Automatica 161 (2024) 111473

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical Communique

Large time behavior of nonautonomous linear differential equations with Kirchhoff coefficients[☆]

Josef Diblík[a], Mihály Pituk[b,c], Gábor Szederkényi[d,e,*]

[a] Brno University of Technology, Faculty of Civil Engineering, Veveří 331/95, 602 00 Brno, Czech Republic
[b] Department of Mathematics, University of Pannonia, Egyetem út 10, H-8200 Veszprém, Hungary
[c] HUN-REN-ELTE Numerical Analysis and Large Networks Research Group, Budapest, Hungary
[d] Faculty of Information Technology and Bionics, Pázmány Péter Catholic University, Práter 50/a, H-1083 Budapest, Hungary
[e] Systems and Control Laboratory, HUN-REN Institute for Computer Science and Control (SZTAKI), Kende u. 13-17, H-1111 Budapest, Hungary

ARTICLE INFO

Article history:
Received 22 August 2023
Accepted 11 November 2023
Available online 23 December 2023

Keywords:
Linear systems
Time-varying systems
Positive systems
Kirchhoff matrix

ABSTRACT

Nonautonomous linear ordinary differential equations with Kirchhoff coefficients are considered. Under appropriate assumptions on the topology of the directed graphs of the coefficients, it is shown that if the Perron vectors of the coefficients are slowly varying at infinity, then every solution is asymptotic to a constant multiple of the Perron vectors at infinity. Our results improve and generalize some recent convergence theorems.

© 2023 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

In this brief paper, we study the nonautonomous linear ordinary differential equation

$$x' = A(t)x, \tag{1}$$

where $A\colon [t_0,\infty) \to \mathbb{R}^{n\times n}$ is a continuous matrix function such that $A(t)$ is a Kirchhoff matrix for $t \geq t_0$. By a *Kirchhoff matrix* (Magyar, Szederkényi, & Hangos, 2018) or $\mathbb{W}$-*matrix* (van Kampen, 2007), we mean a Metzler matrix with zero column sums. As usual, $M \in \mathbb{R}^{n\times n}$ is a *Metzler matrix* if the off-diagonal elements of M are nonnegative.

Eq. (1) arises as a model of first-order chemical reactions with time-dependent coefficients (Glasser, Horn, & Meidan, 1980; Summers & Scott, 1988), a prototype of the average-consensus protocol (Olfati-Saber, Fax, & Murray, 2007; Ren & Beard, 2008), a master equation for non-stationary Markovian jump processes (van Kampen, 2007, Chap. V) and compartmental systems (Haddad, Chellaboina, & Hui, 2010; Jacquez & Simon, 1993).

The recent paper (Garab & Pituk, 2021) features a convergence theorem for Eq. (1) which can be described as follows. Suppose that A is a bounded and uniformly continuous matrix function on $[t_0,\infty)$. Assume also that the directed graphs of the coefficient matrices have a common directed spanning tree and the off-diagonal elements of $A(t)$, $t \geq t_0$, are bounded away from zero along this common directed spanning tree. Then the convergence of the Perron vectors of the coefficient matrices to a positive vector at infinity implies that every solution of (1) is convergent as $t \to \infty$ and its limit can be expressed explicitly in terms of the initial data. By a *Perron vector* of a Metzler matrix, we mean any nonnegative normalized eigenvector which corresponds to the spectral abscissa. Our aim in this note is to complement and generalize the convergence theorem (Garab & Pituk, 2021, Theorem 3.1) to the case when the Perron vectors of the coefficient matrices are not necessarily convergent. Moreover, we will impose substantially weaker conditions on the Kirchhoff coefficients which allow time-dependent lower bounds. Under appropriate assumptions, we will show that if the unique Perron vector $p(t)$ of $A(t)$ is *slowly varying* as $t \to \infty$ (for the definition, see Section 2), then every solution $x(t)$ of Eq. (1) is asymptotically equivalent to a constant multiple of $p(t)$ as $t \to \infty$.

☆ J. Diblík was supported by the project of specific university research FAST-S-22-7867 (Faculty of Civil Engineering, Brno University of Technology). M. Pituk was supported by the Hungarian National Research, Development and Innovation Office grant no. K139346. G. Szederkényi acknowledges the support of the projects NKFIH-OTKA-145934 and RRF-2.3.1-21-2022-00006 of the Hungarian National Research, Development and Innovation Office. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Hernan Haimovich under the direction of Editor André Tits.

* Corresponding author at: Faculty of Information Technology and Bionics, Pázmány Péter Catholic University, Práter 50/a, H-1083 Budapest, Hungary.

E-mail addresses: diblik@vutbr.cz (J. Diblík), pituk.mihaly@mik.uni-pannon.hu (M. Pituk), szederkenyi@itk.ppke.hu (G. Szederkényi).

https://doi.org/10.1016/j.automatica.2023.111473
0005-1098/© 2023 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

2. Preliminaries

Before we formulate our main result, we introduce the notations and summarize some auxiliary results which will be needed in the proof.

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{N} = \{0, 1, 2, \dots\}$. Given a positive integer n, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimensional space of real column vectors and the space of $n \times n$ matrices with real entries, respectively. The symbol $\|\cdot\|$ denotes the l_1-norm on $\mathbb{R}^n$ and the corresponding induced matrix norm on $\mathbb{R}^{n\times n}$, i.e. $\|x\| = \sum_{j=1}^n |x_j|$ for $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and $\|M\| = \max_{1\le j\le n} \sum_{i=1}^n |m_{ij}|$ for $M = (m_{ij}) \in \mathbb{R}^{n\times n}$. As usual, the superscript T indicates the transpose. A vector $x \in \mathbb{R}^n$ is called *normalized* if $\|x\| = 1$. The all-one vector in $\mathbb{R}^n$ is denoted by $e := (1, \dots, 1)^T \in \mathbb{R}^n$. Inequalities between vectors and matrices are to be understood elementwise. Thus, for $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$, we have $x \le y$ if and only if $x_i \le y_i$ for all $i = 1, \dots, n$. A vector $x \in \mathbb{R}^n$ is called *nonnegative* if $x \ge 0$. The set of nonnegative vectors in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. We use the similar notation $\mathbb{R}^{n\times n}_+$ for the set of nonnegative matrices in $\mathbb{R}^{n\times n}$. Thus, $M = (m_{ij}) \in \mathbb{R}^{n\times n}_+$ if and only if $m_{ij} \ge 0$ for all i, j. If $M = (m_{ij}) \in \mathbb{R}^{n\times n}$, then the *directed graph of* M, denoted by $\Gamma(M)$, is the pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes (vertices) and $\mathcal{E}$ is the set of edges defined by $\mathcal{V} = \{1, 2, \ \dots, n\}$ and $\mathcal{E} = \{(i, j) \in \mathcal{V}\times\mathcal{V} \mid i \ne j,\ m_{ji} \ne 0\}$, respectively. We say that $\Gamma(M)$ has a *(rooted) directed spanning tree* if there exists a node $i \in \mathcal{V}$, called the *root*, such that for every other node $j \in \mathcal{V}$, $j \ne i$, there exists a directed path from i to j. By a *directed path* from node i to node j, we mean a sequence of directed edges $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k) \in \mathcal{E}$ with distinct nodes $i_1, \dots, i_k \in \mathcal{V}$ such that $i_1 = i$ and $i_k = j$. For $M \in \mathbb{R}^{n\times n}$, the symbol $\operatorname{diag} M$ will denote the $n \times n$ diagonal matrix having the same diagonal as M. The *spectrum* of $M \in \mathbb{R}^{n\times n}$, denoted by $\sigma(M)$, is the set of all eigenvalues of M and the *spectral abscissa* $s(M)$ is defined by $s(M) := \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(M)\}$.

It follows from the Perron–Frobenius theory (see, e.g., Kato (1982, Chap. I, Theorem 7.5)) that if $M \in \mathbb{R}^{n\times n}$ is a Metzler matrix, then its spectral abscissa $s(M)$ is an eigenvalue of M with a nonnegative eigenvector. Every nonnegative normalized eigenvector of a Metzler matrix M corresponding to its spectral abscissa $s(M)$ will be called a *Perron vector* of M. Thus, every Metzler matrix has at least one Perron vector. The Perron vectors are not necessarily unique. Indeed, the identity matrix $I \in \mathbb{R}^{n\times n}$ is a Metzler matrix which has n linearly independent Perron vectors, the canonical basis vectors in $\mathbb{R}^n$. The following result from Pituk (2023) gives a sufficient condition for the uniqueness of the Perron vectors of Kirchhoff matrices.

Proposition 1 (*Pituk (2023, Theorem 1.1)*). *Let $K \in \mathbb{R}^{n\times n}$ be a Kirchhoff matrix. If $\Gamma(K^T)$ has a directed spanning tree, then $s(K) = 0$ is an algebraically simple eigenvalue of K so that K has a unique Perron vector.*

Recall that a nonnegative matrix $M \in \mathbb{R}^{n\times n}_+$ is *row-allowable* if each row of M contains at least one positive entry. A row-allowable matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}_+$ is called *scrambling* (Hajnal, 1958) if for any two indices $1 \le i_1 < i_2 \le n$, there exists an index $j \in \{1, \dots, n\}$ such that $m_{i_1 j} > 0$ and $m_{i_2 j} > 0$. For every $M = (m_{ij}) \in \mathbb{R}^{n\times n}_+$, the quantity

$$\gamma(M) := \min_{1\le i_1, i_2\le n} \sum_{j=1}^n \min\{m_{i_1 j}, m_{i_2 j}\}$$

is called the *scrambling power* of M (Hajnal, 1958). Evidently, $M \in \mathbb{R}^{n\times n}_+$ is scrambling if and only if $\gamma(M) > 0$. The function $\gamma : \mathbb{R}^{n\times n}_+ \to [0, \infty)$ is monotone and positively homogeneous, i.e.

$$\gamma(M_1) \le \gamma(M_2) \text{ for } M_1, M_2 \in \mathbb{R}^{n\times n}_+, M_1 \le M_2, \tag{2}$$

$$\gamma(\lambda M) = \lambda\gamma(M) \text{ for } \lambda > 0, M \in \mathbb{R}^{n\times n}_+. \tag{3}$$

We shall need the following result from Wu (2006) which provides a sufficient condition for the product of nonnegative matrices to be scrambling.

Proposition 2 (*Wu (2006, Theorem 5.1)*). *If $P_1, P_2, \dots, P_{n-1} \in \mathbb{R}^{n\times n}_+$ are nonnegative matrices with positive diagonal elements such that the directed graph $\Gamma(P_j^T)$ has a directed spanning tree for each $j \in \{1, \dots, n-1\}$, then the product $P_1 P_2 \dots P_{n-1}$ is scrambling.*

A nonnegative matrix $S \in \mathbb{R}^{n\times n}_+$ is called *row-stochastic* (*column-stochastic*) if the row sums (column sums) of S are equal to one. Note that if $S \in \mathbb{R}^{n\times n}_+$ is a column-stochastic matrix, then $\|S\| = 1$. The following known result from Ipsen and Selee (2011) and Seneta (2006, Sec. 3.1 and 4.3) will be fundamental for the proof of our main theorem.

Proposition 3 (*Ipsen and Selee (2011, Corollary 3.9)*). *Suppose that $S \in \mathbb{R}^{n\times n}_+$ is a row-stochastic matrix so that its transpose S^T is column-stochastic. Let V denote the hyperplane in $\mathbb{R}^n$ defined by $V := \{\, v \in \mathbb{R}^n \mid e^T v = 0 \,\}$. Then*

$$\|S^T v\| \le \tau(S)\|v\| \quad \text{whenever } v \in V, \tag{4}$$

where

$$\tau(S) := 1 - \gamma(S). \tag{5}$$

The quantity $\tau(S)$ given by (5) is called the *(one-norm) coefficient of ergodicity* of the row-stochastic matrix S (Ipsen & Selee, 2011). For every row-stochastic matrix $S \in \mathbb{R}^{n\times n}_+$, we have that $0 \le \tau(S) \le 1$, and $\tau(S) < 1$ if and only if $\gamma(S) > 0$. Thus, $\tau(S)$ serves as contraction coefficient in (4) if and only if the row-stochastic matrix S is scrambling.

Let $\mathcal{I} \subset \mathbb{R}$ be an interval. Consider the linear differential equation

$$x' = C(t)x, \tag{6}$$

where $C : \mathcal{I} \to \mathbb{R}^{n\times n}$ is a continuous matrix function. If $\Phi : \mathcal{I} \to \mathbb{R}^{n\times n}$ is any fundamental matrix solution of (6), then the *transition matrix* of Eq. (6), denoted by $T_C(t, s)$, is defined by $T_C(t, s) = \Phi(t)\Phi^{-1}(s)$ for $t, s \in \mathcal{I}$. Evidently, every solution x of (6) can be written in the form $x(t) = T_C(t, s)x(s)$ for $t, s \in \mathcal{I}$.

We will need the following comparison result for differential equations with Metzler coefficients. In the special case $\mathcal{I} = [t_0, \infty)$, the result was proved in Garab and Pituk (2021, Proposition 2.3). If $\mathcal{I} \subset \mathbb{R}$ is an arbitrary interval, then we can use literally the same arguments, therefore the proof is omitted.

Proposition 4. *Consider the linear differential equations*

$$x' = M(t)x$$

and

$$x' = N(t)x,$$

where $M, N : \mathcal{I} \to \mathbb{R}^{n\times n}$ are continuous matrix functions such that $M(t)$ is a Metzler matrix for $t \in \mathcal{I}$. If

$$M(t) \le N(t) \quad \text{for } t \in \mathcal{I},$$

then

$$0 \le T_M(t, s) \le T_N(t, s) \quad \text{whenever } t, s \in \mathcal{I} \text{ and } t \ge s.$$

A function $f : [t_0, \infty) \to \mathbb{R}^n$ is called *slowly varying at infinity (in additive form)* (Seneta, 1976) if for every $s \in \mathbb{R}$,

$$f(t + s) - f(t) \to 0 \qquad \text{as } t \to \infty.$$

As shown in Pituk (2017, p. 30), a continuous function $f:[t_0,\infty)\to\mathbb{R}^n$ is slowly varying at infinity if and only if there exists $t_1\ge t_0$ such that f can be written in the form

$$f(t)=g(t)+h(t),\qquad t\ge t_1,$$

where $g:[t_1,\infty)\to\mathbb{R}^n$ is a continuous function which tends to a finite limit in $\mathbb{R}^n$ as $t\to\infty$ and $h:[t_1,\infty)\to\mathbb{R}^n$ is a continuously differentiable function such that $h'(t)\to 0$ as $t\to\infty$. An example of a scalar function which is slowly varying and has no limit at infinity is $f(t)=\sin\sqrt{t}$ for $t\ge 0$.

Finally, we remark that every solution x of the Kirchhoff system (1) has the *mass conservation property*

$$\sum_{i=1}^n x_i(t)=e^Tx(t)=m\qquad\text{for } t\ge t_0, \tag{7}$$

where m is a constant given by $m=m(x)=e^Tx(t_0)$ (see, e.g., Garab and Pituk (2021, Lemma 3.9) with $v=e$).

3. Main result

Now we can formulate our main theorem which states that, under appropriate assumptions, every solution $x(t)$ of Eq. (1) is asymptotic to a constant multiple of the unique Perron vector $p(t)$ of the coefficient matrix $A(t)$ as $t\to\infty$. The assumptions include the slowly varying property of the Perron vectors $p(t)$ of $A(t)$ as $t\to\infty$ and the uniform continuity of A on $[t_0,\infty)$. Recall that $A:[t_0,\infty)\to\mathbb{R}^{n\times n}$ is *uniformly continuous* on $[t_0,\infty)$ if for every $\epsilon>0$ there exists $\delta>0$ such that $\|A(t_1)-A(t_2)\|<\epsilon$ whenever $t_1,t_2\ge t_0$ and $|t_1-t_2|<\delta$. It follows from the mean value theorem that a sufficient condition for A to be uniformly continuous on $[t_0,\infty)$ is that its derivative is bounded on $[t_0,\infty)$.

Theorem 5. *Let $A:[t_0,\infty)\to\mathbb{R}^{n\times n}$ be a uniformly continuous and bounded matrix function such that $A(t)$ is a Kirchhoff matrix for all $t\ge t_0$. Suppose that*

$$A(t)-\operatorname{diag}A(t)\ge c(t)P \tag{8}$$

for $t\ge t_0$, where $c:[t_0,\infty)\to[0,\infty)$ is a nonnegative, uniformly continuous function and $P\in\mathbb{R}_+^{n\times n}$ is a nonnegative matrix such that $\Gamma(P^T)$ has a directed spanning tree. Assume also that $A(t)$ has a unique Perron vector $p(t)$ for $t\ge t_0$ and $p(t)$ is slowly varying as $t\to\infty$, i.e. for every $s\in\mathbb{R}$,

$$p(t+s)-p(t)\to 0\qquad\text{as } t\to\infty. \tag{9}$$

If

$$\liminf_{t\to\infty}\int_t^{t+h}c(u)\,du>0 \tag{10}$$

for some $h>0$, then for every solution x of Eq. (1),

$$x(t)-mp(t)\to 0\qquad\text{as } t\to\infty, \tag{11}$$

where $m=e^Tx(t_0)$.

Remark 6. Suppose that the hypotheses of Theorem 5 hold and $c(t)>0$ for some $t\ge t_0$. In view of (8), we have that $\Gamma(A^T(t))=\Gamma(P^T)$. Thus, $A(t)$ is a Kirchhoff matrix and its directed graph has a directed spanning tree. Proposition 1 implies that $A(t)$ has a unique Perron vector $p(t)$. Thus, if c in (8) is positive, then the uniqueness of the Perron vector is automatically satisfied. However, if $c(t)=0$ for some $t\ge t_0$, then (8) merely says that the off-diagonal elements of $A(t)$ are nonnegative and the uniqueness of the Perron vector of $A(t)$ in general does not hold.

Proof of Theorem 5. Let x be an arbitrary solution of Eq. (1). Define

$$y(t):=x(t)-mp(t)\qquad\text{for } t\ge t_0. \tag{12}$$

By Garab and Pituk (2021, Lemma 3.9) ($v=e$), the solution x and hence the function y is bounded on $[t_0,\infty)$. Let $w\in\mathbb{R}^n$ be an arbitrary accumulation point of y at infinity, i.e. there exists a sequence $\{t_k\}_{k=1}^\infty$ in $[t_0,\infty)$ with $t_k\to\infty$ and such that

$$w=\lim_{k\to\infty}y(t_k). \tag{13}$$

Since both functions x and p are bounded on $[t_0,\infty)$, without loss of generality, we may (and do) assume that the limits

$$\xi:=\lim_{k\to\infty}x(t_k)\quad\text{and}\quad\eta:=\lim_{k\to\infty}p(t_k) \tag{14}$$

exist, and hence (cf. (12) and (13))

$$w=\xi-m\eta. \tag{15}$$

Otherwise, we pass to an appropriate subsequence of $\{t_k\}_{k=1}^\infty$. Define

$$z_k(t):=x(t_k+t),\quad B_k(t)=A(t_k+t),\quad d_k(t):=c(t_k+t)$$

for each k and t such that $t_k+t\ge t_0$. From Eq. (1), we find that

$$z_k'(t)=B_k(t)z_k(t) \tag{16}$$

for each k and t satisfying $t_k+t\ge t_0$. The functions z_k, B_k and d_k are defined on the interval $[t_0-t_k,\infty)$ for $k\in\mathbb{N}$. If $I=[a_1,a_2]\subset\mathbb{R}$ is an arbitrary compact interval, then, taking into account that $t_k\to\infty$, the above functions are defined on I for all $k\ge k_0$ whenever k_0 is sufficiently large. The boundedness of A and (8) imply that c is bounded on $[t_0,\infty)$. From Eq. (1) and the boundedness of A and x, we obtain that x' is bounded on $[t_0,\infty)$. Hence, x is uniformly continuous on $[t_0,\infty)$. Finally, from the boundedness and the uniform continuity of A, x and c on $[t_0,\infty)$, we conclude that, on the interval I, the functions $\{z_k\}_{k=k_0}^\infty$, $\{B_k\}_{k=k_0}^\infty$ and $\{d_k\}_{k=k_0}^\infty$ are uniformly bounded and equicontinuous, respectively. From this, by the application of a variant of the Arzelà–Ascoli theorem (Garab & Pituk, 2021, Proposition 2.10), we conclude that there exist subsequences $\{z_{k_j}\}_{j=1}^\infty$, $\{B_{k_j}\}_{j=1}^\infty$ and $\{d_{k_j}\}_{j=1}^\infty$ of $\{z_k\}_{k=1}^\infty$, $\{B_k\}_{k=1}^\infty$ and $\{d_k\}_{k=1}^\infty$, respectively, such that, for every $t\in\mathbb{R}$, the limits $z(t)=\lim_{j\to\infty}z_{k_j}(t)$, $B(t)=\lim_{j\to\infty}B_{k_j}$ and $d(t)=\lim_{j\to\infty}d_{k_j}(t)$ exist and the convergence is uniform on every compact subinterval of $\mathbb{R}$. Since $\{t_{k_j}\}_{j=1}^\infty$ is a subsequence of $\{t_k\}_{k=1}^\infty$, we have (cf. (14))

$$z(0)=\lim_{j\to\infty}z_{k_j}(0)=\lim_{j\to\infty}x(t_{k_j})=\xi. \tag{17}$$

By passing to the limit in the integrated form of (16),

$$z_{k_j}(t)=z_{k_j}(0)+\int_0^t B_{k_j}(u)z_{k_j}(u)\,du,$$

we obtain

$$z(t)=z(0)+\int_0^t B(u)z(u)\,du$$

for all $t\in\mathbb{R}$. Hence

$$z'(t)=B(t)z(t) \tag{18}$$

for all $t\in\mathbb{R}$. The boundedness of x and A imply that the functions z and B are bounded on $\mathbb{R}$. As a limit of Kirchhoff matrices, $B(t)$ is a Kirchhoff matrix for all $t\in\mathbb{R}$. By Pituk (2023, Lemma 3.1), this implies that the transition matrix $T_B(t,s)$ of (18) is column-stochastic for all $t\ge s$. Moreover, (8) implies that

$$B(t)-\operatorname{diag}B(t)\ge d(t)P\quad\text{for all } t\in\mathbb{R}. \tag{19}$$

Since $\Gamma(P^T) = \Gamma(P^T + I)$ has a directed spanning tree, applying Proposition 2, we conclude that $(P^T + I)^{n-1}$ is scrambling, i.e. $\gamma((P^T + I)^{n-1}) > 0$. The boundedness of B implies that $\operatorname{diag} B(t) \geq bI$ for all $t \in \mathbb{R}$, where $b = \min_{1\leq i\leq n} \inf_{t\geq t_0} b_{ii}(t)$. Then (cf. (19)) $B(t) \geq d(t)P + bI$ for all $t \in \mathbb{R}$. Hence, by Proposition 4, we have for $s \in \mathbb{R}$,

$$T_B(s + h, s) \geq e^{bh} e^{\alpha(s)P},$$

where $\alpha(s) := \int_s^{s+h} d(u)\,du$. From this, taking into account that

$$e^{Mt} = e^{(M+I)t} e^{-It} = e^{-t} e^{(M+I)t} \geq e^{-t} \frac{[\,(M + I)t\,]^k}{k!}$$

for all $M \in \mathbb{R}_+^{n\times n}$ and $t \geq 0$, we have ($M = P$, $t = \alpha(s)$)

$$T_B(s + h, s) \geq e^{bh-\alpha(s)} \frac{\alpha^{n-1}(s)}{(n-1)!}(P + I)^{n-1}$$

for all $s \in \mathbb{R}$. From this, taking into account that

$$\begin{aligned}\alpha(s) &= \int_s^{s+h} d(u)\,du = \lim_{j\to\infty} \int_s^{s+h} c(t_{k_j} + u)\,du \\ &= \lim_{j\to\infty} \int_{t_{k_j}+s}^{t_{k_j}+s+h} c(v)\,dv \geq \rho\end{aligned}$$

for all $s \in \mathbb{R}$, where $\rho := \liminf_{t\to\infty} \int_t^{t+h} c(u)\,du > 0$ (cf. (10)), we obtain

$$T_B(s + h, s) \geq \mu(P + I)^{n-1}$$

for all $s \in \mathbb{R}$, where $\mu := ((n-1)!)^{-1} e^{bh-\delta} \rho^{n-1}$ with $\delta := \sup_{s\in\mathbb{R}} \alpha(s) < \infty$. (Note that, in view of the boundedness of B and (19), the function d and hence α is bounded on $\mathbb{R}$.) This, together with (2) and (3), implies that

$$\gamma\left(\left(T_B(s + h, s)\right)^T\right) \geq \mu\gamma((P^T + I)^{n-1}) =: \theta > 0 \tag{20}$$

for all $s \in \mathbb{R}$. By the definition of the Perron vector, we have

$$B_k(t)p(t_k + t) = A(t_k + t)p(t_k + t) = 0 \tag{21}$$

for all k and t such that $t_k + t \geq t_0$. From the second limit relation in (14) and the slowly varying condition (9), we have that $p(t_k + t) \to \eta$ as $k \to \infty$ for all $t \in \mathbb{R}$. Therefore, letting $k = k_j \to \infty$ in (21), we obtain that $B(t)\eta = 0$ for all $t \in \mathbb{R}$. Thus, η is an equilibrium of Eq. (18), i.e. $\eta = T_B(t, s)\eta$ for all $t, s \in \mathbb{R}$. Hence

$$z(s + h) - m\eta = T_B(s + h, s)(z(s) - m\eta) \tag{22}$$

for all $s \in \mathbb{R}$. As a limit of Perron vectors, η is nonnegative and normalized so that $e^T\eta = 1$, while the mass conservation property (7) and the definition of z imply that $e^T z(t) = m$ for all $t \in \mathbb{R}$. Therefore, $z(t) - m\eta \in V$ for all $t \in \mathbb{R}$. From this and (22), by the application of Proposition 3, we conclude that

$$\|z(s + h) - m\eta\| \leq \tau\left(\left(T_B(s + h, s)\right)^T\right)\|z(s) - m\eta\|$$

for all $s \in \mathbb{R}$. This, together with (5) and (20), implies that

$$\|z(s + h) - m\eta\| \leq (1 - \theta)\|z(s) - m\eta\| \tag{23}$$

for all $s \in \mathbb{R}$. As noted before, z is a bounded on $\mathbb{R}$, and hence $\Delta := \sup_{s\in\mathbb{R}} \|z(s) - m\eta\| < \infty$. From (23), we find that $\Delta \leq (1 - \theta)\Delta$. Since $\theta > 0$, this implies that $\Delta = 0$. In particular, $z(0) = m\eta$, and hence (cf. (15) and (17))

$$w = \xi - m\eta = z(0) - m\eta = 0.$$

Thus, the function y defined by (12) is bounded and the only accumulation point of y at infinity is $w = 0$. Therefore, $x(t) - mp(t) = y(t) \to 0$ as $t \to \infty$. □

The following example shows the importance of assumption (10) in Theorem 5.

Example 7. Consider Eq. (1) with the Kirchhoff matrix-valued coefficient $A\colon [1, \infty) \to \mathbb{R}^{2\times 2}$ given by

$$A(t) = \begin{pmatrix} -\dfrac{1 - \cos\sqrt{t}}{t} & \dfrac{1 + \cos\sqrt{t}}{t} \\ \dfrac{1 - \cos\sqrt{t}}{t} & -\dfrac{1 + \cos\sqrt{t}}{t} \end{pmatrix}$$

for $t \geq 1$. Assumption (8) is fulfilled with

$$c(t) = \frac{1 - \cos\sqrt{t}}{t} \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Clearly, c is bounded on $[1, \infty)$. Since c' is bounded and $c'(t) \to 0$ as $t \to \infty$, it follows that c is uniformly continuous and slowly varying at infinity. Evidently, $\Gamma(P^T)$ has a directed spanning tree. For every $t \geq 1$, the unique Perron vector $p(t) = (p_1(t), p_2(t))^T$ of $A(t)$ is given by

$$p(t) = \frac{1}{2}\left(1 + \cos\sqrt{t}, 1 - \cos\sqrt{t}\right)^T.$$

Since $p'(t) \to 0$ as $t \to \infty$, it follows that $p\colon [1, \infty] \to \mathbb{R}^2$ is slowly varying at infinity. Let $x = (x_1, x_2)^T$ be an arbitrary solution Eq. (1) with $x_1(1) + x_2(1) = 1$ so that $m = m(x) = 1$ and $x_1(t) + x_2(t) = 1$ for all $t \geq 1$. From Eq. (1), taking into account that $x_2(t) = 1 - x_1(t)$ for all $t \geq 1$, we obtain

$$x_1'(t) = -\frac{2}{t}x_1(t) + \frac{1 + \cos\sqrt{t}}{t}$$

for all $t \geq 1$. It follows by elementary calculations that the general solution of the latter linear differential equation has the form

$$x_1(t) = \frac{1}{2} + \left(\frac{6}{t} - \frac{12}{t^2}\right)\cos\sqrt{t} + \left(\frac{2}{\sqrt{t}} - \frac{12}{t\sqrt{t}}\right)\sin\sqrt{t} + \frac{C}{t^2},$$

where C is a constant. Hence $\lim_{t\to\infty} x_1(t) = 1/2$ and

$$x_1(4k^2\pi^2) - mp_1(4k^2\pi^2) = x_1(4k^2\pi^2) - 1 \longrightarrow -\frac{1}{2} \neq 0$$

as $k \to \infty$. Therefore, the asymptotic relation (11) does not hold. Clearly, all assumptions of Theorem 5 are satisfied with the exception of (10).

Now we present an example which shows that the slowly varying property of the Perron vectors in Theorem 5 cannot be omitted.

Example 8. Consider Eq. (1) with a uniformly continuous and bounded Kirchhoff matrix function

$$A(t) = \begin{pmatrix} -(1 + \sin^2 t) & 1 + \cos^2 t \\ 1 + \sin^2 t & -(1 + \cos^2 t) \end{pmatrix}$$

for $t \geq 0$. Assumption (8) is fulfilled with

$$c(t) \equiv 1 \quad \text{and} \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, $\Gamma(P^T)$ has a directed spanning tree with root $i = 1$. The unique Perron vector of $A(t)$ is given by

$$p(t) = \frac{1}{3}(1 + \cos^2 t, 1 + \sin^2 t)^T \qquad \text{for } t \geq 0.$$

For the solution $x(t) = (x_1(t), x_2(t)^T)$ given by

$$\begin{aligned} x_1(t) &= 13 + 2\sin(2t) + 3\cos(2t), \\ x_2(t) &= 13 - 2\sin(2t) - 3\cos(2t), \end{aligned}$$

we have that $m = m(x) = x_1(0) + x_2(0) = 26$. It is easily verified that

$$x(k\pi) - mp(k\pi) \to \frac{4}{3}(-1, 1)^T \neq (0, 0)^T \qquad \text{as } k \to \infty.$$

Therefore, conclusion (11) of Theorem 5 does not hold. Note that except for the slowly varying condition (9) all assumptions of Theorem 5 are satisfied. Since

$$p\left(t + \frac{\pi}{2}\right) - p(t) = \frac{\cos(2t)}{3}(-1, 1)^T$$

has no limit as $t \to \infty$, the slowly varying condition (9) is violated for $s = \pi/2$.

Remark 9. The slowly varying condition (9) is automatically satisfied if $p(t)$ tends to a finite limit $v \in \mathbb{R}^n_+$ as $t \to \infty$. It is easily seen that in this case the asymptotic relation (11) can be written equivalently as $\lim_{t\to\infty} x(t) = mv$. This convergence criterion may be viewed as an improvement of Garab and Pituk (2021, Theorem 3.1) since it does not require that all components of the limiting vector $v \in \mathbb{R}^n_+$ are positive. Moreover, in contrast to Garab and Pituk (2021), the lower bounds for the coefficients of Eq. (1) are allowed to be time-dependent.

4. Conclusion

We studied a class of nonautonomous linear ordinary differential equations with Kirchhoff coefficients which arises in numerous applications. Under appropriate assumptions, it was shown that every solution is asymptotic to a constant multiple of the time-dependent Perron vector of the coefficient matrix as $t \to \infty$. A key assumption is the slowly varying property of the Perron vectors at infinity. The main result is an improvement of a recent convergence theorem.

References

Garab, Á., & Pituk, M. (2021). Convergence in nonautonomous linear differential equations with Kirchhoff coefficients. *Systems & Control Letters*, *149*, Article 104884.

Glasser, D., Horn, F. J. M., & Meidan, R. (1980). Properties of certain zero column-sum matrices with applications to optimization of chemical reactors. *Journal of Mathematical Analysis and Applications*, *73*, 315–337.

Haddad, W. M., Chellaboina, V., & Hui, Q. (2010). *Nonnegative and compartmental dynamical systems*. Princeton: Princeton University Press.

Hajnal, J. (1958). Weak ergodicity in nonhomogeneous Markov chains. *Mathematical Proceedings of the Cambridge Philosophical Society*, *54*(2), 233–246.

Ipsen, I. C. F., & Selee, T. M. (2011). Ergodicity coefficients defined by vector norms. *SIAM Journal of Mathematical Analysis*, *32*(1), 153–200.

Jacquez, J. A., & Simon, C. P. (1993). Qualitative theory of compartmental systems. *SIAM Review*, *35*(1), 43–79.

Kato, T. (1982). *A short introduction to perturbation theory for linear operators*. New York: Springer.

Magyar, A., Szederkényi, G., & Hangos, K. M. (2018). *Analysis and control of polynomial dynamic models with biological applications*. London: Academic Press.

Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networks of multi-agent systems. *Proceedings of the IEEE*, *95*(1), 215–233.

Pituk, M. (2017). Oscillation of a linear delay differential equation with slowly varying coefficient. *Applied Mathematics Letters*, *73*, 23–36.

Pituk, M. (2023). Global asymptotic stability of nonautonomous master equations: A proof of the Earnshaw–Keener conjecture. *Journal of Differential Equations*, *364*, 456–470.

Ren, W., & Beard, R. W. (2008). *Communications and control engineering series*, *Distributed consensus in multi-vehicle cooperative control*. London: Springer.

Seneta, E. (1976). Regularly varying functions. In *Lecture notes in math. vol. 508*. Berlin: Springer.

Seneta, E. (2006). *Series in statistics*, *Non-Negative matrices and markov chains* (revised reprint of the 1981 edition). Springer.

Summers, D., & Scott, M. W. (1988). Systems of first-order chemical reactions. *Math. Comp. Modelling*, *10*(12), 901–909.

van Kampen, N. G. (2007). Stochastic processes in physics and chemistry. In *Lecture notes in mathematics, vol. 888* (3rd ed.). Amsterdam-New York: North-Holland Publishing Co..

Wu, C. W. (2006). Synchronization and convergence of linear dynamics in random directed networks. *IEEE Transactions on Automatic Control*, *51*, 1207–1210.