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A data based system representation ⋆

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Abstract: The paper proposes a system representation formed by a minimal collection of sufficiently long restricted trajectories generated by an observable discrete time LTI system. Conditions are given under which such a collection is a system representation and also an exhaustive parametrization of these representations is provided. These can be also interpreted as a generalized persistency condition which complements the results encountered for the controllable case. In terms of the proposed representation some system properties are investigated and a controllable–autonomous decomposition is given. Finally it is shown how the representation associated to the inverse system, to the parallel and cascade connection, respectively, can be derived.

Keywords: Modeling, LTI system, representation, data based

1. INTRODUCTION

Starting from Persis and Tesi (2019) there is a revival of the research concerning data-driven modeling and control which is based on the idea that the whole set of trajectories that an LTI system can generate can be represented by a finite set of system trajectories provided that such trajectories come from sufficiently excited dynamics. In particular, the so called Fundamental Lemma, Willems et al. (2005), guarantees that for a controllable system using a sufficiently long data obtained by using an input fulfilling a suitable persistent excitation condition encodes the system's behavior as an image of a Hankel matrix formed from the data.

Most of the results concern the input-state case only. In order to enlarge the potential of the data-driven control a more deep understanding of the modelling issues which link data, more precisely a set of (in some sense) informative data, to a so-called data-based representation and later on to the solution of some analysis or design task is necessary. As a first step in this direction in this paper a candidate representation is fixed by proposing a more compact (e.g., compared to the Hankel matrix based approaches) data based system representation formed by a minimal collection of restricted trajectories, for the general, (possibly) uncontrollable case. At this point it should be emphasised that this is a pure modelling question, related to representation theory. Accordingly, in these investigations data is supposed to be given and fixed a priory and, in contrast to questions connected to identification theory, no data acquisition issues are concerned

We provide conditions under which such a matrix is a representation of an observable LTI system and also a parametrization of these representations. This characterization can be also interpreted as a generalized persistency condition which complements the results for the controllable case. Finally we show how the representation associated to the inverse system, to the parallel and cascade connection can be derived. Also it is shown how a controllable–autonomous decomposition can be performed.

The success of a given system representation (parametrization) highly depends on the possibility to integrate it in a numerical toolchain with reasonable complexity that is able to provide answers to practically interesting questions. It is an interesting question from both theoretical and practical point of view whether data-based approaches could replace the traditional state-space based algorithms in a wide variety of analysis and design problems. Applicability of the proposed representation for the solution of these problems, especially in output feedback problems, is the subject of future research.

2. BASIC FACTS

In the so called data driven context it is convenient to adopt a behavioral perspective which allows for system theory independent of an a priori fixed parametric system representations, see, e.g., Willems (1991); Willems and Polderman (1997) for details. In that approach a dynamical system postulates which signals w, called trajectories of the system, are possible to observe. The set of all

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trajectories, denoted by $\mathfrak{B}$, is called the behavior of the system and we identify the system with its behavior.

Accordingly, we consider the discrete time axis $\mathbb{N}$, the signal space $\mathbb{R}^{n_w}$, and the associated space of all possible trajectories $(\mathbb{R}^{n_w})^{\mathbb{N}}$ consisting of all n_w-variate sequences $(w(0), w(1), \dots)$. Then, the behavior $\mathfrak{B}$ is defined as a subset of the space of trajectories, $\mathfrak{B} \subset (\mathbb{R}^{n_w})^{\mathbb{N}}$, and a system as the triple $(\mathbb{N}, \mathbb{R}^{n_w}, \mathfrak{B})$. In what follows, we denote a system merely by its behavior $\mathfrak{B}$. In contrast to the general behavioral approach, this paper assumes that

$$w(i) = \begin{bmatrix} u(i) \\ y(i) \end{bmatrix}, \quad u(i) \in \mathbb{R}^{n_u},\ y(i) \in \mathbb{R}^{n_y},\ n_y = n_w - n_u,$$

where u and y are free and dependent variables that will later serve as inputs and outputs.

A system is linear if $\mathfrak{B}$ is a subspace of $(\mathbb{R}^{n_w})^{\mathbb{N}}$, and (time) shift-invariant if $\sigma\mathfrak{B} = \mathfrak{B}$, where σ denotes the shift operator with action $\sigma w(t) = w(t+1)$. The restriction of the behavior $\mathfrak{B}$ to the time interval $[t_1, t_2]$, where $t_1 < t_2$, is denoted by $\mathfrak{B}|_{[t_1,t_2]} = \left\{ w \in (\mathbb{R}^{n_w})^{t_2 - t_1 + 1} \mid \text{there are } w_- \right.$ and w_+ such that $\text{col}\,(w_-, w, w_+) \in \mathfrak{B}\}$.

Due to time-invariance, we may take the interval $[0, L]$ for simplicity, i.e., the corresponding restriction $\mathfrak{B}_L$. We consider $w|_L$ as the stacked vector

$$w|_L = \begin{bmatrix} w(0) \\ \vdots \\ w(L-1) \end{bmatrix} \in \mathbb{R}^{n_w L}.$$

A system $\mathfrak{B}$ is complete if

$$w|_{[t_0,t_1]} \in \mathcal{B}|_{[t_0,t_1]}$$

for all $t_0, t_1 \in \mathbb{T}, t_0 \leq t_1$ implies that $w \in \mathcal{B}$.

One typically works with explicit parametric representations (models) of LTI systems. A kernel representation with lag ℓ specifies an LTI behavior as

$$\mathfrak{B} = \text{kernel}(R(\sigma)) = \left\{ w \in (\mathbb{R}^{n_w})^{\mathbb{N}} \,:\, R(\sigma)w = 0 \right\},$$

where $R(\sigma) = R_0 + R_1\sigma + \cdots + R_\ell \sigma^\ell$ is a polynomial matrix of degree ℓ, and the matrices $R_0, R_1, \dots, R_\ell$ take values in $\mathbb{R}^{(n_y) \times n_w}$.

Theorem 1. (Willems (1986)). The following statements are equivalent:

- (i) $\mathfrak{B}$ is linear, time-invariant, and complete.
- (ii) $\mathfrak{B}$ is linear, shift-invariant, and closed in the topology of pointwise convergence, i.e., if $w_i \in \mathfrak{B}$ and $w_i(t) \to w(t)$, for all $t \in \mathbb{N}$, implies $w \in \mathfrak{B}$.
- (iii) There is a polynomial matrix $R \in \mathbb{R}^{\bullet \times n_w}[z]$, such that $\mathfrak{B} = \ker(R(\sigma))$.

Thus, for LTI systems, the completeness property is equivalent to finite dimensionality. For continuous time systems also see, Lomadze (2012).

A behavior is called autonomous if it is a finite dimensional subspace of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{n_w})$, i.e., if $\mathfrak{B} = \ker R$ then R has full column rank. Any behavior admits a direct sum decomposition as $\mathfrak{B} = \mathfrak{B}_{\text{cont}} \oplus \mathfrak{B}_{\text{aut}}$, where $\mathfrak{B}_{\text{cont}}$ is the largest controllable subbehavior of $\mathfrak{B}$ and $\mathfrak{B}_{\text{aut}}$ is an autonomous subbehavior of $\mathfrak{B}$. The controllable part is uniquely determined by $\mathfrak{B}$.

There exists a minimal kernel representation $\mathfrak{B} = \ker(R(\sigma))$, in which the number of equations $p = \text{rowdim}(R)$, the maximum lag $\ell = \max_{i=1,\dots,p} l_i$, and the total lag $n = \sum_{i=1}^{p} l_i$ are simultaneously all minimal over all possible kernel representations, called shortest lag representation, where $R = \begin{bmatrix} r_1 & \cdots & r_p \end{bmatrix}^T$ and $\deg(r_i) = l_i$. It can be shown that the l_i's are the observability indices of the system. The minimal and shortest lag kernel representations correspond to special properties of the R matrix: in a minimal representation, R is full row rank, and in a shortest lag representation, R is row proper.

Alternatively, one can unfold the kernel representation by revealing a latent variable. State variables are special latent variables that specify the memory of the system. Any LTI system admits a representation by an input/state/output representation $\mathfrak{B}_{\text{i/s/o}}(A, B, C, D)$

$$\sigma x = Ax + Bu, \quad y = Cx + Du$$

in which both the input/output and the state structure of the system are explicitly displayed. The minimal state dimension $n(\mathfrak{B}) = \sum_{i=1}^{q-m} \ell_i$ among all i/s/o representations is an invariant of $\mathfrak{B}$.

Due to space constraints, in what follows we will investigate in more details the relation of the proposed data-driven system representation to an observable input/state/output representation. The connections to other representations, e.g., kernel based representation, will be presented in a forthcoming paper.

2.1 Parametrisation of all length-L trajectories

Instead of the stacked vector $w|_L$ it is convenient to consider its rearranged counterpart

$$\tilde{w}_L = \Pi_{u,y} w|_L = \begin{bmatrix} u_{[0,L-1]} \\ y_{[0,L-1]} \end{bmatrix},$$

i.e., by a slight abuse of the notation, instead of $\mathfrak{B}|_L$ the subspace $\Pi_{u,y}\ \mathfrak{B}|_L$ is considered.

The following is a standard result, which is a starting point for the investigations concerning data-based system representations:

Theorem 2. Let us consider an observable i/s/o representation of a finite dimensional LTI system $\mathfrak{B}$:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \quad x(0) = x_{\text{ini}} \in \mathbb{R}^{n_x}, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

and assume that $L \geq \ell$. Then, for any trajectory $w \in \mathfrak{B}|_L$, there is a unique initial state x_{ini} such that

$$w = \mathcal{B}_L(A, B, C, D) \begin{bmatrix} u \\ x_{\text{ini}} \end{bmatrix}, \tag{1}$$

with

$$\mathcal{B}_L(A, B, C, D) = \begin{bmatrix} I_{n_u L} & 0_{n_u L \times n_x} \\ \mathcal{T}_L(\mathfrak{B}) & \mathcal{O}_L(A, C) \end{bmatrix} \in \mathbb{R}^{n_w L \times (n_x + n_u L)},$$

where

$$\mathcal{T}_L(\mathfrak{B}) = \begin{bmatrix} M_0 & 0 & \cdots & 0 \\ M_1 & M_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ M_{L-1} & \cdots & M_1 & M_0 \end{bmatrix} \in \mathbb{R}^{n_y L \times n_u L}$$

is the Toeplitz (convolution) matrix with L block rows constructed from the impulse response (Markov) parameters $M_0 = D$, $M_k = CA^{k-1}B$ of the system and

$$\mathcal{O}_L(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{bmatrix} \in \mathbb{R}^{n_y L \times n_x}$$

is the extended observability matrix.

Note, that

$$\mathcal{T}_{L+1} = \begin{bmatrix} D & 0 \\ \mathcal{O}_L B & \mathcal{T}_L \end{bmatrix}, \quad \mathcal{O}_{L+1} = \begin{bmatrix} C \\ \mathcal{O}_L A \end{bmatrix}. \tag{2}$$

Recall, that the lag ℓ is equal to the observability index of the state-space representation, i.e., the smallest integer i, for which the extended observability matrix with i block rows becomes full column rank.

At this point it is important to stress that an i/s/o realization of $\mathfrak{B}$ is minimal if and only if it is observable, see Willems (1983). Thus, the following assertion, which appears, e.g., in Willems (2009), and whose representation-free version can be found in Markovsky and Dörfler (2023), is not completely trivial:

Lemma 3. (Characterization of $\mathfrak{B}|_L$). For $L \geq \ell$ the dimension of the subspace $\mathfrak{B}|_L$ is equal to $n_u L + n$.

Note, that in contrast to the image representation of the entire behavior $\mathfrak{B}$, which exists only if the behavior is controllable, a representation of the type (1) exists regardless to controllability for the finite dimensional restricted behaviors $\mathfrak{B}|_L$ provided that the window length L is sufficiently long.

Given a trajectory w of a system $\mathfrak{B}$ with length T_d, by the shift-invariance property, multiple short L-samples-long trajectories ($L < T_\text{d}$) can be created and organised in a Hankel matrix

$$\mathcal{H}_L(w) = \begin{bmatrix} w(1) & w(2) & \cdots & w(T_d - L + 1) \\ w(2) & w(3) & \cdots & w(T_d - L + 2) \\ \vdots & \vdots & & \vdots \\ w(L) & w(L+1) & \cdots & w(T_d) \end{bmatrix}.$$

For any $w \in \mathfrak{B}|_{T_\text{d}}$ and $L \in [1, T_\text{d}]$, we have that

$$\text{image } \mathcal{H}_L(w) \subseteq \mathfrak{B}|_L.$$

For controllable systems, under the condition called persistency of excitation,

$$\mathfrak{B}|_L = \text{ image } \mathcal{H}_L(w), \text{ i.e., } \operatorname{rank} \mathcal{H}_L(w) = mL + n,$$

for $L > \ell$, see Willems et al. (2005).

Maupong and Rapisarda (2017) have used the formulation that $w \in \mathfrak{B}$ is sufficiently informative about $\mathfrak{B}$ if $\operatorname{colspan}(\mathfrak{H}_L(\tilde{w})) = \mathfrak{B}_L$. In the controllable case persistency provides a practical sufficient condition to generate (simulate) such data. However, instead of a single trajectory one can use multiple trajectories $w^1, \ldots, w^N$ and the mosaic-Hankel matrix

$$\mathcal{H}_L(w^1, \ldots, w^N) = \begin{bmatrix} \mathcal{H}_L(w^1) & \cdots & \mathcal{H}_L(w^N) \end{bmatrix}$$

to obtain the same result under the generalized persistency of excitation condition, van Waarde et al. (2020). Along the same chain of ideas, other data structures, like the page matrix and the trajectory matrix which are special cases of the mosaic-Hankel matrix, can also be used. In van Waarde et al. (2020) the notion, i.e., informativity of the data, is extended to a whole branch of standard analysis and design problems.

In what follows we relax the desire to use a single trajectory. Accordingly, in this context the term informativity refers to a given data-set, regardless to the provenience of that particular data. The sole requirement is that the different pieces should be trajectories of a given LTI system, justifying the use of the term "data-driven".

3. A DATA DRIVEN REPRESENTATION

If otherwise not stated, in what follows we silently identify the restricted behavior with the later one, i.e., $w|_L \to \tilde{w}|_L$. It will also be assumed, that $L \geq \ell$. After selecting a basis V in $\mathfrak{B}|_L$ we can obtain a matrix $\mathcal{B}_L(V) \in \mathbb{R}^{n_u L + n_x}$. Note that the elements (columns) v_i of the basis corresponds to certain trajectories of the system.

3.1 A canonical representation

First let us consider the special case when the basis V is directly related to an impulse response approach. In order to be able to exploit Theorem 2 for data generation condition $L > \ell$ is needed. Then data is partitioned in the past and future part according to $L = L_p + L_f$, $L_p \geq \ell$, $L_f \geq 1$ and

$$\begin{bmatrix} U_p \\ U_f \\ Y_p \\ Y_f \end{bmatrix} = \begin{bmatrix} I_{n_u L_p} & 0_{n_u L_p \times n_u L_f} & 0_{n_u L_p \times n_x} \\ 0_{n_u L_f \times n_u L_p} & I_{n_u L_f} & 0_{n_u L_f \times n_x} \\ \mathcal{T}_{L_p}(\mathfrak{B}) & 0_{n_y L_p \times n_u L_f} & \mathcal{O}_{L_p}(A, C) \\ \mathcal{H}_{L_f}(\mathfrak{B}) & \mathcal{T}_{L_f}(\mathfrak{B}) & \mathcal{S}_{L_f}(A, C) \end{bmatrix} \begin{bmatrix} U_p \\ U_f \\ x_\text{ini} \end{bmatrix}, \tag{3}$$

with the obvious meaning of $\mathcal{H}_{L_f}$ and $\mathcal{S}_{L_f}$. Since (U_p, Y_p) is a trajectory by assumption, it follows that

$$Y_f = (\mathcal{H}_{L_f} - \mathcal{S}_{L_f} \mathcal{O}_{L_p}^l \mathcal{T}_{L_p}) U_p + \mathcal{S}_{L_f} \mathcal{O}_{L_p}^l Y_p + \mathcal{T}_{L_f} U_f, \tag{4}$$

where $\mathcal{O}_{L_p}^{(l)} = (\mathcal{O}_{L_p}^T \mathcal{O}_{L_p})^{-1} \mathcal{O}_{L_p}^T$ is the left inverse of $\mathcal{O}_{L_p}$.

Note that in contrast to $\mathcal{T}_L$ which does not depend on the choice of the i/s/o representation, $\mathcal{O}_L$ depends solely on the actual choice of the basis in the state space. The behavior is completely determined by $\mathcal{M}_L \times \mathcal{O}_L$ the set $\mathcal{M}_L = \{M_0, M_1, \ldots M_L\}$ of Markov parameters and the extended observability matrix $\mathcal{O}_L$. It is a matter of convenience to write

$$\mathfrak{B}|_L \sim \mathcal{B}_L(A, B, C, D) \sim \mathcal{M}_L \times \mathcal{O}_L \sim \mathcal{T}_L \times \mathcal{O}_L \tag{5}$$

in order to emphasise that the first part is related to the "transfer function" or convolutional kernel (zero state response) while the second to the autonomous part (zero input response).

In contrast to other representations, e.g., kernel or state space, in this case it is not granted that arbitrarily given parameters, i.e., $\mathcal{M}_L \times \mathcal{O}_L$ represent any system. The state of the autonomous part should also be the (possibly non minimal) state which corresponds to the given Markov parameter sequence.The following Lemma provides a constructive test for this condition.

Lemma 4. The data $\mathcal{M}_L \times \mathcal{O}_L$ with $L < \ell$ is a representation of a system if and only if $\mathcal{M}_L$ and the extended observability matrix $\mathcal{O}_L$ is compatible, i.e.,

$$\mathcal{V}_{L-1} = \begin{bmatrix} M_1 \\ \vdots \\ M_{L-1} \end{bmatrix} = \mathcal{O}_{L-1} B \tag{6}$$

for some B.

Thus, if $L > \ell$ and the compatibility condition holds then there is a one to one map from $\mathcal{B}_L(A, B, C, D)$ to the observable realization (A, B, C, D).

In what follows, we will show how the representation can be extended to have arbitrary length. For the sake of simplicity consider the case with $L_f = 1$. To make the shift $L \to L + 1$, i.e., to $L_f = 2$, we first compute the prolongation of the available trajectories, i.e., compute $\tilde{Y}_f$ for the shifted block columns (padded with zero for the inputs), which for the first column reads as $\tilde{U}_p = 0$, $\tilde{U}_f = 0$ and $\tilde{Y}_p = [0 \ I_{L-1}] \begin{bmatrix} \mathcal{T}_{L_p}(:, 1 : n_u) \\ \mathcal{H}_{L_f}(:, 1 : n_u) \end{bmatrix}$. This results in

$$X_{\text{ini}} = B = \mathcal{O}_{L_p}^{(l)} \begin{bmatrix} CB \\ \vdots \\ CA^{L_p - 1} B \end{bmatrix},$$

and from (4), in the output $\tilde{Y}_f = \mathcal{S}_{L_f} B = CA^L B$, as expected. Analogously, considering the same inputs but taking as $\tilde{Y}_P$ the shifted extended observability matrix $\mathcal{O}_+$, which corresponds to $\mathcal{O}_{L_p}$, we first get $X_{\text{ini}} = A = \mathcal{O}_{L_p}^l \mathcal{O}_+$ and then the corresponding output $\tilde{S}_f = CA^{L+1}$. This procedure can be also repeated for the rest of the block columns: in those case we always get $X_{\text{ini}} = 0$ giving $\tilde{Y}_f = \mathcal{H}_{L_f} \tilde{U}_p + \mathcal{S}_{L_f} \tilde{U}_f$. Finally, the above procedure will end in $\tilde{U}_p = 0$ and $\tilde{U}_f = \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix}$, and observe that arriving at this point we need a new trajectory. This fact introduces some freedom in the extension procedure. A natural choice is to obtain an extension of the same "kind", that we already have, i.e., one that starts from $x_{\text{ini}} = 0$ and keeps the initial structure intact. This makes us to choose the shifted old trajectory, which finally ends in the new row data

$$\begin{bmatrix} \tilde{Y}_f & \mathcal{H}_{L_f} & \mathcal{T}_{L_f} & \tilde{S}_f \end{bmatrix}. \tag{7}$$

Then, we are in the position to form the augmented representation matrix as

$$\mathcal{B}_{L+1}(A, B, C, D) = \begin{bmatrix} I_{n_u(L+1)} & 0_{n_u(L+1) \times n_x} \\ \mathcal{T}_{L+1} & \mathcal{O}_{L+1} \end{bmatrix}. \tag{8}$$

The process can be iterated, obtaining representations of arbitrary length. It is also obvious, that the partition $L = L_p + L_f$ is a matter of choice, till the standing assumption on L_p holds. While in this case the extension process sketched above is quite obvious, it gives the opportunity for data driven interpretation of the entire procedure.

Finally, note that since having the one to one map from $\mathcal{B}_L(A, B, C, D)$ and the observable realization (A, B, C, D), all the system properties, like controllability, stability, stabilizability can be tested by using the classical methods like Kalman rank condition, PHB test, LMI based Lyapunov or stabilizability checks, etc.

3.2 A general representation

In the general case observe, that for any other choice of the trajectory based basis V in $\mathfrak{B}|_L$, there is a nonsingular transformation matrix T such that

$$\mathcal{B}_L(A, B, C, D) = \mathcal{B}_L(V) T.$$

For convenience, let us consider the row partitioning

$$\mathcal{B}_L(V) = \begin{bmatrix} V_{up} \\ V_{uf} \\ V_{yp} \\ V_{yf} \end{bmatrix}.$$

Then the following structural conditions must hold:

$$V_u = \begin{bmatrix} V_{up} \\ V_{uf} \end{bmatrix} \text{ is of full row rank,} \tag{C1}$$

$$V_x = \begin{bmatrix} V_{up} \\ V_{uf} \\ V_{yp} \end{bmatrix} \text{ is of full column rank.} \tag{C2}$$

Since the columns of V, hence, those of V_x are trajectories, the prediction (simulation) formula is

$$Y_f = V_{yf} V_x^{(l)} \begin{bmatrix} U_p \\ U_f \\ Y_p \end{bmatrix}, \tag{9}$$

where $V_x^{(l)} = (V_x^T V_x)^{-1} V_x^T$ is the left inverse of V_x.

As a first step it is convenient to consider a simplified representation constructed as follows: with the right inverse $V_u^{(r)} = V_u^T (V_u V_u^T)^{-1}$ let us consider a nonsingular matrix

$$T_V = \begin{bmatrix} V_u^{(r)} & \Omega^T \end{bmatrix},$$

where Ω is orthogonal to V_u and $\Omega \Omega^T = I_x$. Then, applying this transformation, we can obtain the following description:

$$\begin{bmatrix} U_p \\ U_f \\ Y_p \\ Y_f \end{bmatrix} = \mathcal{B}_L(V) T_V \begin{bmatrix} U_p \\ U_f \\ \omega \end{bmatrix} = \mathcal{B}_L(\bar{V}) \begin{bmatrix} U_p \\ U_f \\ \omega \end{bmatrix} = \\ = \begin{bmatrix} I_{up} & 0 & 0 \\ 0 & I_{uf} & 0 \\ \bar{V}_{pp} & \bar{V}_{pf} & \bar{V}_{p\omega} \\ \bar{V}_{fp} & \bar{V}_{ff} & \bar{V}_{f\omega} \end{bmatrix} \begin{bmatrix} U_p \\ U_f \\ \omega \end{bmatrix}. \tag{10}$$

It turns out, that this $\mathcal{B}_L(\bar{V})$ is a convenient candidate for a closer study of the data based system representation.

It follows, that $\bar{V}_{p\omega}$ is of full column rank, i.e.,

$$V_{yp} \Omega^T \quad \text{is of full column rank.} \tag{C3}$$

Then, in the general case T should be of the form

$$T = T_V \begin{bmatrix} I_{up} & 0 & 0 \\ 0 & I_{uf} & 0 \\ Z_p & Z_f & Z \end{bmatrix}, \quad Z \text{ nonsingular.} \tag{11}$$

Since Z is actually a state transformation matrix, we can set it to identity without restricting the generality, i.e.,

$$\mathcal{O}_{L_p} = \bar{V}_{p\omega} \text{ and } \mathcal{S}_{L_f} = \bar{V}_{f\omega}. \tag{12}$$

With this choice and $\mathcal{Z} = [Z_p \ Z_f]$ we have

$$T = \begin{bmatrix} V_u^{(r)} + \Omega^T \mathcal{Z} & \Omega^T \end{bmatrix} \tag{13}$$

and

$$\bar{V}_y = \begin{bmatrix} \bar{V}_{pp} & \bar{V}_{pf} \\ \bar{V}_{fp} & \bar{V}_{ff} \end{bmatrix} = \begin{bmatrix} V_{yp} \\ V_{yf} \end{bmatrix} (V_u^{(r)} + \Omega^T \mathcal{Z}), \quad \mathcal{O}_L = \begin{bmatrix} V_{yp} \\ V_{yf} \end{bmatrix} \Omega^T \tag{14}$$

$$\mathcal{T}_L = \begin{bmatrix} \mathcal{T}_{L_p} & 0 \\ \mathcal{H}_{L_f} & \mathcal{T}_{L_f} \end{bmatrix} = \begin{bmatrix} \bar{V}_{pp} & \bar{V}_{pf} \\ \bar{V}_{fp} & \bar{V}_{ff} \end{bmatrix} + \begin{bmatrix} \mathcal{O}_{L_p} \\ \mathcal{S}_{L_f} \end{bmatrix} [Z_p \ Z_f] = = \bar{V}_y + \mathcal{O}_L \mathcal{Z}, \tag{15}$$

$$\mathcal{B}_L(V) = \operatorname{Im} \begin{bmatrix} V_u \\ \bar{V}_y V_u + \mathcal{O}_L \Omega \end{bmatrix}. \tag{16}$$

Observe that Z_p and Z_f depend entirely on the past. We also get

$$Y_p = \bar{V}_{pp} U_p + \bar{V}_{pf} U_f + \bar{V}_{p\omega}\omega = \bar{V}_{pp} U_p + \bar{V}_{p\omega}(\omega - Z_f U_f),$$

i.e., $x_{\text{ini}} = \omega - Z_p U_p - Z_f U_f = \omega - Z_p U_p + \bar{V}_{p\omega}^{(l)} \bar{V}_{pf} U_f$. This formula reveals the initial conditions that corresponds to each column (trajectory).

Proposition 5. Starting from an observable i/s/o reprezentation of $\mathfrak{B}$ and assuming $L > \ell$, the trajectory based basis V for which the corresponding restricted behavior is $\mathcal{B}_L(V)$, can be parametrized as

$$\mathcal{B}_L(V) = \operatorname{Im} \begin{bmatrix} V_u \\ \mathcal{T}_L V_u + \mathcal{O}_L(-\mathcal{Z} V_u + \Omega) \end{bmatrix}, \tag{17}$$

where V_u is of full row rank, $V_u \Omega^T = 0$, $\Omega\Omega^T = I$ and $\mathcal{Z}$ is arbitrary.

Conversely, for a given representation $\mathcal{Z}$ is provided by the equation

$$\mathcal{T}_L - \bar{V}_y = \mathcal{O}_L \mathcal{Z}. \tag{18}$$

By definition persistency of the input does not depend on the system and for controllable LTI behaviors guarantees the required rank for the relevant Hankel matrices for sufficiently long trajectories. For uncontrollable systems we always need different trajectories in order to exhaust the behavior and conditions of the Theorem provides a selection criteria for the corresponding initial states. Thus, they can be interpreted as generalized persistency conditions, i.e., condition for the data to be informative, and complement the results of Willems et al. (2005); van Waarde et al. (2020) and Markovsky et al. (2022) to the uncontrollable case.

If we are searching for conditions that ensure that a given matrix is a representation of a system, we need to check that there exists a set of parameters, bearing the required structure imposed by shift invariance, such that (17) holds. Assuming (12) we have C and A. Then we should check that $\begin{bmatrix} \bar{V}_{p\omega} \\ \bar{V}_{f\omega} \end{bmatrix}$ is indeed an extended observability matrix. Let us refer to this test as (C_{obs}).

If we assume that the given data encode a system, starting from $x_{\text{ini}} = 0$, i.e., $U_p = 0$, $Y_p = 0$ and $U_f = \begin{bmatrix} I_{n_u} \\ 0 \end{bmatrix}$, by a recursive application of (9) one can obtain a sequence of Markov parameters of arbitrary length. Recall, that the prediction formula (9) reads as

$$Y_f = \bar{V}_{yf} \bar{V}_x^{(l)} \begin{bmatrix} U_p \\ U_f \\ Y_p \end{bmatrix} =$$
$$(\bar{V}_{fp} - \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)}\bar{V}_{pp})U_p + \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)} Y_p + (\bar{V}_{ff} - \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)}\bar{V}_{pf})U_f. \tag{19}$$

Then, assuming $L_f = 1$, we get

$$M_0 = (\bar{V}_{ff} - \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)}\bar{V}_{pf}) \begin{bmatrix} I_{n_u} \\ 0 \end{bmatrix}$$
$$M_1 = (\bar{V}_{fp} - \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)}\bar{V}_{pp}) \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} + \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)} \begin{bmatrix} 0 \\ M_0 \end{bmatrix}$$
$$\vdots$$
$$M_{L_p} = (\bar{V}_{fp} - \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)}\bar{V}_{pp}) \begin{bmatrix} I_{n_u} \\ 0 \end{bmatrix} + \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)} \begin{bmatrix} M_0 \\ \vdots \\ M_{L_p-1} \end{bmatrix}$$
$$M_{L_p+k} = \bar{V}_{f\omega}\bar{V}_{p\omega}^{(l)} \begin{bmatrix} M_k \\ \vdots \\ M_{k+L_p-1} \end{bmatrix}.$$

Thus, $\mathcal{T}_{L_p}$ can be computed and Z_p, hence $\mathcal{Z}$ can be determined.

Putting these facts together, we can formulate the data based representation result of the paper, i.e., the condition for a subspace $\mathcal{B}_L(V)$ to be a restricted behavior:

Proposition 6. Let us assume that for given $L > \ell$ and n_x the matrix $\mathcal{B}_L(V)$ is of full column rank of $n_u L + n_x$. Then,

$$\mathfrak{B}|_L = \mathcal{B}_L(V) = \operatorname{image} \mathcal{B}_L(A,B,C,D)$$

for an observable realization, if and only if the conditions (C1) – (C3), equation (17) and finally the tests (C_{obs}) and (6) hold.

Finally, we would like to obtain an extension formula. At this point we can repeat the argument to obtain the one step augmented representation made previously. Starting from (7) and (17) we obtain the following equations:

$$0 = \bar{V}_{y,\text{pn}} + \mathcal{O}_L Z_n$$
$$[\tilde{Y}_f \ \mathcal{H}_{L_f} \ \mathcal{T}_{L_f}] = \bar{V}_{y,\text{fn}} + \tilde{S}_f [\mathcal{Z} \ Z_n]$$

to obtain the extension

$$\begin{bmatrix} [\bar{V}_y \ \bar{V}_{y,\text{pn}}] \\ \bar{V}_{y,\text{fn}} \end{bmatrix}.$$

With an arbitrary Z_n we get valid augmented matrices. Thus, in increasing the prediction horizont L_f with fixed L_p, there is a freedom in the choice of Z_n. Concerning the initial basis, there is another freedom in selecting a full row rank extension of V_u, hence in the corresponding T_V.

We conclude this section with the analogue of (5), formulated as

$$\mathcal{B}_L(\bar{V}) \sim \bar{V}_y \times \mathcal{O}_L \sim \mathcal{T}_L \times \mathcal{O}_L \times \mathcal{Z} \tag{20}$$
$$\mathcal{B}_L(V) \sim \bar{V}_y \times \mathcal{O}_L \times V_u \times \Omega \sim \mathcal{T}_L \times \mathcal{O}_L \times \mathcal{Z} \times V_u \times \Omega. \tag{21}$$

Clearly with $\mathcal{Z} = 0$ we regain (5) in (20).

3.3 System properties

As a consequence, without restricting the generality, it can be assumed that the data based representation of a finite dimensional system is either of the form $\mathcal{B}_L(A,B,C,D)$ or $\mathcal{B}_L(\bar{V})$, where $L = 2L_p$, $L_p \geq \max\{\ell_o, \ell_c\}$, with ℓ_o the observability and ℓ_c the controllability index. For convenience, if otherwise not stated, in what follows we consider this setting.

There are a lot of possibilities to test controllability, starting from the classical conditions, like Kalman rank condition, and ending in the data driven (re)formulations of the PHB test, e.g., Mishra et al. (2021); Yu et al. (2021).

In case of uncontrollability there is a representation $\mathcal{B}_L(A,B,C,D)$ or even of type (10) in which all the block elements of the extended observability matrix splits according the controllable–uncontrollable mode, according to

$$[C_c\ C_{uc}]\begin{bmatrix}A_c & A_{cu}\\ 0 & A_{uc}\end{bmatrix}^l = \begin{bmatrix}C_cA_c^l & \star + C_{uc}A_{uc}^l\end{bmatrix}.$$

On the data based representation this corresponds to the controllable – uncontrollable (autonomous) decomposition met in the behavioral framework and also to the corresponding Kalman decomposition of the observable i/s/o representation. Since it is a standard knowledge how to compute this decomposition, the details are left out for brevity.

Recall that the controllable part is completely determined by $\mathcal{M}_{2L}$ according to the full rank factorization

$$\begin{bmatrix}M_1 & \cdots & M_L\\ \vdots & \ddots & \vdots\\ M_L & \cdots & M_{2L-1}\end{bmatrix} = \mathcal{O}_L^c\mathcal{C}_L^c, \tag{22}$$

i.e., the information content of $\mathcal{M}_{2L}$ and that of the representation $\mathcal{M}_L \times \mathcal{O}_L^c$ is the same.

Recall, that

$$\mathcal{T}_L = \bar{V}_y + \mathcal{O}_L\left[Z_p\ Z_f\right],$$

and consider the controllable – uncontrollable partitioning

$$[\mathcal{O}_L^c\ \mathcal{O}_L^{uc}]\begin{bmatrix}Z_p^c & Z_f^c\\ Z_p^{uc} & Z_f^{uc}\end{bmatrix} = \mathcal{O}_L^c\left[Z_p^c\ Z_f^c\right] + \mathcal{O}_L^{uc}\left[Z_p^{uc}\ Z_f^{uc}\right].$$

Then, we have

$$\bar{V}_y^c = \mathcal{T}_L - \mathcal{O}_L^c\left[Z_p^c\ Z_f^c\right], \tag{23}$$
$$\bar{V}_y^{uc} = \bar{V}_y - \bar{V}_y^c = -\mathcal{O}_L^{uc}\left[Z_p^{uc}\ Z_f^{uc}\right], \tag{24}$$

Proposition 7. Using the notations as above we have the following controllable–autonomous decomposition:

$$\mathcal{B}_L^{\text{cont}}(\bar{V}^c) \sim \bar{V}_y^c \times \mathcal{O}_L^c \sim \mathcal{T}_L \times \mathcal{O}_L^c \times \mathcal{Z}^c, \tag{25}$$
$$\mathcal{B}_L^{\text{aut}}(\bar{V}^{uc}) \sim \bar{V}_y^{uc} \times \mathcal{O}_L^{uc} \sim \mathcal{T}_L \times \mathcal{O}_L^{uc} \times \mathcal{Z}^{uc}. \tag{26}$$

Stability, stabilizability, detectability can be verified by using the standard Lyapunov equations obtaining LMIs in the vein of De Persis and Tesi (2020). However, more interesting results can be formulated along dissipativity arguments, e.g., van Waarde and Camlibel (2021) for stabilizability. Due to lack of space this topic will be treated elsewhere.

4. SYSTEM ALGEBRA

In the i/s/o representation it is trivial to obtain the (possibly non-minimal) representation for the sum, product and inverse (if exists) systems in terms of the original parameters. In the data driven case this is not necessarily trivial, due to the observability property imposed by the i/o data.

Let us consider first the inverse of the system (A,B,C,D), i.e., $(A-BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$. Since on the level of the behavior only the order of the i/o signals changes, i.e., we should consider $\begin{bmatrix}V_y\\ V_u\end{bmatrix}$ with $n_u = n_y$, the set of output data should be of full row rank, where

$$V_y = \begin{bmatrix}V_{yp}\\ V_{yf}\end{bmatrix}.$$

In the impulse response based representation this condition automatically holds and we have

$$\begin{bmatrix}\mathcal{T}_L & \mathcal{O}_L\\ I & 0\end{bmatrix} = \begin{bmatrix}I & 0\\ \mathcal{T}_L^{-1} & -\mathcal{T}_L^{-1}\mathcal{O}_L\end{bmatrix}\begin{bmatrix}\mathcal{T}_L & \mathcal{O}_L\\ 0 & I\end{bmatrix}.$$

It is a matter of easy computations to verify that $\mathcal{T}_L^{-1}$ is the Toeplitz matrix and $-\mathcal{T}_L^{-1}\mathcal{O}_L$ is indeed the extended observability matrix of the inverse system. In the general case, with the notation and parameters of Proposition 5 we have

$$\begin{bmatrix}V_y\\ V_u\end{bmatrix} = \begin{bmatrix}V_y\\ \mathcal{T}_L^{-1}V_y - \mathcal{T}_L^{-1}\mathcal{O}_L(-\mathcal{Z}V_u + \Omega)\end{bmatrix} \tag{27}$$

as the candidate parametrization of the representation for the inverse. Note, that from this equation we have

$$\mathcal{T}_L = V_yV_u^{(r)} + \mathcal{O}_L\mathcal{Z}.$$

and $V_y = \mathcal{T}_LV_u + \mathcal{O}_L(-\mathcal{Z}V_u + \Omega)$ should be of full row rank. Thus, for generalized persisyency in this case input selection and the proper choice of the initial conditions interfere. Observe, that while nonsingularity of $\mathcal{T}_L - \mathcal{O}_L\mathcal{Z}$ is a sufficient condition, $\mathcal{Z} = 0$ being obviously a good choice, it remains an interesting research question how to guarantee this condition, in general.

For parallel and cascade interconnections even the impulse response case is nontrivial. As a starting point observe, that having representations of length L_1 and L_2, respectively, the length $L = L_1 + L_2$ suffice for the resulting representation. It is also immediate, that $\mathcal{T}_L = \mathcal{T}_L^1 + \mathcal{T}_L^2$ and $\mathcal{T}_L = \mathcal{T}_L^2\mathcal{T}_L^1$ provides the required Toeplitz part. The nontrivial issue occurs for the computation of the extended observability matrix, whose dimension $n_x \le n_{x_1} + n_{x_2}$ is not known, in general. A possible shortcut is to compute only the controllable part of the representation from the Markov parameters, as it was shown in the previous section. This corresponds to the case when the "transfer function" approach is used, i.e., we are dealing with controllable behaviors only.

For parallel connections we have

$$\begin{bmatrix}I & 0 & 0\\ \mathcal{T}_L & \mathcal{O}_{L,1} & \mathcal{O}_{L,2}\end{bmatrix}$$

and recall that we can compute the corresponding (possibly) unobservable pair (C,A). Applying classical ideas, by computing the QR factorization we can select an orthogonal basis T_o of the observable subspace according to

$$\begin{bmatrix}\mathcal{O}_{L,1}^T\\ \mathcal{O}_{L,2}^T\end{bmatrix} = [T_o\ T_{uo}]\begin{bmatrix}R\\ 0\end{bmatrix}.$$

Then, the observable pair is $(CT_o, T_o^TAT_o)$ while the desired extended observability matrix $\mathcal{O}_L = R^T$. Finally, the representation of the parallel composition is

$$\begin{bmatrix}I & 0\\ \mathcal{T}_L^1 + \mathcal{T}_L^2 & [\mathcal{O}_{L,1}\ \mathcal{O}_{L,2}]\,T_o\end{bmatrix} = \begin{bmatrix}I & 0\\ \mathcal{T}_L & \mathcal{O}_L\end{bmatrix}. \tag{28}$$

For the general case, the corresponding formula to (17) reads as

$$\mathcal{B}_L(V) = \text{Im}\begin{bmatrix} V_u \\ \mathcal{T}_L V_u + \mathcal{O}_L(-T_o^T \mathcal{Z} V_u + T_o^T \Omega) \end{bmatrix}, \tag{29}$$

where

$$\mathcal{Z} = \begin{bmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix}$$

are formed using the parameters which corresponds to each of the component system.

For cascade connections we have

$$\begin{bmatrix} I & 0 & 0 \\ \mathcal{T}_L^2 \mathcal{T}_L^1 & \mathcal{T}_L^2 \mathcal{O}_{L,1} & \mathcal{O}_{L,2} \end{bmatrix}.$$

Analogously, as above, we can obtain a representation for the cascade connection as

$$\begin{bmatrix} I & 0 \\ \mathcal{T}_L^2 \mathcal{T}_L^1 & \left[\mathcal{T}_L^2 \mathcal{O}_{L,1} \ \ \mathcal{O}_{L,2}\right] T_o \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathcal{T}_L & \mathcal{O}_L \end{bmatrix}, \tag{30}$$

where

$$\begin{bmatrix} \mathcal{O}_{L,1}^T \mathcal{T}_{L,2}^T \\ \mathcal{O}_{L,2}^T \end{bmatrix} = [T_o \ T_{uo}] \begin{bmatrix} R \\ 0 \end{bmatrix}.$$

Accordingly, for the general case we have the formula

$$\mathcal{B}_L(V) = \text{Im}\begin{bmatrix} V_u \\ \mathcal{T}_L V_u + \mathcal{O}_L(-T_o^T \mathcal{Z} V_u + T_o^T \Omega) \end{bmatrix}, \tag{31}$$

where

$$\mathcal{Z} = \begin{bmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2(\mathcal{T}_L^1 - \mathcal{O}_{L,1}\mathcal{Z}_1) \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 - \mathcal{Z}_2 \Omega_1 \end{bmatrix}$$

are formed using the parameters which corresponds to each of the component system.

5. CONCLUSION

A new system representation has been proposed formed by a minimal collection of sufficiently long restricted trajectories generated by an observable discrete time LTI system. We give conditions under which such a collection is a system representation and also an exhaustive parametrization of these representations was provided.

This characterization can be also interpreted as a generalized persistency condition (informativity) which complements the results for the controllable case.

In terms of the proposed representation some system properties were investigated and a controllable–autonomous decomposition was given. Finally it was provided the reprezenation associated to the inverse system, to the parallel and cascade connection.

Applicability of the proposed representation in analysis and design problems, especially in output feedback problems, is the subject of future research.

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