Characterizing ranking stability in interval pairwise comparison matrices

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ARTICLE HISTORY
Journal of Control and Decision, https://doi.org/10.1080/23307706.2023.2179551

ABSTRACT
Relative pairwise comparisons represent the cornerstone for several decision-making methods. Such approaches aim to support complex decision-making situations with multiple alternatives and are essential in order to provide an overall absolute evaluation of the alternatives despite the presence of experts and/or decision-makers with conflicting opinions. Moreover, when decision-maker's opinions are affected by uncertainty, there is the need to analyze the effect of such uncertain measures on the result of the decision-making process. We propose an approach based on a multi objective optimization problem, able to identify the presence of rank reversal issues in order to evaluate the stability of the final outcome of the decision-making process and a metric able to support experts in evaluating the effects of their uncertainty. We characterize the robustness of the ranking with respect to rank reversal by identifying a perturbation that is as small as possible, while causing the maximum number of ordinal swaps.

KEYWORDS
Decision-Making; Pairwise comparison matrix; Rank reversal

1. Introduction

Pairwise comparisons and relative judgements are frequently adopted in Multi-Criteria decision-making (MCDM) methods, such as the Analytic Hierarchy Process (AHP), with the aim to reduce the complexity of evaluating several alternatives according to multiple criteria or in identifying the best alternative among multiple and often hardly comparable ones. The Eigenvector Method (EM) proposed by T. L. Saaty (1977), and the Logarithmic Least Squares approach (LLS, Rabinowitz (1976)) are two of the main approaches used in order to define an absolute ranking on the basis of multiple relative comparisons about several alternatives, provided by multiple experts or decision-makers according to a number of criteria. Such approaches are successfully adopted in a

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wide variety of application fields, such as energy optimization (Ali, Rasheed, Muhammad, and Yousaf (2018)), infrastructures protection (Faramondi, Oliva, and Setola (2020)), and plant construction (Ali, Butt, Sabir, Mumtaz, and Salman (2018)).

Usually, relative judgements are defined according to the well known Saaty's scale (T. L. Saaty (1988)) and are organized in comparison matrices $Y \in \mathbb{R}^{n \times n}$ where the entry Y_{ij} represents the relative score (or utility) of the i-th alternative with respect to the j-th alternative. In the literature, two key issues related to the comparison matrices are the inconsistency of the judgements and the presence of rank reversal. The consistency of data, provided by a decision-maker or an expert, is a classical issue in AHP, and several approaches have been proposed in order to ensure that the decision-maker's relative judgements are not incoherent or at least have a negligible level of incoherence.

Rank reversal, which means that the ordinal positions of two or more decision alternatives in the final ranking could be reversed, has been widely studied in the literature. Such phenomenon can occur due to multiple factors, for example when a relative score is changed or a new alternative is added or deleted. Recent studies highlight that such phenomenon should be considered also when the relative judgements provided by the decision-makers are uncertain. In this context, relative scores are usually defined as fuzzy numbers (Wang, Luo, and Hua (2008)), uncertainty intervals (T. L. Saaty and Vargas (1987)), or associated to a confidence level (Durbach, Lahdelma, and Salminen (2014)). When relative judgements are affected by uncertainty, alternative methods may be used in order to identify the absolute ranking. As highlighted by T. L. Saaty and Vargas (1987) and Faramondi, Oliva, Setola, and Bozóki (2022b), in the presence of uncertainty, such comparison matrices can lead to multiple rankings. The first pioneering study about such phenomenon is presented in T. L. Saaty and Vargas (1987), where the probability of rank reversal is estimated. A similar approach, the *Stochastic Multicriteria Acceptability Analysis* (SMAA-AHP), based on the Monte Carlo simulation, is defined by Durbach et al. (2014) and proposes the definition of indices able to measure the probability of rank reversal. Such approach analyzes how the consistency of judgements and the ability of the AHP model to discern the best alternative deteriorates as uncertainty increases. Cavallo and Brunelli (2018) provide a general unified framework for dealing with comparison matrices whose entries are intervals on real continuous Abelian linearly ordered groups. Such work is useful to unify several approaches proposed in the literature, such as multiplicative, additive and fuzzy comparison matrices. Faramondi et al. (2022b) propose an optimization approach able to identify multiplicative perturbations of the comparison matrices able to compromise the stability of the ordinal ranking and cause a rank reversal phenomenon. The authors extend their work by integrating their results with the SMAA-AHP approach (Faramondi, Oliva, Setola, and Bozóki (2022a)). The stability of the final outcome with respect to the perturbation of a single uncertain relative judgement is studied by Mazurek, Perzina, Ramík, and Bartl (2021).

As mentioned above, the definition of matrices of intervals is one of the possible approaches for the definition of comparison matrices affected by uncertainty (see Cavallo and Brunelli (2018)). Hence, in this case, one of the most popular approaches for the identification of an ordinal ranking is to consider the average values of such intervals and apply EM or LLS. In this work, we propose an optimization approach, derived from a multiobjective optimization problem, able to identify the presence of rank reversal phenomenon in interval pairwise comparison matrices compatible with the given uncertainty intervals. In more detail, we consider a scenario where the utilities of a set of alternatives must be estimated based on the relative preferences of multiple decision-makers. In this view, we aim to identify a perturbation of the nominal pairwise comparisons which is as small as possible in a logarithmic least squares sense, while causing the maximum possible number of ordinal swaps among the alternatives. Notably, the number of swaps (rank reversals) is an intuitive and well understandable index, while this is not necessarily true for cardinal (continuous) measures especially when thresholds of acceptability should be determined.

As discussed above, in previous literature the research of rank reversal phenomenon is usually addressed by randomly sampling the uncertainty intervals in order to identify a set of matrices leading to different ordinal rankings. Notably, although multiobjective optimization has been used in the context of AHP (see, among others, the work by Siraj, Mikhailov, and Keane (2012), where preferences are obtained by simultaneously minimizing deviations from both direct and indirect judgments), to the best of our knowledge, such a framework has not been adopted to assess robustness with respect to rank reversal. In our approach we identify a nominal ranking by applying LLS on the average values of the uncertainty intervals and solve an optimization problem able to obtain a new ordinal ranking characterized by rank reversal. In other words, our approach is able to provide the experts involved in the decision-making process with an evaluation about their uncertainty and and the related possibility to obtain rank reversal issues which involves two or more alternatives. A negative evaluation of the experts' uncertainty can push them towards revising their judgements, while a positive evaluation is a guarantee of the absence, or low probability, of rank reversal issues. Notice that our approach is applicable when multiple decision-makers are involved in the decision-making process and when some alternative are not directly compared (i.e., the sparse setting). The outline of the paper is as follows: In Section 2 we provide some preliminary definitions, while in Section 3 we review some fundamental result on pairwise comparison matrices. In Section 4 we describe our approach for the identification of rank reversal phenomenon and propose a metric able to evaluate the probability of rank reversal with respect to the expert's uncertainty. Results and discussions are collected in Section 5 with the aim to validate the proposed framework; finally some conclusive remarks are collected in Section 6.

2. Preliminaries

2.1. Notation

We denote vectors by boldface lowercase letters and matrices with uppercase letters and we refer to the (i, j)-th entry of a matrix A by A_{ij}. We represent by $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively the column vector with n components all equal to zero and all equal to one. We denote by $\|A\|_F$ the Frobenius norm of a matrix A.

2.2. Graph-theoretical preliminaries

Let $G = \{V, E\}$ be a *graph* with $|V| = n$ nodes, $V = \{v_1, v_2, \dots, v_n\}$, and $e = |E|$ edges, $E \subseteq V \times V$, where $(v_i, v_j) \in E$ represents the existence of a link from node v_i to node v_j. A graph G is *undirected* if $(v_i, v_j) \in E$ whenever $(v_j, v_i) \in E$. Let G be an undirected graph, the *neighborhood* $\mathcal{N}_i$ of a node v_i is the set of nodes v_j, such that $(v_i, v_j) \in E$. For a given node v_i, the *degree* d_i of the node is the number of edges incident on it; i.e., $d_i = |\mathcal{N}_i|$. Given a simple graph $G = \{V, E\}$ with n nodes, we define the *Laplacian matrix* $\mathcal{L}$ as the $n \times n$ matrix such that

$$\mathcal{L}_{ij} = \begin{cases} d_i, & \text{if} \quad i = j \\ -1, & \text{if} \quad (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

2.3. Ordinal relations and rankings distance measures

We denote by $a_i \succ a_j$ the *ordinal relation* between two alternatives from an ordinal perspective. Let $\mathbf{r}$ be a vector in $\mathbb{R}^n$ which represents the utilities of a set of alternatives, the relation $a_i \succ a_j$ is verified if and only if $r_i > r_j$.

Definition 2.1. Let $\mathbf{r}$ be a vector in $\mathbb{R}^n$ which represents the utilities of a set of alternatives. A *matrix ordinal representation* of a ranking $\mathbf{r}$, is an upper triangular matrix $X \in \mathbb{R}^{n \times n}$, defined as follows:

$$X_{ij} = \begin{cases} 1 & \text{if} \quad r_i > r_j \quad \text{and} \quad i < j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2. Let $\mathbf{r}, \mathbf{t}$ be vectors in $\mathbb{R}^n$, each representing the utilities of a set of alternatives. Let $\mathbf{r}$ and $\mathbf{t}$ be two rankings in $\mathbb{R}^n$. An *ordinal swap*, or more simply a *swap*, between two alternatives a_i and a_j occurs whenever the ordinal relation between the two alternatives in $\mathbf{r}$ is reverted in $\mathbf{t}$.

The *Kendall distance* metric $\tau(\mathbf{r}, \mathbf{t})$ is a popular rank-based coefficient for comparing two vectors, $\mathbf{r}$ and $\mathbf{t}$, representing rankings of n elements (see Abdi (2007)). It is based on the number of consecutive pairwise ordinal swaps required to transform one ordinal ranking vector into the other. The metric is defined as:

$$\tau(\mathbf{r}, \mathbf{t}) = \frac{2}{n(n-1)} \sum_{\forall (i,j) s.t., i<j} K_{ij}(\mathbf{r}, \mathbf{t})$$

where $K_{ij}(\mathbf{r}, \mathbf{t}) = 1$ in presence of an ordinal swap (i.e., the i-th and j-th alternatives are in the opposite order (from the ordinal point of view) in $\mathbf{r}$ and $\mathbf{t}$), while $K_{ij}(\mathbf{r}, \mathbf{t}) = 0$ otherwise (i.e., if the two alternatives are in the same order in the two rankings). In this work, we normalize such metric with respect to the maximum value in order to obtain values in the range $[0, 1]$. Considering two rankings $\mathbf{r}$ and $\mathbf{t}$, the lower bound of their distance is $\tau(\mathbf{r}, \mathbf{t}) = 0$, when the two rankings exhibit the same ordinal relations among their entries, while the upper bound is $\tau(\mathbf{r}, \mathbf{t}) = 1$ (when the two rankings are completely reverted). Moreover, such distance can attain only a finite number of values in the range $[0, \dots, 1]$. More precisely, given a ranking $\mathbf{r}$ it is possible to obtain $n!$ reverted rankings, the distance of each reverted ranking from $\mathbf{r}$, computed via Kendall distance, can attain $\frac{n(n-1)}{2} + 1$ values in the range.

For the sake of clarity consider the following absolute ranking $\mathbf{r} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \end{bmatrix}$, which corresponds to an ordinal ranking where $a_1 \succ a_2 \succ a_3$. In Table 1 we summarize the Kendall distances between $\mathbf{r}$ and a set of six rankings $(\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(6)})$ which encompass three alternatives.

Notice that $\tau(\mathbf{s}, \mathbf{r}^{(1)}) = 0$ in fact, despite the numerical differences about the entries of such rankings, the two utility vector reflect the same ordinal rankings where

	$\mathbf{s}^{(1)} = [0.7\ 0.2\ 0.1]$	$\mathbf{s}^{(2)} = [0.7\ 0.1\ 0.2]$	$\mathbf{s}^{(3)} = [0.2\ 0.1\ 0.7]$	$\mathbf{s}^{(4)} = [0.2\ 0.7\ 0.1]$	$\mathbf{s}^{(5)} = [0.1\ 0.7\ 0.2]$	$\mathbf{s}^{(6)} = [0.1\ 0.2\ 0.7]$
$\mathbf{r} = [0.5\ 0.3\ 0.2]$	0 (0)	0.33 (1)	0.66 (2)	0.33 (1)	0.66 (2)	1 (3)

Table 1. Kendall distances among rankings. The associated numbers of ordinal swap are given in brackets.

$a_1 \succ a_2 \succ a_3$. According to Table 1, $\tau(\mathbf{r}, \mathbf{s}^{(2)}) = \tau(\mathbf{r}, \mathbf{s}^{(4)}) = 0.33$, in fact, both $\mathbf{s}^{(2)}$ and $\mathbf{s}^{(4)}$ differs from $\mathbf{r}$ for one ordinal swap. With respect to the ordinal relations in $\mathbf{s}$, in $\mathbf{r}^{(2)}$ the swap is related to the alternatives a_2 and a_3, while in $\mathbf{r}^{(4)}$ the swap involves alternatives a_1 and a_2. Concerning $\mathbf{r}^{(3)}$ and $\mathbf{r}^{(5)}$, both the ranking differ from $\mathbf{s}$ for two ordinal swaps, while $\mathbf{r}^{(6)}$ is completely reverted with respect to $\mathbf{s}$ (i.e., three ordinal swaps occur). Hence, in the presence of s ordinal swaps between two rankings $\mathbf{r}$ and $\mathbf{t} \in \mathbb{R}^n$, the corresponding Kendall distance is $\frac{2s}{n(n-1)}$.

2.4. Multiobjective optimization preliminaries

A multiobjective optimization problem can be expressed as:

$$\min \mathbf{f}(\mathbf{x}) = \{f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\}, \quad \text{subject to} \quad \mathbf{x} \in \mathscr{F},$$

where $k \geq 2$ and the i-th objective is given by

$$f_i(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}, \quad \text{for} \quad i = 1, \ldots, k,$$

while $\mathbf{f}(\mathbf{x})$ is the multiobjective function. The set $\mathscr{F}$ represents the set of admissible solutions for the problem at hand. In multiobjective problems, the aim is to identify multiple admissible solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(p)}$ that represent trade-offs among the minimization of the different objectives, as discussed next.

Definition 2.3. The *objective-space* is defined as: $\mathscr{S} = \{\mathbf{s} \in \mathbb{R}^k : \exists \mathbf{x} \in \mathscr{F}, \mathbf{s} = \mathbf{f}(\mathbf{x})\}$.

Definition 2.4. Let $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)} \in \mathscr{F}$, $\mathbf{s}^{(2)}$ is *Pareto-dominated* by $\mathbf{s}^{(1)}$ ($\mathbf{s}^{(1)} \leq_P \mathbf{s}^{(2)}$) if $\mathbf{s}_i^{(1)} \leq \mathbf{s}_i^{(2)}$ for each $i = 1, \ldots, k$, and $\mathbf{s}_j^{(1)} < \mathbf{s}_j^{(2)}$ at least for a value of $j = 1, \ldots, k$.

Definition 2.5. A solution $\mathbf{x}^* \in \mathscr{F}$ is a *Pareto optimal solution* if there is no other solution $\mathbf{x} \in \mathscr{F}$ such that $\mathbf{f}(\mathbf{x}) \leq_P \mathbf{f}(\mathbf{x}^*)$.

Definition 2.6. The *Pareto front* $\mathscr{P}$ is the set of all possible Pareto optimal solutions $\mathbf{x}^*$ for the problem at hand.

3. Comparison matrices and ranking identification

In this section we briefly summarize the main aspects of decision-making process based on relative judgements. The aim is the identification of an absolute ranking $\mathbf{y} \in \mathbb{R}^n$ whose components $\mathbf{y}_i$ represent the relevance or the utility of the alternative a_i. Such process is based on relative judgements usually collected in *pairwise comparison matrices* (PCMs). A PCM $Y \in \mathbb{R}^{n \times n}$ is a matrix whose elements Y_{ij} are usually defined according to the well known Saaty's scale (see Table 2). Each entry Y_{ij} describes the relative relevance of the alternative a_i with respect to the alternative a_j (e.g. if

$Y_{12} = 3$, hence the alternative a_1 is three times better than the alternative a_2). The PCM is *locally consistent* if $Y_{ij} = \frac{1}{Y_{ji}}$ for each $i, j = 1, \ldots, n$. A PCM Y is *consistent* if $Y_{ik} = Y_{ij}Y_{jk}$ for each $i, j, k = 1, \ldots, n$. A *sparse* (or incomplete) comparison matrix (SPCM) is a particular PCM where Y_{ij} is unknown for some i, j. In the sequel, this will be technically denoted by $Y_{ij} = 0$. The evaluation of the inconsistency degree of the given comparison matrices represents a fundamental preliminary step in decision-making process. In particular, in the complete information context (i.e., when Y_{ij} is known $\forall i, j$), according to Ágoston and Csató (2022), highly inconsistent PCM results in unreliable rankings and should not be considered. The *Consistency Index* is the most frequently used approach for the evaluation of consistency degree of a given instance, it is based on the dominant eigenvalue of the comparison matrix $Y \in \mathbb{R}^{n \times n}$:

$$CI(Y) = \frac{\lambda_n\{Y\} - n}{n - 1}, \tag{1}$$

where n represents the number of considered alternatives. Moreover, Saaty proposed to normalize such index with respect to the so-called *Random Index* RI_n which is the average $CI(Y)$ computed by considering a large number of random pairwise comparison matrices of degree n, thus obtaining the *Consistency Ratio* as in Equation (2).

$$CR(Y) = \frac{CI(Y)}{RI_n} \tag{2}$$

If CR is smaller or equal to 10%, the inconsistency is considered acceptable and the absolute utilities can be computed (e.g. via EM or LLS), if instead CR is greater than such threshold, it is suggested to revise the subjective relative judgment in order to reduce such inconsistency (R. W. Saaty (1987)). A similar approach is applied also in the sparse setting. The *Sparse Consistency Index* (Ágoston and Csató (2022)) directly derives from the consistency index proposed by Saaty in Equation (1). In the sparse setting it is defined as:

$$\widetilde{CI}(Y) = \frac{\lambda_n\{\mathcal{M}\} - n}{n - 1}, \tag{3}$$

where $\mathcal{M}$ is an *auxiliary matrix* obtained from the sparse comparison matrix Y according to one of the procedures described in Ágoston and Csató (2022) and Bozóki, Fülöp, and Rónyai (2010). Similarly to the complete information setting, the proposed index requires a normalization. In fact, in Ágoston and Csató (2022) the authors define an alternative Random Index $\widetilde{RI}_{n,m}$ also for the sparse setting which is a function of matrix size n and the number of missing relative comparisons m. As a result, the *Consistency Ratio for sparse comparison matrices* $\widetilde{CR}$ is defined as:

$$\widetilde{CR}(Y) = \frac{\widetilde{CI}(Y)}{\widetilde{RI}_{n,m}}. \tag{4}$$

Without any changes, as in the complete information setting, the rule about the 10% threshold for the ratio $\widetilde{CI}/\widetilde{CR}_{n,m}$ can be adopted.

The aim of MCDM methods is the identification of the absolute utilities $\mathbf{y}_i$ on the basis of the given relative judgements Y_{ij}. In the literature, two main classes of ap-

Y_{ij}	**Definition**
1	Equal importance
3	Moderate importance of one over another
5	Essential or strong importance
7	Very strong importance
9	Extreme importance
2, 4, 6, 8	Intermediate values between the two adjacent judgements

Table 2. Saaty's scale for comparing the importance of criteria in AHP (for comparing alternatives, *importance* is replaced by *preference*).

proaches for the estimation of absolute utilities have been defined: *extremal methods* and *eigenvector methods*. The class of extremal methods amounts to a set of optimization problems aiming to minimize a distance function between the relative judgements given by the experts Y_{ij} and the unknown absolute utilities $\mathbf{y}_i$ (Chu, Kalaba, and Spingarn (1979); Cook and Kress (1988); Crawford and Williams (1985)). Concerning the class of eigenvector methods, in T. L. Saaty and Hu (1998), Saaty proposes the Eigenvector Method (EM), i.e., the adoption of the dominant eigenvector of the PCM as the desired vector of utilities. As mentioned above, in the literature, there is no universal consent on how to estimate the utilities $\mathbf{y}_i$, see for instance the debate in Dyer (1990); Menci, Oliva, Papi, Setola, and Scala (2018); T. L. Saaty (1990).

3.1. Logarithmic Least Squares

The Logarithmic Least Squares (LLS) method (Crawford and Williams (1985); Rabinowitz (1976)) represents a valid approach for the identification of the absolute utilities $\mathbf{y}_i$ on the basis of the given relative judgements Y_{ij} provided by the experts. Such approach is just one of the many procedures that belong the class of extremal methods: the *Weighted Least Squares* (Blankmeyer (1987)), the *Direct Least Squares*, (Barzilai and Golany (1990); Chu et al. (1979)), and the *Logarithmic Least Absolute Values* (Cook and Kress (1988)), just to name a few. The LLS problem is defined as follows:

$$\mathbf{y}^* = \operatorname*{argmin}_{\mathbf{x}\in\mathbb{R}^{\mathbf{n}}_{+}} \left\{ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln(Y_{ij}) - \ln\left(\frac{x_i}{x_j}\right)\right)^2 \right\}.$$

In this work we rely on the Incomplete Logarithmic Least Squares (ILLS, (Kwiesielewicz, 1996)) approach, which extends the Logarithmic Least Squares (LLS) approach including also the case of sparse comparison matrices.

Specifically, the aim of ILLS is to find a vector $\mathbf{y}^*$, such that

$$\mathbf{y}^* = \operatorname*{argmin}_{\mathbf{x}\in\mathbb{R}^{\mathbf{n}}_{+}} \left\{ \frac{1}{2} \sum_{i=1}^{n} \sum_{v_j\in\mathcal{N}_i} \left(\ln(Y_{ij}) - \ln\left(\frac{x_i}{x_j}\right)\right)^2 \right\}. \quad (5)$$

Notice that, the notation $v_j \in \mathcal{N}_i$ is based on the *graph representation* of the sparse comparison matrix. The graph $G = \{V, E\}$ represents the graph underlying Y, the nodes $V = v_1, \dots, v_n$ correspond to the alternatives $a_1, \dots, a_n$, while the edges in E are associated to the given relative judgements (outside the diagonal), hence $(v_i, v_j) \in$

$E \iff Y_{ij}$ is known and $i \neq j$. Differently from the complete LLS method, the notation $v_j \in \mathcal{N}_i$ allows us to consider only the defined ratios in the sparse pairwise comparison matrix.

An effective strategy to solve ILLS is to operate the substitution $\mathbf{z} = \ln(\mathbf{x})$, where $\ln(\cdot)$, is the component-wise logarithm, so that Equation (5) can be rearranged as:

$$\mathbf{y}^* = \exp\left(\underset{\mathbf{z}\in\mathbb{R}^\mathbf{n}}{\operatorname{argmin}}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{v_j\in\mathcal{N}_i}\Big(\ln(Y_{ij}) - z_i + z_j\Big)^2\right\}\right),$$

where $\exp(\cdot)$ is the component-wise exponential. This allows to handle an equivalent problem which is convex, thus greatly simplifying the solution.

3.1.1. Multiple experts evaluations

With the notation $Y_{ij}^{(k)}$ we indicate the relative importance of the alternative a_i with respect to the alternative a_j expressed by the expert k. As mentioned above, in this work we want to characterize the stability of the priority vector $\mathbf{y}$ when m multiple experts provide their uncertain relative judgements by defining SPCMs $(Y^{(1)}, \dots, Y^{(m)})$ about the same n alternatives.

Usually, when multiple experts are involved in the decision-making process and preferences are devoid of uncertainty, ILLS is extended as follows[1]:

$$\mathbf{y}^* = \exp\left(\underset{\mathbf{z}\in\mathbb{R}^\mathbf{n}}{\operatorname{argmin}}\left\{\frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{v_j\in\mathcal{N}_i}\Big(\ln(Y_{ij}^{(k)}) - z_i + z_j\Big)^2\right\}\right).$$

where we denote by $Y^{(i)}$ the SPCM defined by the i-th expert. We now define

$$\kappa(\mathbf{z}) = \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i}\Big(\ln(Y_{ij}^{(k)}) - z_i + z_j\Big)^2. \tag{6}$$

Thanks to the substitution $\mathbf{z} = \ln(\mathbf{x})$, the problem is now convex[2] and unconstrained, hence its global minimum is in the form $\mathbf{y}^* = \exp(\mathbf{z}^*)$, where $\mathbf{z}^*$ satisfies for each entry z_i:

$$\left.\frac{\partial\kappa(\mathbf{z})}{\partial z_i}\right|_{\mathbf{z}=\mathbf{z}^*} = \sum_{k=1}^{m}\sum_{j\in\mathcal{N}_i}\Big(\ln(Y_{ij}^{(k)}) - z_i^* + z_j^*\Big) = 0. \tag{7}$$

We now define the $n \times n$ matrix $P^{(k)}$ such that $P_{ij}^{(k)} = \ln(Y_{ij}^{(k)})$ if $Y_{ij}^{(k)} > 0$ and $P_{ij}^{(k)}$=0 otherwise; we can express Equation (7) as

[1]Also in this case, we operate the substitution $\mathbf{z} = \ln(\mathbf{x})$; we only report the equivalent problem formulation with respect to the variable $\boldsymbol{z}$ for the sake of brevity.

[2]It can be easily shown that the Hessian matrix of $\kappa(\cdot)$ is the Laplacian matrix $\mathcal{L}$, which is positive semidefinite, thus implying convexity of $\kappa(\cdot)$.

$$\sum_{k=1}^{m} \left(P^{(k)} \mathbf{1}_n - \mathcal{L}^{(k)} \mathbf{z}^* \right) = \mathbf{0}_n \tag{8}$$

where $\mathcal{L}^{(k)}$ is the Laplacian matrix of the graph $G = \{V, E\}$ underlying the PCM $Y^{(k)}$.

3.2. Handling uncertainty in pairwise comparisons

As highlighted in T. L. Saaty and Vargas (1987), it is essential to extend classical approaches based on pairwise comparisons by including the concept of expert's uncertainty or indecision. In this view, T. L. Saaty and Vargas (1987) introduce the *interval pairwise comparison matrix* (IPCM). An IPCM is defined as a matrix of uncertainty intervals:

$$\widetilde{Y} = \begin{bmatrix} 1 & [l_{12}, u_{12}] & \dots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \dots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \dots & 1 \end{bmatrix},$$

where $l_{ij} \leq u_{ij}$, $l_{ij} = 1/u_{ji}$, and $u_{ij} = 1/l_{ji}$ for each $i, j = 1, \dots, n$. In this way, this formalism allows to account for the indecision of the decision-maker. Notice that, due to consistency issues, it is required that either $(l_{ij}, u_{ij}) \subseteq [1, 9]$, or $(l_{ij}, u_{ij}) \subseteq [1/9, 1]$. In other words, the entire interval must either be above or equal to one or below or equal to one. Such requirement is essential to avoid intervals where $l_{ij} < 1$ and $u_{ij} > 1$. This case would include conflicting evaluations for which a_i is better than a_j, and at the same time a_j is better than a_i (Cavallo and Brunelli (2018)).

Notice that, in the presence of uncertainty, approaches such as ILLS are not directly applicable. In this case, according to T. L. Saaty and Vargas (1987), suggested approaches randomly define traditional PCMs by choosing entries within the bounds $[l_{ij}, u_{ij}]$. Alternative approaches amount to the definition of a PCM $\overline{Y}$ based on the average values of the uncertainty intervals and the formulation of ILLS on such instance. More precisely, in this work we define such matrix as:

$$\overline{Y}_{ij} = \begin{cases} \frac{(l_{ij}+u_{ij})}{2} & \text{if} \quad l_{ij} \geq 1 \\ \frac{1}{\overline{Y}_{ji}} & \text{otherwise.} \end{cases} \tag{9}$$

As highlighted by T. L. Saaty and Vargas (1987), when absolute utilities, computed via eigenvector methods, are obtained on the basis of uncertain relative judgements it is essential verify the stability of such solution and the presence of rank reversal phenomena.

4. Ranking robustness identification

In this section we propose a multiobjective optimization problem with the aim to identify the presence of rank reversal phenomena in ranking estimation based on interval pairwise comparison matrices $(\widetilde{Y}^{(1)}, \ldots, \widetilde{Y}^{(m)})$ provided by m experts. Let $\mathbf{y}$ be the nominal ranking obtained via ILLS on the basis of the average values of each interval $[l_{ij}^{(k)}, u_{ij}^{(k)}]$ for each expert k, according to Equation (9). The aim of the proposed problem is the identification of m PCMs $(\widehat{Y}_1, \ldots, \widehat{Y}_m)$ compatible with the indecision intervals, which correspond to an absolute ranking $\widehat{\mathbf{y}}$ different from $\mathbf{y}$ from the ordinal point of view, hence characterized by rank reversal.

Such problem represents the cornerstone for the definition of an index able to measure the robustness of the given instances with respect to the rank reversal phenomenon.

We point out that instances characterized by a high degree of uncertainty (i.e., large indecision intervals $[l_{ij}^{(k)}, u_{ij}^{(k)}]$) are more prone to rank reversal with respect to instances characterized by a small uncertainty. Based on this intuition, our formulation is characterized by two (usually conflicting) objective functions:

- *Comparison matrices distance:* we want to minimize the distance between the identified *perturbed matrices* $(\widehat{Y}^{(1)}, \ldots, \widehat{Y}^{(m)})$ and the corresponding comparison matrices $(\overline{Y}^{(1)}, \ldots, \overline{Y}^{(m)})$ defined according to the average value of each interval $[l_{ij}^{(k)}, u_{ij}^{(k)}]$ (see Equation (9)). More precisely, we rely on the Frobenius norm as a measure of the distances between the matrices. Hence, we minimize the following function:

$$f_1(\widehat{Y}^{(1)}, \ldots, \widehat{Y}^{(m)}) = \frac{1}{2} \sum_{k=1}^{m} \| \ln(\overline{Y}^{(k)}) - \ln(\widehat{Y}^{(k)}) \|_F^2.$$

- *Ranking distance:* the second objective function is related to the maximization of the distance between the nominal ranking $\mathbf{y}$, obtained considering the average values of the intervals $[l_{ij}^{(k)}, u_{ij}^{(k)}]$, and the ranking $\widehat{\mathbf{y}}$ obtained via ILLS on the basis of the identified perturbed matrices $\widehat{Y}^{(1)}, \ldots, \widehat{Y}^{(m)}$. In this objective function we rely on the Kendall Distance.

Let us now formulate the problem at hand.

Problem 1. Let an interval comparison matrix $\widetilde{Y}^{(k)}$ be given for each decision-maker k, and let $\overline{Y}^{(k)}$ be the comparison matrix which entries are defined as the average value of each uncertainty interval of $\widetilde{Y}^{(k)}$ as in Equation (9). Finally, let $\mathbf{y}$ be the nominal absolute ranking based on such average ratio matrices. Find the matrices $\widehat{Y}^{(k)} \in \mathbb{R}^{n \times n}$

and the vector $\widehat{\mathbf{y}} \in \mathbb{R}^n$ that solve:

$$\underset{\widehat{Y}^{(1)},\ldots,\widehat{Y}^{(m)},\widehat{\mathbf{y}}}{\text{minimize}} \quad \left\{ f_1(\cdot) = \frac{1}{2}\sum_{k=1}^{m} \| \ln(\overline{Y}^{(k)}) - \ln(\widehat{Y}^{(k)}) \|_F^2, \quad f_2(\cdot) = -\tau(\widehat{\mathbf{y}}, \mathbf{y}) \right\} \tag{10a}$$

subject to

$$\widehat{\mathbf{y}} = \min_{\mathbf{x}\in\mathbb{R}_+^{\mathbf{n}}} \left\{ \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j\in N_i} \left(\ln(\widehat{Y}_{ij}^{(k)}) - \ln\left(\frac{x_i}{x_j}\right)\right)^2 \right\}, \tag{10b}$$

$$\widehat{Y}_{ij}^{(k)} \frac{1}{\widehat{Y}_{ji}^{(k)}} = 1 \qquad \forall\, (k,i,j) \text{ s.t. } \overline{Y}_{ij}^{(k)} \neq 0, \tag{10c}$$

$$l_{ij}^{(k)} \leq \widehat{Y}_{ij}^{(k)} \leq u_{ij}^{(k)} \qquad \forall\, (k,i,j) \text{ s.t. } \overline{Y}_{ij}^{(k)} \neq 0, \tag{10d}$$

$$\widehat{Y}^{(1)},\ldots,\widehat{Y}^{(m)} \in \mathbb{R}^{n\times n}, \tag{10e}$$

$$\widehat{\mathbf{y}} \in \mathbb{R}^n. \tag{10f}$$

The proposed formulation is a multi-objective problem characterized by three set of constraints. The constraint (10b) is necessary in order to define the new absolute ranking $\widehat{\mathbf{y}}$ with respect to the identified comparison matrices $\widehat{Y}^{(1)},\ldots,\widehat{Y}^{(m)}$. Such identified matrices are locally consistent as required by (10c) and, moreover, the matrices are compatible with respect to the intervals defined by the expert in the IPCMs $\widetilde{Y}^{(1)},\ldots,\widetilde{Y}^{(m)}$ as required by the constraints (10d). Notice that, the solution of Problem 1 consists of a set of non-dominated solutions $\{s_1,\ldots,s_l\}$ where each solution is characterized by a couple of values, i.e., the values attained for the two objective functions $f_1(\cdot)$ and $f_2(\cdot)$. Notice further that, as mentioned above, $f_2(\cdot)$ represents the normalized Kendall distance between the nominal ranking $\mathbf{y}$ and the obtained altered ranking $\widehat{\mathbf{y}}$ and, as suggested in Section 2.3, such function can attain only a finite and discrete number of values. On the basis of such characteristic about the values that $f_2(\cdot)$ can attain, we propose a new formulation of Problem 1. In the following new formulation, the distance between the nominal ranking $\mathbf{y}$ and the new altered ranking $\widehat{\mathbf{y}}$ is now considered as a constraint in Constraint (11e). On the basis of such transformation and similarly to the approach proposed by Bérubé, Gendreau, and Potvin (2007), we can obtain the Pareto front of Problem 1, by iteratively solving Problem 2 for multiple values of the parameter d, where $d \in [-1,\ldots,0]$.

Problem 2. Let an interval comparison matrix $\widetilde{Y}^{(k)}$ be given for each decision-maker k, and let $\overline{Y}^{(k)}$ be the comparison matrix which entries are defined as the average value of each uncertainty interval of $\widetilde{Y}^{(k)}$ as in Equation (9). Finally, let $\mathbf{y}$ be the nominal absolute ranking based on such average ratio matrices. Find the matrices $\widehat{Y}^{(k)}$ and the vector $\widehat{\mathbf{y}}$ that solve:

$$\underset{\widehat{Y}^{(1)},\dots,\widehat{Y}^{(m)},\widehat{\mathbf{y}}}{\text{minimize}} \quad f_1(\cdot) = \frac{1}{2}\sum_{k=1}^{m} \| \ln(\overline{Y}^{(k)}) - \ln(\widehat{Y}^{(k)})\|_F^2 \tag{11a}$$

subject to

$$\widehat{\mathbf{y}} = \min_{\mathbf{x}\in\mathbb{R}^{\mathbf{n}}_+} \left\{ \frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j\in N_i} \left(\ln(\widehat{Y}_{ij}^{(k)}) - \ln\left(\frac{x_i}{x_j}\right)\right)^2 \right\}, \tag{11b}$$

$$\widehat{Y}_{ij}^{(k)} \frac{1}{\widehat{Y}_{ji}^{(k)}} = 1 \qquad \forall\,(k,i,j) \text{ s.t. } \overline{Y}_{ij}^{(k)} \neq 0, \tag{11c}$$

$$l_{ij}^{(k)} \leq \widehat{Y}_{ij}^{(k)} \leq u_{ij}^{(k)} \qquad \forall\,(k,i,j) \text{ s.t. } \overline{Y}_{ij}^{(k)} \neq 0, \tag{11d}$$

$$-\tau(\widehat{\mathbf{y}}, \mathbf{y}) = d, \tag{11e}$$

$$\widehat{Y}^{(1)},\dots,\widehat{Y}^{(m)} \in \mathbb{R}^{n\times n}, \tag{11f}$$

$$\widehat{\mathbf{y}} \in \mathbb{R}^n. \tag{11g}$$

With the aim to propose a linear formulation of the proposed problem, we now operate the following substitutions:

$$\widehat{\mathbf{z}}_i = \ln(\widehat{\mathbf{y}}_i), \quad \mathbf{z}_i = \ln(\mathbf{y}_i) \tag{12}$$

and

$$\overline{P}_{ij}^{(k)} = \begin{cases} \ln(\overline{Y}_{ij}^{(k)}) & \text{if } \overline{Y}_{ij}^{(k)} > 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \widehat{P}_{ij}^{(k)} = \begin{cases} \ln(\widehat{Y}_{ij}^{(k)}) & \text{if } \widehat{Y}_{ij}^{(k)} > 0 \\ 0 & \text{otherwise} \end{cases}.$$

According to such substitutions we propose the following formulation.

Problem 3. On the basis of the uncertain expert's evaluations $\widetilde{Y}^{(1)},\dots,\widetilde{Y}^{(m)}$, let $\overline{Y}^{(1)},\dots,\overline{Y}^{(m)}$ be the PCMs obtained according to Equation (9). Let $\overline{P}^{(1)},\dots,\overline{P}^{(m)}$ be the $n \times n$ matrices computed as in Equation (12). Let X and $\widehat{X}$ be respectively the ordinal representation of the absolute rankings $\mathbf{y}$ and $\widehat{\mathbf{y}}$. Find the matrices $\widehat{P}^{(k)}$ and the vector $\widehat{\mathbf{z}}$ that solve:

$$\underset{\widehat{P}^{(1)},\dots,\widehat{P}^{(m)},\widehat{\mathbf{z}}}{\text{minimize}} \quad f_1(\cdot) = \frac{1}{2}\sum_{k=1}^{m} \|\overline{P}^{(k)} - \widehat{P}^{(k)}\|_F^2 \tag{13a}$$

subject to

$$\sum_{k=1}^{m} \mathcal{L}^{(k)}\,\widehat{\mathbf{z}} = \sum_{k=1}^{m} \widehat{P}^{(k)}\mathbf{1}_n, \tag{13b}$$

$$\widehat{P}_{ij}^{(k)} = -\widehat{P}_{ji}^{(k)} \qquad \forall\,(k,i,j) \text{ s.t. } \overline{P}_{ij}^{(k)} \neq 0, \tag{13c}$$

$$\ln(l_{ij}^{(k)}) \le \widehat{P}_{ij}^{(k)} \le \ln(u_{ij}^{(k)}) \qquad \forall\,(k,i,j) \text{ s.t. } \overline{P}_{ij}^{(k)} \neq 0, \tag{13d}$$

$$\widehat{z}_i - \widehat{z}_j \le M\widehat{X}_{ij} \qquad \forall\, i<j, \tag{13e}$$

$$M(\widehat{X}_{ij} - 1) \le \widehat{z}_i - \widehat{z}_j - \epsilon \qquad \forall\, i<j, \tag{13f}$$

$$R_{ij} \ge X_{ij} - \widehat{X}_{ij} \qquad \forall\, i<j, \tag{13g}$$

$$R_{ij} \ge \widehat{X}_{ij} - X_{ij} \qquad \forall\, i<j, \tag{13h}$$

$$R_{ij} \le \widehat{X}_{ij} + X_{ij} \qquad \forall\, i<j, \tag{13i}$$

$$2 - R_{ij} \ge \widehat{X}_{ij} + X_{ij} \qquad \forall\, i<j, \tag{13j}$$

$$-\frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=2\\ i<j}}^{n} R_{ij} = d, \tag{13k}$$

$$\widehat{P}^{(1)},\dots,\widehat{P}^{(m)} \in \mathbb{R}^{n\times n}, \tag{13l}$$

$$\widehat{\mathbf{z}} \in \mathbb{R}^{n}., \tag{13m}$$

$$X, R \in \{0,1\}^{n\times n} \tag{13n}$$

The formulation of Problem 3 directly comes from Problem 2, hence, also in this case the objective is the minimization of the distance among the identified matrices $\widehat{P}^{(k)}$ and the matrices $\overline{P}^{(k)}$, derived by $\overline{Y}^{(u)}$ according to Equation (12). On the basis of Equation (8) we can write the constraint (11b) as

$$\sum_{k=1}^{m} \mathcal{L}^{(k)}\,\widehat{\mathbf{z}} = \sum_{k=1}^{m} \widehat{P}^{(k)}\mathbf{1}_n.$$

The local consistency for each couple of alternatives (a_i, a_j) and each expert k is required by Constraint (13c) which directly come from Constraint (11c). The limits about the entries of the perturbed matrices $\widehat{P}^{(k)}$, compatible with the expert uncertainty, are defined by Constraint (13d) similarly to Constraint (11d) in Problem 2. Let M be a large positive constant and let ϵ be a small constant close to 0, Constraints (13e) and (13f) are necessary in order to define the matrix ordinal representation $\widehat{X}$ of $\widehat{\mathbf{z}}$. The set of constraints from (13g) to (13j) corresponds to the definition of the $n \times n$ matrix R which entries R_{ij} depends on the values of $\widehat{X}$ and X as follows:

$$R_{ij} = \begin{cases} 1 & \text{if} \quad \widehat{X}_{ij} \neq X_{ij} \quad \text{and} \quad i<j \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Finally, Constraint (13k) corresponds to the definition of the Kendall distance between $\mathbf{z}$ and $\widehat{\mathbf{z}}$. In this constraint it is required to consider feasible solution where the Kendall distance is equal to an arbitrary value d. We reiterate that for two given rankings with n components, the Kendall distance can attain only $\frac{n(n-1)}{2}+1$ different values.

To conclude the section, let us now discuss a useful approach to normalize the objective function values in order to provide to the decision-makers information about rank reversal issues due to their uncertainty. As discussed above, since the set of possible values attained by $f_2(\cdot)$ is finite, our problem can be equivalently formulated as a family of single-objective problems (one for every possible choice of the number of swaps, or a particular choice of the parameter d) is solved in lieu of a single multi-objective problem. By construction, the largest value attained by $f_1(\cdot)$ corresponds to the case where the maximum number of feasible swaps is identified. Notice that, there is not guarantee about the presence of feasible solutions in Problem 3 for each value of d, hence for each number of ordinal swaps.

We now refer to the optimal value for $f_1(\cdot)$ when a Kendall distance d is considered as $f_1^*[d]$. We observe that the maximum possible value attained by $f_1(\cdot)$ is obtained when all terms $\widehat{Y}_{ij}^{(k)}$ correspond to the lower limits $l_{ij}^{(k)}$ for each (i,j) such that $l_{ij}^{(k)} \geq 1$. On the basis of such assumption

$$f_{UB} = 2\sum_{k=1}^{m} \sum_{\substack{\forall (i,j) \text{ s.t.} \\ l_{ij}^{(k)} \geq 1}} \left(\ln(l_{ij}^{(k)}) - \ln(\overline{Y}_{ij}^{(k)}) \right)^2 \tag{15}$$

represents an upper bound (which may or may not be attained by one of the non-dominated solutions) on $f_1^*[d]$ for all d. Therefore, a possible normalization is given by

$$f_1^{\dagger}[d] = \frac{f_1^*[d]}{f_{UB}} \in [0,1].$$

Notably, the larger is $f_1^{\dagger}[d]$, the closer a non-dominated solution is to a case where, to be able to obtain d rank reversals, we need to apply the maximum admissible perturbation according to the expert uncertainty. In this case, the obtained ranking is robust to rank reversal with respect to the given uncertainty intervals. On the contrary, a small value of $f_1^{\dagger}[d]$ implies that the perturbations required in order to obtain a new altered ranking (such that the Kendall distance from the nominal ranking is equal to $-d$) are small with respect to the given uncertainty intervals, hence the ranking exhibits no robustness to rank reversal. It is possible to exclude a rank reversal, characterized by a Kendall distance equal to $-d$, only if for such specific choice of d, Problem 3 is not feasible, hence perturbation matrices able to provide a new perturbed ranking (such that the Kendall distance from the nominal ranking is equal to $-d$) does not exist.

5. Illustrative Examples

In this section we provide experimental results able to prove the efficiency of our framework for the estimation of the rank robustness in multi-expert decision-making process. We now consider $m = 3$ decision-makers who express their preferences about $n = 3$ alternatives (a_1, a_2, and a_3) in terms of interval pairwise comparison matrices:

$$\widetilde{Y}^{(1)} = \begin{bmatrix} 1 & [1,2] & [1,2] \\ [1/2,1] & 1 & [1,2] \\ [1/2,1] & [1/2,1] & 1 \end{bmatrix} \quad \widetilde{Y}^{(2)} = \begin{bmatrix} 1 & [1/3,1/2] & [1/8,1/4] \\ [2,3] & 1 & [1/3,1/2] \\ [4,8] & [2,3] & 1 \end{bmatrix} \tag{16}$$

$$\widetilde{Y}^{(3)} = \begin{bmatrix} 1 & [1,2] & [2,4] \\ [1/2,1] & 1 & [3,6] \\ [1/4,1/2] & [1/6,1/3] & 1 \end{bmatrix}.$$

Let $\overline{Y}^{(1)}$, $\overline{Y}^{(2)}$, and $\overline{Y}^{(3)} \in \mathbb{R}^{n\times n}$ be the comparison matrices, whose entries $\overline{Y}_{ij}^{(k)}$ represent the average value of the interval $[l_{ij}^{(k)}, u_{ij}^{(k)}]$ according to Equation (9). Notice that the priority vector of such instance, computed according to ILLS is

$$\overline{\mathbf{y}} = \begin{bmatrix} 0.3234 & 0.3750 & 0.3016 \end{bmatrix}^T$$

which represents an ordinal ranking such that $a_2 \succ a_1 \succ a_3$. Considering Problem 3, our aim is the identification of a set of solutions which consist in three perturbed comparison matrices $\widehat{Y}^{(1)}$, $\widehat{Y}^{(2)}$, and $\widehat{Y}^{(3)} \in \mathbb{R}^{n\times n}$, compatible with the experts uncertainty. Such matrices lead to an altered ordinal ranking $\widehat{\mathbf{y}}$ such that $\tau(\widehat{\mathbf{y}}, \mathbf{y}) = -d$. The instance has been analyzed by solving Problem 3 for each possible value of d. According to Section 2.3 we reiterate that the Kendall distance can attain $\frac{n(n-1)}{2} + 1$ values, where n is the size of the ranking vector. Moreover, in the presence of s ordinal swaps, the corresponding Kendall distance is $\frac{2s}{n(n-1)}$.

Since $n = 3$ we iteratively solve Problem 3 for $d \in \{0, -1/3, -2/3, -1\}$, which corresponds to the case of an increasing number of ordinal swaps from 0 to 3.

In Table 3 we collect the details of each solution $(\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(4)})$ of Problem 3 for each possible value of the parameter d.

Notably, given the complexity of the problem at hand, we seek an approximated solution resorting to an approximate solver, namely, Mosek in Python environment (see Andersen and Andersen (2000) for additional detail about the solver).

The results of Problem 3 show that the experts' uncertainty reflects in rank reversal phenomena for any value of d, moreover such phenomena could severely compromise the ordinal ranking by causing a complete rank inversion (i.e., in solution $\mathbf{s}^{(1)}$, where $f_1 = 0.2232$ and $d = -1$). Hence, for each value of d, it is possible to identify three matrices which lead to the altered ordinal ranking. For example when it is required to completely revert the ranking (i.e., when $d = -1$) we have $\widehat{\mathbf{y}} = \begin{bmatrix} 0.3333 & 0.3332 & 0.3334 \end{bmatrix}^T$ where $a_3 \succ a_1 \succ a_2$. Notice that, in the last column of Table 3 we provide the objective function value of Problem 3, normalized with respect to the upper bound $f_{UB} = 2.5008$ as defined by Equation (15). Notice that, for each value of the parameter d, the values of $f_1^\dagger[d]$ are close to zero. Such values suggest that the expert uncertainty is extremely large therefore the rank exhibits no robustness to rank reversal for any value of d, in fact numerical results highlight that, given the experts' uncertainty, it is possible to completely revert the ordinal ranking considering small perturbations of the PCM $\overline{Y}^{(k)}$. Moreover, notice that, for each expert k, the perturbed matrices, summarized in Table 5, are compatible with the his/her uncertainty (i.e. $\widehat{Y}_{ij}^{(k)} \in [l_{ij}^{(k)}, u_{ij}^{(k)}]$).

We can conclude that the experts' uncertainty is sufficiently large to identify a set

Sol.	d	f_1	Swaps	$f_1^\dagger[d]$
$\mathbf{s}^{(1)}$	-1	0.2232	3	0.0893
$\mathbf{s}^{(2)}$	$-2/3$	0.2140	2	0.0893
$\mathbf{s}^{(3)}$	$-1/3$	0.0220	1	0.0088
$\mathbf{s}^{(4)}$	0	0	0	-

Table 3. Solutions of Problem 3 for each value of d.

of compatible matrices who completely revert the ranking (solution $\mathbf{s}^{(1)}$). Moreover, for the sake of completeness, in Table 5, we collect the consistency ratios for all the perturbed matrices obtained as solution of Problem 3. Notice that, all the consistency ratios are less than the critical threshold of 0.1, hence such instances are considered valid from the consistency perspective (Ágoston and Csató (2022)).

	Expert 1			Expert 2			Expert 3			
Sol	$\widehat{Y}_{12}^{(1)}$	$\widehat{Y}_{13}^{(1)}$	$\widehat{Y}_{23}^{(1)}$	$\widehat{Y}_{12}^{(2)}$	$\widehat{Y}_{13}^{(2)}$	$\widehat{Y}_{23}^{(2)}$	$\widehat{Y}_{12}^{(3)}$	$\widehat{Y}_{13}^{(3)}$	$\widehat{Y}_{23}^{(3)}$	$\widehat{\mathbf{y}}$
$\mathbf{s}^{(1)}$	1.7394	1.3986	1.2060	0.4831	0.1748	0.3350	1.7394	2.7972	3.6182	$\begin{bmatrix}0.3333 & 0.3332 & 0.3334\end{bmatrix}^T$
$\mathbf{s}^{(2)}$	1.6728	1.3450	1.2060	0.4646	0.1681	0.3350	1.6728	2.6900	3.6182	$\begin{bmatrix}0.3248 & 0.3375 & 0.3376\end{bmatrix}^T$
$\mathbf{s}^{(3)}$	1.4483	1.3986	1.4483	0.4023	0.1748	0.4023	1.4483	2.7973	4.3451	$\begin{bmatrix}0.3123 & 0.3753 & 0.3124\end{bmatrix}^T$

Table 4. Perturbed matrices and perturbed ordinal ranking for each solution of Problem 3. The trivial solution $\mathbf{s}^{(4)}$ is ignored. For space reasons, we report only the upper triangular entries of the matrices.

Sol.	$\widehat{Y}^{(1)}$	$\widehat{Y}^{(2)}$	$\widehat{Y}^{(3)}$
$\mathbf{s}^{(1)}$	0.0158	0.0005	0.0634
$\mathbf{s}^{(2)}$	0.0158	0.0005	0.0634
$\mathbf{s}^{(3)}$	0.0158	0.0005	0.0634

Table 5. Consistency ratios for each perturbed matrix identified as a solution of Problem 3

5.1. Evaluating experts' uncertainty

In this section we analyze the expert uncertainty on the basis of the proposed normalization of the objective function of Problem 3. More precisely, we consider a well known case study in the literature proposed by Liu (2009), featuring a single decision-maker, with the aim to validate our approach by comparing the results of Problem 3 with the results identified in Liu (2009). We consider the IPCM $\widetilde{Y}$ as follows:

$$\widetilde{Y} = \begin{bmatrix} 1 & [2,5] & [2,4] & [1,3] \\ [1/5, 1/2] & 1 & [1,3] & [1,2] \\ [1/4, 1/2] & [1/3, 1] & 1 & [1/2, 1] \\ [1/3, 1] & [1/2, 1] & [1,2] & 1 \end{bmatrix}.$$

Let $\overline{Y}$ be the 4×4 comparison matrix, whose entries $\overline{Y}_{ij}$ represent the average value of the interval $[l_{ij}, u_{ij}]$ according to Equation (9). Notice that the nominal ranking, computed according to ILLS, is

$$\overline{\mathbf{y}} = \begin{bmatrix}0.4735 & 0.2128 & 0.1277 & 0.1860\end{bmatrix}^T$$

which represents an ordinal ranking such that $a_1 \succ a_2 \succ a_4 \succ a_3$. According to the literature (Liu (2009)), such instance is affected by rank reversal phenomenon. In fact, the author identifies two matrices A and B, compatible with the uncertainty intervals and locally consistent:

$$A = \begin{bmatrix} 1 & 5 & 4 & 3 \\ 1/5 & 1 & 3 & 2 \\ 1/4 & 1/3 & 1 & 1 \\ 1/3 & 1/2 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1/2 & 1 & 1 & 1 \\ 1/2 & 1 & 1 & 1/2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

which lead to two different absolute rankings from the ordinal point of view:

$$\mathbf{a} = \begin{bmatrix}0.5560 & 0.2091 & 0.1073 & 0.1276\end{bmatrix}^T, \quad \mathbf{b} = \begin{bmatrix}0.3407 & 0.2026 & 0.1703 & 0.2865\end{bmatrix}^T.$$

Notice that, from the ordinal point of view, $\mathbf{a}$ exhibits the same ordinal ranking of $\overline{\mathbf{y}}$, while $\mathbf{a}$ is characterized by an ordinal swap which involves the alternatives a_2 and a_4. On the basis of such results, we now consider our approach in order to validate its outcome. Since the instance considers $n = 4$ alternatives, we can consider 7 different values for the parameter d in the following set $\{0, -1/6, -1/3, -1/2, -2/3, -5/6, -1\}$. As summarized in Table 6, we iteratively solve Problem 3 for each value of d obtaining seven solutions $\mathbf{s}_1, \ldots, \mathbf{s}_7$. Notice that, as confirmed by Liu (2009), the instance is affected by rank reversal issues due to the expert's uncertainty. The results show that when $d \in \{0, -1/6, -1/3, -1/2\}$, i.e., when we consider from 0 to 3 ordinal swaps, we obtain the feasible solutions from $\mathbf{s}_1$ to $\mathbf{s}_4$, while the given uncertainty is not large enough to obtain more than 3 swaps, in fact, solutions $\mathbf{s}_5$, $\mathbf{s}_6$, and $\mathbf{s}_7$ are infeasible solutions. Moreover, with the aim to analyze the expert uncertainty, we normalize the objective function with respect to $f_{UB} = 3.5346$. We reiterate that, on the basis of the nominal ranking $\mathbf{y}$, $f_1^\dagger[d] \in [0, 1]$ is a measure about the existence of altered ranking $\widehat{\mathbf{y}}$ characterized by $\tau(\mathbf{y}, \widehat{\mathbf{y}}) = -d$. Moreover, values of $f_1^\dagger[d]$ close to 1 correspond to a robust instance, while values of $f_1^\dagger[d]$ close to zero imply that the uncertainty is excessively large and there are many matrices that correspond to a different ordinal ranking. With the aim to further investigate about the presence of rank reversal and validate our approach, we randomly generate comparison matrices by sampling the indecision intervals $[l_{ij}, u_{ij}]$. In the Monte Carlo analysis we define 10 Million of matrices, for each matrix we compute the ordinal ranking via ILLS. Notice that the 73.91% of such matrices lead to the same ordinal ranking $\mathbf{y}$, the 25.75% of the matrices lead to an ordinal ranking characterized by 1 swap (i.e., $d = -1/6$), the 0.31% of the matrices lead to an ordinal ranking characterized by 2 swaps (i.e., $d = -1/3$), the 0.03% of the matrices leads to an ordinal ranking characterized by 3 swaps (i.e., $d = -1/2$), while the uncertainty is not sufficiently large to obtain a random matrix who lead to an altered ranking $\widehat{\mathbf{y}}$ characterized by a higher number of swaps. Hence, we conclude that by randomly sampling the intervals there is a high probability (25.75%) to identify a matrix who lead to an altered ranking with one ordinal swap. While the probability to obtain 2 or 3 swap is significantly reduced. Such information about the rank reversal probability can be obtained without the computational effort required to analyze such a large number of random instances. Notice that the values of $f_1^\dagger[d = -1/6]$ confirm that the probability to obtain a comparison matrix who lead to an altered ranking characterized by one swap is higher with respect to the case characterized by 2 o 3 ordinal swaps. Hence, the normalized value of the objective function of Problem 3 can be considered as a valid alternative to the Monte Carlo approach for the identification of rank reversal issues characterized by a precise number of ordinal swaps.

Sol.	d	f_1	Swaps	$f_1^\dagger[d]$	Randomly gen.	Ordinal Ranking
$\mathbf{s}^{(1)}$	0	0	0	-	73.91%	$a_1 \succ a_2 \succ a_4 \succ a_3$
$\mathbf{s}^{(2)}$	$-1/6$	0.0363	1	0.0103	25.75%	$a_1 \succ a_4 \succ a_2 \succ a_3$
$\mathbf{s}^{(3)}$	$-1/3$	0.5217	2	0.1476	0.31%	$a_1 \succ a_4 \succ a_3 \succ a_2$
$\mathbf{s}^{(4)}$	$-1/2$	0.5629	3	0.1593	0.03%	$a_1 \succ a_3 \succ a_4 \succ a_2$
$\mathbf{s}^{(5)}$	$-2/3$	infeasible	4	-	0%	-
$\mathbf{s}^{(6)}$	$-5/6$	infeasible	5	-	0%	-
$\mathbf{s}^{(7)}$	-1	infeasible	6	-	0%	-

Table 6. Comparison between the solutions of Problem 3 and a Monte Carlo approach for the research of rank reversal issues.

6. Conclusion

In this work, consider a scenario where the utility of a set of alternatives must be assessed based on the relative preferences of multiple decision-makers, each affected by uncertainty. In this context, we develop a methodology to quantify robustness to rank reversal by identifying a perturbation of the nominal pairwise comparisons which is as small as possible in a logarithmic least squares sense, while causing the maximum possible number of ordinal swaps among the alternatives. Our framework can support decision-makers in evaluating the reliability of the decision-making process outcome. In fact, a negative evaluation of the experts' uncertainty can push them revise their judgements, while a positive evaluation is a guarantee of the absence, or low probability, of rank reversal issues. Future work includes the possibly different voting powers of the decision makers. Moreover, we will investigate the possibility to apply this methodology to decision-making problems over social networks (e.g., see Gai et al. (2022); Zhang and Li (2021) and references therein).

Acknowledgements

This work was supported by the Italian National Project "DRIVERS: Approccio combinato data-driven ed experience-driven all'analisi del rischio sistemico" funded by INAIL under grant n. BRIC2021-ID03.

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