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Pole identification using discrete Laguerre expansion and variable projection ⋆

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Abstract: We propose a novel algorithm for identifying the poles of transfer functions describing SISO-LTI (single input single output, linear time invariant) systems. Our identification method works in the frequency domain and consists of two parts. In the first part, we extend a discrete Laguerre expansion based method with an automatic parameter selection scheme. This allows us to find an initial estimate of the poles of SISO-LTI transfer functions without the need for human intuition. Then, in the second part, we propose a novel optimization problem to improve our initial estimates. The proposed optimization aims to reduce the least squared error of a parameterized model, which can be interpreted as an orthogonal projection of the system's frequency response onto a subspace spanned by Generalized Orthogonal Rational Basis functions (GOBFs). We solve the corresponding nonlinear optimization task using gradient based methods, where we can analytically calculate the gradient of the error functional. Through robust numerical experiments, we investigate the behavior of the developed methods and show that they work even in scenarios, when the transfer function has a high number of poles.

Keywords: Frequency domain identification; Recursive identification; Linear systems; Variable Projection; Generalized Orthogonal Basis Functions, Malmquist–Takenaka functions

1. INTRODUCTION

System identification is a data-driven approach to model the behavior of dynamical systems. Algorithms of this type have a black-box view of the system, i.e., they rely on the observed input/output measurements, rather than utilizing the underlying physics.

In the frequency domain, the identification can be formulated as an approximation problem, where the system's transfer function is to be derived from the quotient of the input/output spectrum. Besides the standard trigonometric basis, the so-called generalized orthogonal basis functions (GOBF) have been introduced in the 90s to increase the freedom for incorporating a priori knowledge about the system. Since then, the GOBFs have been utilized in many applications including control theory, model approximation, adaptive control problems, and signal processing (see e.g., Heuberger et al. (2005)). One representative of this approach is the Schi-So algorithm (Schipp and Soumelidis (2011); Bokor et al. (2011, 2013); Schipp and Soumelidis (2012); Soumelidis et al. (2013, 2015)), which utilizes discrete Laguerre functions to represent the frequency response of single input single output linear time invariant (SISO-LTI) systems. The Schi-So method allows for the identification of a single pole of the system's transfer function, however it can be applied in a recursive manner to identify the rest of the poles as well (Soumelidis et al. (2013, 2015)). Furthermore, this algorithm has been shown to perform well in scenarios where there is a large number of poles located close to each other and to the unit circle (Soumelidis et al. (2015)).

One of the main limitations of the Schi-So algorithm thus far has been the fact, that the parameters of the used discrete Laguerre-expansions had to be selected manually. This is a direct numerical method in the sense that the estimated poles cannot be systematically improved in

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contrast to iterative approaches. Also, as a side effect of recursive algorithms, the estimation error of the poles identified at earlier stages propagates and accumulates during the progress of the recursion.

In this paper we address the previously mentioned limitations of the Schi-So algorithm. First, we propose an automatic parameter selection scheme for the discrete Laguerre expansions used by the Schi-So algorithm. This can be viewed as an important step towards real-world applications, since the proposed identification method no longer relies on human intuition. Second, we introduce a novel approach to simultaneously correct all the estimated pole positions provided by the automatized Schi-So algorithm. The correction step is discussed in the framework of a general mathematical model called variable projection (VP). In this approach, we use the rational orthonormal parameterization of the corresponding transfer function whose poles are to be identified. As a case study, we consider the identification of SISO LTI systems, where the criteria for modeling and identification are formulated in terms of the mean square error (MSE) between the ground truth and the estimated transfer functions. By means of comparison tests performed on several randomly initialized zero/pole configurations, we demonstrate that the rational VP method initialized by the pole estimates of the Schi-So algorithm is able to precisely identify the poles of transfer functions.

The rest of this paper is organized as follows. In section 2 we introduce the pole identification problem, and discuss the Schi-So algorithm. In section 3 we consider numerical problems regarding the Schi-Scho algorithm and extend it with our new, automatic parameter selection scheme. In section 4 we formulate the pole identification task using the VP approach and introduce a novel algorithm to correct the pole estimates produced by the automatized Schi-So algorithm. Section 5 details our numerical experiments. Finally, in section 6 we draw our conclusions and discuss possible future research directions.

2. AUTOMATIZED POLE IDENTIFICATION USING DISCRETE LAGUERRE SYSTEMS

2.1 Problem specification

In this paper we address the problem of identifying the transfer function H of a discrete SISO-LTI system. Such a system can be expressed in the frequency domain by

$$X(z) = H(z) \cdot U(z),\ H(z) = \frac{P(z)}{Q(z)}, \quad (z \in \mathbb{D} \cup \mathbb{T}), \quad (1)$$

where $U(z), X(z)$ denote the $\mathcal{Z}$-transforms of the input and output sequences, $\mathbb{D} := \{z : z \in \mathbb{C}, |z| < 1\}$, $\mathbb{T} := \{z : z \in \mathbb{C},\ |z| = 1\}$, P and Q are the numerator and the denumerator polynomials of H, respectively. The rational function H is the so-called transfer function of the SISO system. In practice, we usually only have access to an equidistant sampling of H on $\mathbb{T}$ known as the frequency response of the system.

To adequately describe the transfer function we introduce the elementary rational functions

$$r_\alpha(z) := \frac{1}{1 - \overline{\alpha} z} \quad (\alpha \in \mathbb{D}, |z| < 1). \quad (2)$$

In (2), the parameter α will be referred to as the *inverse pole* of r_α, since $1/\overline{\alpha}$ is the pole in the usual sense and $\overline{\alpha}$ denotes the complex conjugate. Consider the linear space $\mathcal{R}_A := \text{span}\{r_\alpha : \alpha \in A \subset \mathbb{D}\} \subset H^2(\mathbb{D})$, where A is a finite subset of $\mathbb{D}$, and $H^2(\mathbb{D})$ denotes the Hardy-space, whose elements are square-integrable complex functions analytic within $\mathbb{D}$. In this work, we assume that the transfer function H, whose inverse poles are to be identified, belongs to $\mathcal{R}_A$. Then, our objective is to estimate the inverse poles $\alpha_0, \ldots, \alpha_{N-1} \in \mathbb{D}$, which define H. We note that a one-to-one mapping between poles in the usual sense and inverse poles exists (Heuberger et al. (2005)), however using inverse poles greatly simplifies notations.

2.2 Identifying a single inverse pole

Schipp and Soumelidis et al. introduced an inverse pole identification method in a series of papers (Schipp and Soumelidis (2011); Bokor et al. (2011, 2013); Schipp and Soumelidis (2012); Soumelidis et al. (2013, 2015)). We begin by briefly describing this method, which we will refer to as the Schi-So algorithm. The discrete Laguerre system (Heuberger et al. (2005)) is a complete and orthogonal rational function system in $H^2(\mathbb{D})$. The nth discrete Laguerre function can be given by

$$L_{n,a}(z) := \frac{\sqrt{1 - |a|^2}}{1 - \overline{a} z} B_a^n(z) \quad (a \in \mathbb{D}, z \in \overline{\mathbb{D}}, n \in \mathbb{N}), \quad (3)$$

where $B_a(z) := \frac{z-a}{1-\overline{a}z}$ is a so-called Blaschke-function. These functions have many interesting properties and contribute greatly to a number of theoretical results (Mashreghi et al. (2013); Pap and Schipp (2006); Schipp and Bokor (2003)). In this paper we make use of the fact, that Blaschke functions are one-to-one maps on both $\mathbb{D}$ and $\mathbb{T}$, furthermore $B_a^{-1}(z) = B_{-a}(z)$ $(z \in \overline{\mathbb{D}}, a \in \mathbb{D})$. The following theorem is a key result for identifying a single inverse pole $\alpha \in A$ of a transfer function $H \in \mathcal{R}_A$ by the Schi-So algorithm.

Theorem 1. Let $H \in \mathcal{R}_A$ have a number $N \in \mathbb{N}$ of inverse poles. Suppose that every inverse pole $A = \{\alpha_0, \ldots, \alpha_{N-1}\}$ of H is simple. Consider the coefficients $c_{n,a}$ of the discrete Laguerre expansion for a priori chosen $a \in \mathbb{D}$ parameter:

$$H(z) := \sum_{n=0}^{\infty} c_{n,a} L_{n,a}(z), \quad (4)$$

where $c_{n,a} := \langle H, L_{n,a} \rangle$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^2(\mathbb{D})$. Then, the sequence

$$q_{n,a} := \frac{c_{n+1,a}}{c_{n,a}} \quad (5)$$

satisfies

$$q_{n,a} = B_a(\alpha^*) + O(s^n), \quad (6)$$

where $s := \max_{\alpha \in A, \alpha \neq \alpha^*} |B_a(\alpha)| / |B_a(\alpha^*)|$ and α^* is an inverse pole of H, i.e., $\alpha^* \in A$.

A proof of theorem 1 can be found in (Schipp and Soumelidis (2011)). Using the limit $B_a(\alpha^*)$, we can easily find α^* by $B_{-a}(B_a(\alpha^*))$. It is important to mention, that the theorem still holds if the inverse poles α were not simple, however in this case the sequence described in (6) converges more slowly. We note that the identified inverse pole α^* can be described using the so-called pseudo hyperbolic distance

$$\rho(a,b) := |B_a(b)| = \left|\frac{b-a}{1-\overline{a}b}\right| \quad (a,b \in \mathbb{D}). \tag{7}$$

Namely, the inverse pole α^* can be found by the Schi-So algorithm if and only if there exists some $a \in \mathbb{D}$ for which

$$\rho(a,\alpha^*) > \rho(a,\alpha) \quad (\alpha \in A, \alpha \neq \alpha^*). \tag{8}$$

If the above condition holds for the parameter $a \in \mathbb{D}$ of the Laguerre-expansion in (4), then the ratios in (5) tend to $B_a(\alpha^*)$ as $n \to \infty$. We note that condition (8) holds for any $a \in \mathbb{D}$, disregarding a zero measure set.

2.3 Identifying every pole using Laguerre coefficients

Given a transfer function H, whose inverse poles belong to the finite set $A \subset \mathbb{D}$, it cannot be guaranteed that for each $\alpha \in A$ condition (8) holds. In other words, some $\alpha \in A$ inverse poles may exist which cannot be identified by the Schi-So algorithm. To overcome this limitation, a recursive approach has been proposed in (Soumelidis et al. (2013)). This approach is based on the so-called Malmquist-Takenaka (MT) functions

$$\Phi_n(z) := \frac{\sqrt{1-|a_n|^2}}{1-\overline{a}_n z} \prod_{k=0}^{n-1} B_{a_k}(z) \quad (a_k \in \mathbb{D}) \tag{9}$$

which contain the Laguerre system (3) as a special case. The MT functions were introduced by (Takenaka (1925)) and (Malmquist (1925)), and form an orthonormal and complete system in $H^2(\mathbb{D})$ if and only if the so-called Szász condition $\sum_{k=0}^{\infty}(1-|a_k|) = \infty$ is satisfied for the parameters $a_k \in \mathbb{D}$ (see section 2.2.5 in (Heuberger et al. (2005))).

Proposition 1. ((Soumelidis (2002))). If the cardinality of A is N, the subspace $\mathcal{R}_A$ is spanned by the MT functions $\Phi_0, \ldots, \Phi_{N-1}$ parameterized by $a_0, \ldots, a_{N-1} \in A$. Consequently, if $H \in \mathcal{R}_A$, then

$$H(z) := \sum_{k=0}^{n-1} \langle H, \Phi_k \rangle \Phi_k(z) \quad (z \in \mathbb{T}). \tag{10}$$

In (Soumelidis et al. (2013, 2015)), this property along with Theorem 1 was exploited to construct a recursive pole identification scheme. Suppose that the first $m \leq N$ inverse poles of $H \in \mathcal{R}_A$ in (10) have already been identified. Then, these inverse poles can be used to construct the mth MT-Fourier partial sum:

$$H(z) \approx S_m H(z) := \sum_{k=0}^{m-1} \langle H, \Phi_k \rangle \Phi_k(z) \quad (z \in \mathbb{T}).$$

Observe that $S_m H$ can be used to "eliminate" the inverse poles which are already identified. In fact,

$$H_m(z) := (H(z) - S_m H(z)) \cdot \overline{A_{m-1}}(z), \tag{11}$$

where the conjugate of $A_{m-1}(z) := \prod_{k=0}^{m-1} B_{a_k}(z)$ simplifies out the inverse poles a_k $(k = 0, \ldots, m-1)$ from the residual $H(z) - S_m H(z)$. Thus, the Schi-So algorithm can be applied recursively by identifying one pole in each step, which is followed by the elimination of this pole from the residual transfer function. In practice, problems can occur if the true inverse pole α^* was not identified precisely enough, as it will not be fully eliminated by (11). We resolve this issue in section 4 by introducing a nonlinear optimization method initialized by the pole estimates of the Schi-So algorithm.

3. AUTOMATIC PARAMETER SELECTION

In this section, we investigate the challenges concerning the practical application of the Schi-So algorithm and propose an automatic parameter selection scheme. Basically, we consider the following two problems:

- For any $f \in H^2(\mathbb{D})$, the Laguerre-Fourier coefficients $c_{n,a} := \langle f, L_{n,a} \rangle$ tend quickly to zero as n increases. This means that calculating the ratios $q_{n,a}$ in (5) for large n introduces numerical errors. Figure 1 illustrates this phenomenon.
- According to Theorem 1, the rate of convergence at which the quantities $q_{n,a}$ approach $B_a(\alpha^*)$ depends heavily on the choice of the parameter $a \in \mathbb{D}$ (see (6) and (8)). Since we can only calculate the ratios (5) up to some finite index, choosing a for which $q_{n,a}$ converges slowly may lead to bad approximations of the true pole α^*. In figure 2 we illustrate different choices for parameter a of the Laguerre system, and show the corresponding sequences $q_{n,a}$.

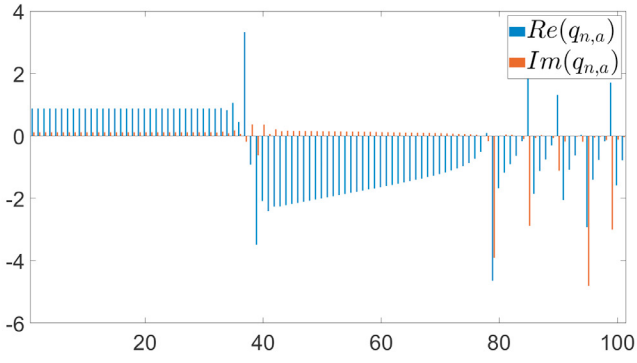


Fig. 1. Since $|c_{n,a}|$ tend quickly to 0 as $n \to \infty$, calculating the ratios $q_{n,a}$ becomes numerically unstable.

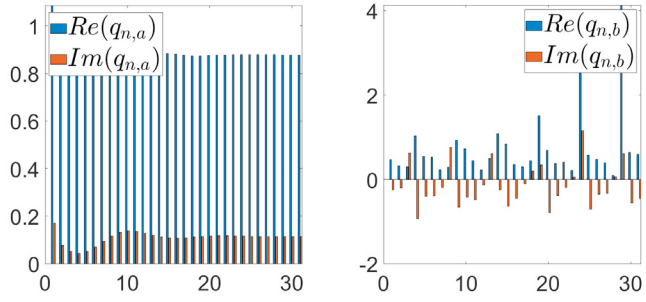


Fig. 2. Convergence properties of the ratios (5) with respect to the same transfer function H, but different parameter setups for the Laguerre expansion. A good choice of a speeds up the convergence (left), whereas a poor parameter selection decreases the rate of convergence (right).

To overcome these issues, we introduce the following heuristics. First, for any fixed parameter a, we should find a subsequence of $q_{n,a}$ where these ratios are nearly constant. We can express this, by fixing a window size $L \in \mathbb{N}$, and considering the values of

$$u(J,a) := \max_{i,j\in J} |q_{i,a} - q_{j,a}|, \tag{12}$$

where $J := [k, k+L]$ is an L-long index window. Then, we determine the window J^* that includes the "most constant" part of the ratio sequence:

$$J^* = \arg\min_J u(J,a). \tag{13}$$

This step is followed by the estimation of the limit

$$\lim_{n\to\infty} q_{n,a} \approx \frac{1}{L}\sum_{j\in J^*} q_{j,a}, \tag{14}$$

and by the estimation of the true inverse pole

$$\alpha^* \approx B_{-a}\left(\frac{1}{L}\sum_{j\in J^*} q_{j,a}\right) =: \alpha(a). \tag{15}$$

Next, we introduce a method to find a suitable Laguerre parameter a for which (15) provides a good estimate of α^*. In doing so, we exploit the observation that the location of the peaks in $|H(z)|$ ($z \in \mathbb{T}$) correspond to the argument of the dominant poles (see e.g., Gőzse and Soumelidis (2015)). Following this idea, we consider the parameter a "good", if the peaks of $|H(z)|$ can be approximated well by means of MT functions defined with the previous and the current inverse pole estimates. Formally, let $\alpha_0, \alpha_1, \ldots, \alpha_{m-2} \in \mathbb{D}$ be already identified inverse poles of the transfer function H and $\alpha_{m-1}(a) \in \mathbb{D}$ be the next estimate. Then, finding a "good" parameter a can be posed as a nonlinear optimization problem

$$\min_{a\in\mathbb{D}} \|H - S_m H\|_\infty, \tag{16}$$

where $S_m H$ denotes the mth MT-Fourier partial sum defined by the parameters $\alpha_0, \ldots, \alpha_{m-2}, \alpha_{m-1}(a) \in \mathbb{D}$. In order to solve the optimization problem (16), we propose to use the so-called hyperbolic variant of the Nelder–Mead algorithm (Lócsi (2013)). The classical Nelder–Mead algorithm (Nelder and Mead (1965)) applies geometric transformations to an initial simplex defined by the function to be minimized. The hyperbolic variant of the algorithm (Lócsi (2013)) replaces these geometric transformations with their corresponding variants from hyperbolic geometry. In our case this is important, because it naturally guarantees, that the parameter a stays inside $\mathbb{D}$ throughout the optimization.

4. SYSTEM IDENTIFICATION WITH VARIABLE PROJECTION

In section 3, we proposed an algorithm to estimate the inverse poles of SISO-LTI transfer functions in the frequency-domain. This could be used as an initialization to the following nonlinear optimization problem:

$$\min_{\boldsymbol{\eta}\in\Gamma} F(\boldsymbol{\eta}) := \min_{\boldsymbol{\eta}\in\Gamma} \|\mathbf{H} - \mathbf{H}(\boldsymbol{\eta})\|_2^2, \tag{17}$$

where $\mathbf{H}$ is the frequency response of H evaluated over M points, i.e., $\mathbf{H}_k = H(e^{2\pi i k/M})$ ($k = 0, \ldots, M-1$). This is a deterministic approach to identify the system by minimizing the least squares error of an a priori chosen model $\mathbf{H}(\boldsymbol{\eta})$ with respect to the parameters $\boldsymbol{\eta}$ over a feasible set Γ. Typically, $\Gamma \subset \mathbb{R}^{2N}$, and $\mathbf{H}(\boldsymbol{\eta}) = \mathbf{P}(\boldsymbol{\eta})/\mathbf{Q}(\boldsymbol{\eta})$, where the coefficients $\boldsymbol{\eta}$ of the numerator $\mathbf{P}$ and the denumerator $\mathbf{Q}$ polynomials are to be determined.

In our setting, we choose

$$\mathbf{H}(\boldsymbol{\eta}) = \mathbf{P}_{\boldsymbol{\Phi}(\boldsymbol{\eta})}\mathbf{H} := \boldsymbol{\Phi}(\boldsymbol{\eta})\boldsymbol{\Phi}(\boldsymbol{\eta})^+\mathbf{H}, \tag{18}$$

where $\boldsymbol{\Phi}(\boldsymbol{\eta})$ is a predefined $M \times N$ matrix parametrized by $\boldsymbol{\eta}$, and $\boldsymbol{\Phi}(\boldsymbol{\eta})^+$ denotes its pseudoinverse. Note that $\mathbf{P}_{\boldsymbol{\Phi}(\boldsymbol{\eta})}$ defines an orthogonal projection to the column space of $\boldsymbol{\Phi}(\boldsymbol{\eta})$ depending on $\boldsymbol{\eta}$, which implies the name variable projection (VP). By utilizing VP models of the transfer function, (17) becomes a separable nonlinear least squares (SNLLS) problem.

In order to make use of the a priori knowledge on the pole structure estimated by the Schi-So algorithm in section 2, we express the VP model of the transfer function in terms of GOBFs. This can be done by utilizing the Malmquist–Takenaka (MT) basis functions (9). In this case, the columns of $\boldsymbol{\Phi}(\boldsymbol{\eta})$ are MT functions sampled uniformly on the unit circle, which are parameterized by $\boldsymbol{\eta} = (r_0, t_0, \ldots, r_{N-1}, t_{N-1}) \in \Gamma \subset \mathbb{R}^{2N}$, where $a_k = r_k e^{it_k}$ ($k = 0, \ldots, N-1$) are the inverse poles, and $(r_k, t_k) \in [0,1) \times [-\pi, \pi]$. For an actual implementation, we refer to the RAIT MatLab Toolbox by (Kovács and Lócsi (2012)), that includes several routines to calculate $\boldsymbol{\Phi}(\boldsymbol{\eta})$, $\mathbf{P}_{\boldsymbol{\Phi}}\mathbf{H}$, etc.

Note that the MT-VP model of the transfer function can provide the exact representation of causal SISO-LTI systems, which is a necessary condition for model consistency. In fact, according to the Titchmarsh theorem, causality implies that the corresponding transfer function belongs to $H^2(\mathbb{D})$, where the MT functions form an orthonormal and complete function system provided that the so-called Szász condition is satisfied (see e.g., section 2.3). Therefore, the approximation error in (17) can be made arbitrarily small by choosing a large enough value for N. By Proposition 1, if the transfer function is also defined with the same finite number N of poles, then it can be exactly represented by MT functions as well.

One of the main advantage of the SNLLS formulation of the identification problem is that the exact gradient can be derived according to the work of (Golub and Pereyra (1973)), which in our case simplifies a lot due to the orthonormality of the MT functions. In fact, the kth coordinate of the gradient F in case of (18) can be calculated as

$$\frac{1}{2}\nabla F_k = \left[-(\mathbf{P}_{\boldsymbol{\Phi}}\mathbf{D}_k\boldsymbol{\Phi}^+ + (\mathbf{P}_{\boldsymbol{\Phi}}\mathbf{D}_k\boldsymbol{\Phi}^+)^T)\mathbf{H})\right]^T \mathbf{P}_{\boldsymbol{\Phi}}^{\perp}\mathbf{H},$$

where we omitted $\boldsymbol{\eta}$ for the sake of simplicity, $\mathbf{P}_{\boldsymbol{\Phi}}^{\perp} = \mathbf{I} - \mathbf{P}_{\boldsymbol{\Phi}}$, and $\mathbf{D}_k = \partial\boldsymbol{\Phi}(\boldsymbol{\eta})/\partial\eta_k$ represents the matrix of partial derivatives of the MT functions with respect to a single parameter η_k. Due to the orthonormality of the MT functions, $\boldsymbol{\Phi}(\boldsymbol{\eta})^+$ is simply equal to the complex conjugate of $\boldsymbol{\Phi}(\boldsymbol{\eta})$. This heavily speeds up the computation, since no singular value decomposition is needed to calculate the pseudoinverse. Additionaly, the existance of ∇F enables the use of fast gradient based numerical optimization techniques, such as the trust-region method, which we utilized in our experiments. Finally we note that SNLLS approaches have several other applications (see e.g., Golub and Pereyra (2003), Kovács et al. (2021)), although this is the first time they are used for frequency domain identification in combination with GOBFs.

5. EXPERIMENTS

In order to demonstrate the effectiveness of the proposed method, we conducted a set of numerical experiments as described below. In each experiment, we generated a number N of random inverse poles. Then, we constructed a transfer function H using the MT system corresponding to the generated inverse poles:

$$H(z) := \sum_{k=0}^{N-1} c_k \cdot \Phi_k(z), \tag{19}$$

where c_k were randomly generated complex coefficients with $|c_k| < 5$. In order to mimic the behavior of real-world systems, the random inverse poles defining the MT functions Φ_k in (19) were chosen as complex conjugate pairs. Furthermore, in practice the poles often fall close to the edge of the unit circle (e.g., in systems describing flexible aircraft wing behavior Soumelidis et al. (2015)). To ensure similar testing scenarios, we generated inverse poles α such that the condition $0.7 \leq |\alpha| < 1$ is satisfied.

A favourable property of the proposed method is that it performs well for large N, i.e., when a high number of inverse poles need to be identified. To demonstrate this, we conducted experiments with transfer functions defined by a number $N = 2, 4, 6, \ldots, 20$ of inverse poles. Specifically, for each value of N, we generated 100 number of different transfer functions, and measured the performance of the following identification approaches:

- The Schi-So algorithm in itself complemented by the automatic parameter selection scheme proposed in section 3.
- The Malmquist-Takenaka based variable projection (MT-VP) approach proposed in section 4, with random intialization.
- The hybrid identification scheme, that is, the optimization of MT-VP model initialized by the automatized Schi-So algorithm.

We evaluated the performance of each method according to the two criteria, which are the accuracy of the estimated poles and the transfer function. The first criterion was measured in the average (Euclidean) distance between each estimated inverse pole and the true inverse pole closest to it (table 1). The second was measured in terms of the mean squared error (MSE):

$$MSE := \frac{1}{M} \sum_{k=1}^{M} |\mathbf{H}_k - \widehat{\mathbf{H}}_k|^2, \tag{20}$$

where $\mathbf{H} \in \mathbb{C}^M$ denotes the frequency response of H sampled over $M \in \mathbb{N}$ points, and $\widehat{\mathbf{H}} \in \mathbb{C}^M$ denotes the MT approximation of $\mathbf{H}$ generated by the estimated inverse poles (table 2).

N	Schi-So	MT-VP	Schi-So + MT-VP
4	0.1417	0.0001	**0.0000**
8	0.1531	0.0160	**0.0025**
16	0.1359	0.3166	**0.0468**
20	0.1382	0.3928	**0.0982**

Table 1. Average distance of each estimated inverse pole from the nearest true inverse pole.

N	Schi-So	MT-VP	Schi-So + MT-VP
4	0.3815	**0.0000**	**0.0000**
8	5.5951	0.1980	**0.0010**
16	6.4826	2.8154	**0.0284**
20	6.7320	4.4282	**0.1442**

Table 2. Mean Squared Error (20) of the generated model.

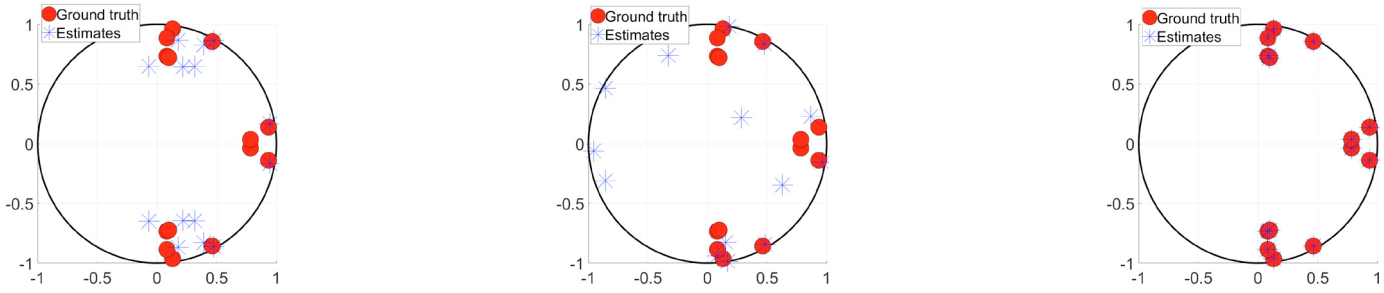
The results in tables 1-2 are averaged over 100 experiments for each N. Our results show that the proposed method, in which the variable projection based optimization is initialized by the automatized Schi-So algorithm provides very precise estimates of the inverse poles (and the transfer function H) even for large N. If H is defined by only a few number of inverse poles (i.e., N is small), the randomly initialized variable projection method also performs well, however as N increases random initialization becomes less and less effective. The automatized Schi-So algorithm's performance remains consistent even for large N. Although the estimates provided by the Schi-So algorithm are not entirely accurate, the estimated inverse poles, in general, lie in the neighborhoods of the true inverse poles and thus can provide a good starting point for the variable projection based optimization.

The results of an example experiment are illustrated in figure 3, which captures the behavior of the investigated methods well. The automatized Schi-So algorithm finds the inverse poles but is not accurate enough, the MT-VP based method with random initialization finds some inverse poles with great accuracy, but may also produce some very far off estimates. However, starting the optimization of the MT-VP model from the neighborhood of the true inverse poles (i.e., from the estimated inverse poles provided by the automatized Schi-So method) improves the final pole estimation a lot.

6. CONCLUSION

In this paper we presented a novel pole identification scheme that is applicable for discrete SISO LTI systems in the frequency domain. To this end, we extended a Laguerre expansion based identification method of (Schipp and Soumelidis (2011)) called the Schi-So algorithm with an automatic parameter selection scheme. Furthermore, we introduced a new optimization approach based on variable projections to improve the estimated pole locations. For the first time, a gradient based optimization procedure was introduced to minimize the error of the frequency response's projection onto a subspace spanned by GOBFs. The gradients of this error functional are analytically calculated during the optimization procedure. Through robust numerical experiments we asserted the effectiveness of the proposed method. In particular, we showed that our hybrid Schi-So + VP approach can successfully identify transfer functions consisting of a large number of poles.

In the next phase of this research, we will further develop the proposed algorithms to solve real-world system identification problems. One interesting direction of inquiry will be to investigate the noise tolerance of the proposed identification techniques. In addition, we note that even though in our current experiments, we assumed the number of inverse poles to be known, in the next steps of the research,



(a) Schi-So (b) MT-VP (c) Schi-So + MT-VP

Fig. 3. An example experiment. Pole estimates produced by the automatized Schi-So algorithm (3a), by the randomly initialized MT-VP model (3b), and by the combination of the proposed approaches (3c).

we plan to exploit the *nonparametric* nature of the Schi-So algorithm. In particular, if the pole estimates provided by the Schi-So algorithm are precise, then the $H_2(\mathbb{D})$ norm of the residual frequency response (11) decreases with each identified pole (Soumelidis et al. (2015)). This behavior can be exploited to define stopping conditions in the initialization part of the proposed method.

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