \pi_x \mathfrak{B}_w(w \equiv w_*)$. This implies that Equation (6) holds for all $x \in \pi_x \mathfrak{B}_w(w \equiv w_*)$ and for all equilibrium points $(x_*, w_*) \in \pi_{x_*,w_*}\mathscr{E}$. Hence, by Theorem 1, Equation (1) is USS. USAS follows similarly by changing Equation (A.13) to a strict inequality.

A.6 | Proof of Theorem 8

Let the system given by Equation (1) be velocity dissipative w.r.t. a supply function s_v. For this supply function, Equation (25) holds for all $z_v \in \mathbb{R}^{n_z}$. Therefore, it holds that

$$\mathcal{V}_v(x_\Delta(t_1+1)) - \mathcal{V}_v(x_\Delta(t_0)) \leq \sum_{t=t_0}^{t_1} s_v(w_\Delta(t), z_\Delta(t)) \leq 0 \tag{A.14}$$

for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$ and $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_v$. This gives that

$$\mathcal{V}_v(x_\Delta(t+1)) - \mathcal{V}_v(x_\Delta(t)) \leq 0, \tag{A.15}$$

for all $t \in \mathbb{Z}_0^+$ and $x_\Delta \in \pi_{x_\Delta}\mathfrak{B}_{v,\mathscr{W}}$. Moreover, the storage function $\mathcal{V}_v$ satisfies the conditions for the function V_v in Theorem 7. Hence, by Theorem 7, Equation (A.15) implies that the system is USS.

In case of USAS, the supply function satisfies Equation (25), but with strict inequality for all $z_v \in \mathbb{R}^{n_z}$, except when $x_\Delta = 0$. Therefore, Equation (A.15) holds, but with strict inequality except when $x_\Delta(t) = 0$, which by Theorem 7 implies USAS.

A.7 | Proof of Theorem 9

If the nonlinear system given by Equation (26) is velocity dissipative w.r.t. the supply function $s_v(w_\Delta, z_\Delta) = w_\Delta^\top Q w_\Delta + z_\Delta^\top R z_\Delta$, then there exists a function $\mathcal{V}_v$, such that for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$

$$\begin{aligned} &\mathcal{V}_v(x_\Delta(t_1+1)) - \mathcal{V}_v(x_\Delta(t_0)) \\ &\quad \leq \sum_{t=t_0}^{t_1} w_\Delta(t)^\top Q w_\Delta(t) + z_\Delta(t)^\top R z_\Delta(t), \end{aligned} \tag{A.16}$$

for all $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_v$, corresponding to $(x, w, z) \in \mathfrak{B}$. Note that by consideration of the theorem, $x_\Delta(0) = 0$. Hence, as $\mathcal{V}_v(x_\Delta(0)) = \mathcal{V}_v(0) = 0$ and $\mathcal{V}_v(x_v) \geq 0$, $\forall x_v \in \mathbb{R}^{n_x}$, this implies that

$$0 \leq \sum_{t=0}^{T} w_\Delta(t)^\top Q w_\Delta(t) + z_\Delta(t)^\top R z_\Delta(t), \tag{A.17}$$

for all $T \geq 0$ and $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_v$. Defining $\tilde{Q} := \frac{1}{\|Q\|} Q$ and $\tilde{R} := \frac{1}{\|Q\|} R$, it also holds that

$$0 \leq \sum_{t=0}^{T} w_\Delta(t)^\top \tilde{Q} w_\Delta(t) + z_\Delta(t)^\top \tilde{R} z_\Delta(t), \tag{A.18}$$

Next, using Equations (26)–(28) and as $x_\Delta(t) = x(t+1) - x(t)$, we have that, omitting dependence on time for brevity,

$$\begin{aligned} z_\Delta^\top \tilde{R} z_\Delta &= x_\Delta^\top C^\top \tilde{R} C x_\Delta, \\ &= (\star)^\top \tilde{R} C (f(x) + Bw - x), \\ &= (\star)^\top \tilde{R} C (f(x) + Bw - x + \overbrace{x_* - (f(x_*) + Bw_*)}^{=0}), \\ &= (\star)^\top \tilde{R} C (f(x) - f(x_*) - (x - x_*) + B(w - w_*)). \end{aligned} \tag{A.19}$$

Through the fundamental theorem of calculus [24], we have that

$$\begin{aligned} f(x) - f(x_*) &= \left(\int_0^1 \frac{\partial f}{\partial x}(x_* + \lambda(x - x_*))\, \mathrm{d}\lambda \right)(x - x_*), \\ &= \underbrace{\left(\int_0^1 A_v(x_* + \lambda(x - x_*))\, \mathrm{d}\lambda \right)}_{\bar{A}_v(x, x_*)}(x - x_*), \end{aligned} \tag{A.20}$$

hence,

$$f(x) - f(x_*) - (x - x_*) = (\bar{A}_v(x, x_*) - I)(x - x_*). \tag{A.21}$$

Combining this with Assumption 2, we can write (A.19) as

$$z_\Delta^\top \tilde{R} z_\Delta = (\star)^\top \tilde{R} C (\bar{A}_v(x, x_*) - I)(x - x_*). \tag{A.22}$$

Next, by satisfying Proposition 1 for $T = \tilde{R} \leq 0$, we have that, for every $x_* \in \mathscr{X}$,

$$\begin{aligned} z_\Delta^\top \tilde{R} z_\Delta &= (\star)^\top \tilde{R} C (\bar{A}_v(x, x_*) - I)(x - x_*) \\ &\leq \alpha^{-1} (\star)^\top \tilde{R} C (x - x_*) = \alpha^{-1} (\star)^\top \tilde{R} (z - z_*). \end{aligned} \tag{A.23}$$

Moreover, by Assumption 3 and using that $w_\Delta(t) = w(t+1) - w(t)$, we have that, for a given $(x_*, w_*, z_*) \in \mathscr{E}$,

$$\begin{aligned} w(t+1) &= A_w(w(t) - w_*) + w_*, \\ w(t+1) - w(t) + w(t) &= A_w(w(t) - w_*) + w_*, \\ w(t+1) - w(t) &= A_w(w(t) - w_*) - (w(t) - w_*), \\ w_\Delta(t) &= (A_w - I)(w(t) - w_*), \end{aligned} \tag{A.24}$$

and hence,

$$w_\Delta(t)^\top \tilde{Q} w_\Delta(t) = (\star)^\top \tilde{Q}(A_\mathrm{w} - I)(w(t) - w_*) \leq \beta^2 (\star)^\top \tilde{Q}(w(t) - w_*), \tag{A.25}$$

where $w \in \mathfrak{W}_{(w_*,\beta)}$ and $0 \leq \tilde{Q} \leq I$. Combining Equations (A.18), (A.23) and (A.25), we obtain that, for every $(x_*, w_*, z_*) \in \mathscr{E}$,

$$\sum_{t=0}^{T} \beta^2 (\star)^\top \tilde{Q}(w(t) - w_*) + \alpha^{-1} (\star)^\top \tilde{R}(z(t) - z_*) \geq 0, \tag{A.26}$$

for all $T \geq 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}$ with $w \in \mathfrak{W}_{(w_*,\beta)}$. Hence, also

$$\sum_{t=0}^{T} \beta^2 (\star)^\top Q(w(t) - w_*) + \alpha^{-1} (\star)^\top R(z(t) - z_*) \geq 0, \tag{A.27}$$

for all $T \geq 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}$ with $w \in \mathfrak{W}_{(w_*,\beta)}$.

A.8 | Proof of Theorem 11

The system given by Equation (1) is differentially dissipative w.r.t. a supply function s_δ and for a storage function $\mathcal{V}_\delta$, if Equation (38) holds for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$ and for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$. This condition is equivalent to

$$\mathcal{V}_\delta\big(\bar{x}(t+1), x_\delta(t+1)\big) - \mathcal{V}_\delta\big(\bar{x}(t), x_\delta(t)\big) \leq s_\delta\big(w_\delta(t), z_\delta(t)\big), \tag{A.28}$$

holding for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$ and for all $t \in \mathbb{Z}_0^+$. Substituting the differential dynamics Equation (35), the considered supply function Equation (39), and storage function Equation (40) in Equation (A.28) results in

$$\begin{aligned}
&(\star)^\top M(\bar{x}(t+1))\big(A_\delta(\bar{x}(t), \bar{w}(t)) x_\delta(t) + B_\delta(\bar{x}(t), \bar{w}(t)) w_\delta(t)\big) \\
&- x_\delta(t)^\top M(\bar{x}(t)) x_\delta(t) \leq w_\delta(t)^\top Q w_\delta(t) \\
&+ 2 w_\delta(t)^\top S\big(C_\delta(\bar{x}(t), \bar{w}(t)) x_\delta(t) + D_\delta(\bar{x}(t), \bar{w}(t)) w_\delta(t)\big) \\
&+ (\star)^\top R\big(C_\delta(\bar{x}(t), \bar{w}(t)) x_\delta(t) + D_\delta(\bar{x}(t), \bar{w}(t)) w_\delta(t)\big),
\end{aligned} \tag{A.29}$$

holding for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$ and for all $t \in \mathbb{Z}_0^+$. If it holds for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$, $x_\mathrm{v} \in \mathcal{D}$, $x_\delta \in \mathbb{R}^{n_\mathrm{x}}$, and $w_\delta \in \mathbb{R}^{n_\mathrm{w}}$ that

$$\begin{aligned}
&(\star)^\top M(\bar{x} + \bar{x}_\mathrm{v})\big(A_\delta(\bar{x}, \bar{w}) x_\delta + B_\delta(\bar{x}, \bar{w}) w_\delta\big) \\
&- x_\delta^\top M(\bar{x}) x_\delta \leq w_\delta^\top Q w_\delta + 2 w_\delta^\top S\big(C_\delta(\bar{x}, \bar{w}) x_\delta \\
&+ D_\delta(\bar{x}, \bar{w}) w_\delta\big) + (\star)^\top R\big(C_\delta(\bar{x}, \bar{w}) x_\delta + D_\delta(\bar{x}, \bar{w}) w_\delta\big),
\end{aligned} \tag{A.30}$$

then, Equation (A.29) holds. Finally, Equation (41) is equivalent to Equation (A.30) by pre- and post multiplication of Equation (41) with $\mathrm{col}(x_\delta, w_\delta)^\top$ and $\mathrm{col}(x_\delta, w_\delta)$, respectively.

A.9 | Proof of Theorem 12

The proof follows in a similar manner as Lemma 8. Namely, let the system given by Equation (1) be differentially dissipative w.r.t. a supply function s_δ. For this supply function, Equation (43) holds for all $z_\delta \in \mathbb{R}^{n_\mathrm{z}}$. Therefore, it holds that

$$\mathcal{V}_\delta(\bar{x}(t_1+1), x_\delta(t_1+1)) - \mathcal{V}_\delta(\bar{x}(t_0), x_\delta(t_0)) \leq \sum_{t=t_0}^{t_1} s_\delta(w_\delta(t), z_\delta(t)) \leq 0 \tag{A.31}$$

for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$ and $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_\mathrm{v}$. This gives that

$$\mathcal{V}_\delta(\bar{x}(t+1), x_\delta(t+1)) - \mathcal{V}_\delta(\bar{x}(t+1), x_\delta(t+1)) \leq 0, \tag{A.32}$$

for all $t \in \mathbb{Z}_0^+$ and $\bar{x} \in \pi_x \mathfrak{B}_\mathrm{w}(w)$ under all measurable and bounded $w \in (\mathbb{R}^{n_\mathrm{w}})^{\mathbb{Z}_0^+}$. Moreover, the storage function $\mathcal{V}_\delta$ satisfies the conditions for the function V_δ in Lemma 1. Hence, by Lemma 1, Equation (A.32) implies that the system is IS.

In case of IAS, the supply function satisfies Equation (43), but with strict inequality for all $z_\delta \in \mathbb{R}^{n_\mathrm{z}}$ when $x(t) \neq \tilde{x}(t)$. Therefore, Equation (A.15) holds, but with strict inequality when $x(t) \neq \tilde{x}(t)$, which by Lemma 1 implies USAS.

A.10 | Proof of Theorem 13

To prove our result, we will make use of the results in the proof of [9, Theorem 6]. As the system is differentially dissipative, it implies that by writing out the λ-dependence and integrating over λ,

$$\int_0^1 \left[\mathcal{V}_\delta\big(\bar{x}(t_1+1,\lambda), x_\delta(t_1+1,\lambda)\big) - \mathcal{V}_\delta\big(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda)\big) - \sum_{t=t_0}^{t_1} s_\delta\big(w_\delta(t,\lambda), z_\delta(t,\lambda)\big) \right] \mathrm{d}\lambda \leq 0, \tag{A.33}$$

holds for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$, $\lambda \in [0, 1]$, and for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$. Let us first consider the storage function part of this inequality. Let us define a minimum energy path between x and $\tilde{x}$:

$$\chi_{(x,\tilde{x})}(\lambda) := \underset{\hat{x} \in \Gamma(x,\tilde{x})}{\arg\inf} \int_0^1 \mathcal{V}_\delta\left(\hat{x}(\lambda), \frac{\partial \hat{x}(\lambda)}{\partial \lambda}\right) \mathrm{d}\lambda. \tag{A.34}$$

As $V_\delta(\bar{x}, x_\delta) = x_\delta^\top M(\bar{x}) x_\delta$, the path $\chi_{(x,\tilde{x})}$ corresponds to the geodesic connecting x and $\bar{x}$ under the Riemannian metric $M(\bar{x})$, see also [40, 41]. By the Hopf–Rinow theorem, this implies, for any $x, \bar{x} \in \mathbb{R}^{n_\mathrm{x}}$, that $\chi_{(x,\tilde{x})}$ is a unique, smooth function [40, 42]. Based on this minimum energy path, we define the incremental storage function as:

$$\mathcal{V}_\mathrm{i}(x, \tilde{x}) := \int_0^1 \mathcal{V}_\delta\left(\chi_{(x,\tilde{x})}(\lambda), \frac{\partial \chi_{(x,\tilde{x})}(\lambda)}{\partial \lambda}\right) \mathrm{d}\lambda. \tag{A.35}$$

Note that $\mathcal{V}_\delta(\bar{x},\cdot) \in \mathcal{Q}_0$, $\forall \bar{x} \in \mathbb{R}^{n_x}$. Therefore, $\mathcal{V}_i(x,x) = 0$ for all $x \in \mathbb{R}^{n_x}$ as $\chi_{(x,x)}(\lambda) = x$, hence, $\frac{\partial \chi_{(x,\tilde{x})}(\lambda)}{\partial \lambda} = 0$ and by definition $\mathcal{V}_\delta(\cdot,0) = 0$. Moreover, for all $x, \tilde{x} \in \mathbb{R}^{n_x}$ for which $x \neq \tilde{x}$, we have that $\mathcal{V}_i(x,\tilde{x}) > 0$, as in that case there exists a set of $\lambda \in [0,1]$ for which $\frac{\partial \chi_{(x,\tilde{x})}(\lambda)}{\partial \lambda} \in \mathbb{R}\backslash\{0\}$ (as it can only be zero for all λ if $x = \tilde{x}$) and by definition $\mathcal{V}_\delta(\bar{x}, x_\delta) > 0$, $\forall x_\delta \in \mathbb{R}^{n_x}\backslash\{0\}$. Consequently, we have that $\mathcal{V}_i \in \mathcal{Q}_i$.

Based on the definition of the incremental storage function, it follows that

$$\mathcal{V}_i(x(t_1+1), \tilde{x}(t_1+1)) \leq \int_0^1 \mathcal{V}_\delta\big(\bar{x}(t_1+1,\lambda), x_\delta(t_1+1,\lambda)\big)\,\mathrm{d}\lambda, \tag{A.36}$$

for any $(\lambda \mapsto \bar{x}(t_1+1,\lambda)) \in \Gamma(x(t_1+1), \tilde{x}(t_1+1))$ with $x(t_1+1)$, $\tilde{x}(t_1+1) \in \mathbb{R}^{n_x}$, $t_1 \in \mathbb{Z}_0^+$, and $(t \mapsto \bar{x}(t,\lambda)) \in \pi_x\mathfrak{B}$ for any $\lambda \in [0, 1]$. Moreover, we parameterize the initial condition as $\bar{x}(t_0,\lambda) = \bar{x}_0(\lambda) = \chi_{(x_0,\tilde{x}_0)}(\lambda)$, from which it follows that

$$-\mathcal{V}_i(x(t_0), \tilde{x}(t_0)) = -\int_0^1 \mathcal{V}_\delta\big(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda)\big)\,\mathrm{d}\lambda. \tag{A.37}$$

Combining Equations (A.36) and (A.37) gives that

$$\begin{aligned}
&\mathcal{V}_i\big(x(t_1+1), \tilde{x}(t_1+1)\big) - \mathcal{V}_i\big(x(t_0), \tilde{x}(t_0)\big)\\
&\leq \int_0^1 \mathcal{V}_\delta\big(\bar{x}(t_1+1,\lambda), x_\delta(t_1+1,\lambda)\big)\\
&\quad - \mathcal{V}_\delta\big(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda)\big)\,\mathrm{d}\lambda.
\end{aligned} \tag{A.38}$$

Subsequently, we consider the supply function part of Equation (A.33). This follows in the same manner as in [9, 22], which we will briefly repeat. By changing summation and integration operations, the supply function part of Equation (A.33) is given by

$$\sum_{t=t_0}^{t_1} \int_0^1 (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta(t,\lambda) \\ z_\delta(t,\lambda) \end{bmatrix} \mathrm{d}\lambda. \tag{A.39}$$

Parameterizing our input as $\bar{w}(t,\lambda) = \tilde{w}(t) + \lambda(w(t) - \tilde{w}(t))$, it follows that $w_\delta(t) = \frac{\partial \bar{w}(t,\lambda)}{\partial \lambda} = w(t) - \tilde{w}(t)$. Therefore, we have that $\int_0^1 (\star)^\top Q\, w_\delta(t,\lambda)\mathrm{d}\lambda = (\star)^\top Q(w(t) - \tilde{w}(t))$ and

$$\begin{aligned}
\int_0^1 2\, w_\delta(t,\lambda)^\top S\, z_\delta(t,\lambda)\mathrm{d}\lambda &= 2(w(t) - \tilde{w}(t))^\top S \int_0^1 \frac{\partial \bar{z}(t,\lambda)}{\partial \lambda}\,\mathrm{d}\lambda,\\
&= 2(w(t) - \tilde{w}(t))^\top S\,(\bar{z}(t,1) - \bar{z}(t,0)),\\
&= 2(w(t) - \tilde{w}(t))^\top S\,(z(t) - \tilde{z}(t)).
\end{aligned} \tag{A.40}$$

As we consider $R \preceq 0$, i.e. $-R \succeq 0$, we have by [22, Lemma 16]

$$\begin{aligned}
\int_0^1 (\star)^\top R\, z_\delta(t,\lambda)\mathrm{d}\lambda &= \int_0^1 (\star)^\top R\, \frac{\partial \bar{z}(t,\lambda)}{\partial \lambda}\,\mathrm{d}\lambda\\
&\leq (\star)^\top R\left(\int_0^1 \frac{\partial \bar{z}(t,\lambda)}{\partial \lambda}\,\mathrm{d}\lambda\right) = (\star)^\top R\,(z(t) - \tilde{z}(t)).
\end{aligned} \tag{A.41}$$

Combining this, results in the following inequality to hold

$$\begin{aligned}
&\sum_{t=t_0}^{t_1} \int_0^1 (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta(t,\lambda) \\ z_\delta(t,\lambda) \end{bmatrix} \mathrm{d}\lambda\\
&\leq \sum_{t=t_0}^{t_1} (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w(t) - \tilde{w}(t) \\ z(t) - \tilde{z}(t) \end{bmatrix}.
\end{aligned} \tag{A.42}$$

Combining Equations (A.38) and (A.42) with Equation (A.33) results in

$$\begin{aligned}
&\mathcal{V}_i\big(x(t_1+1), \tilde{x}(t_1+1)\big) - \mathcal{V}_i\big(x(t_0), \tilde{x}(t_0)\big)\\
&\leq \sum_{t=t_0}^{t_1} s_i\big(w(t), \tilde{w}(t), z(t), \tilde{z}(t)\big),
\end{aligned} \tag{A.43}$$

for all $t_0, t_1 \in \mathbb{Z}_0^+$ with $t_0 \leq t_1$ and any two trajectories $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ where $\mathcal{V}_i$ is given by Equation (A.35), which is the condition for incremental dissipativity in Definition 4.

A.11 | Proof of Theorem 14

We have that Equation (46) is a DPV embedding of Equation (1) on the region $\mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$. Therefore, $A(\eta(\bar{x},\bar{w})) = A_\delta(\bar{x},\bar{w}), \ldots, D(\eta(\bar{x},\bar{w})) = D_\delta(\bar{x},\bar{w})$ for all $(\bar{x},\bar{w}) \in \mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$. Moreover, we have Equation (47) holds for all $p \in \mathcal{P}$ and $v \in \Pi$. Hence, it straightforwardly follows that Equation (41) holds for all $(\bar{x},\bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\bar{x}_v \in \mathcal{D}$.