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Spectrum of 3-uniform 6- and 9-cycle systems over $K_v^{(3)} - I$

Anita Keszler [a], Zsolt Tuza [b,c,*]

[a] Machine Perception Research Laboratory, Institute for Computer Science and Control (SZTAKI), Kende u. 13–17, Budapest, 1111, Hungary
[b] Alfred Renyi Institute of Mathematics, Realtanoda u. 13–15, Budapest, 1053, Hungary
[c] Department of Computer Science and Systems Technology, University of Pannonia, Egyetem u. 10, Veszprem, 8200, Hungary

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ABSTRACT

We consider edge decompositions of $K_v^{(3)} - I$, the complete 3-uniform hypergraph of order v minus a set of $v/3$ mutually disjoint edges (1-factor). We prove that a decomposition into tight 6-cycles exists if and only if $v \equiv 0, 3, 6 \pmod{12}$ and $v \geq 6$; and a decomposition into tight 9-cycles exists for all $v \geq 9$ divisible by 3. These results are complementary to the theorems of Akin et al. [Discrete Math. 345 (2022)] and Bunge et al. [Australas. J. Combin. 80 (2021)] who settled the case of $K_v^{(3)}$.

1. Introduction

A *decomposition* of a (hyper)graph $\mathcal{H}$ is a collection of edge-disjoint sub(hyper)graphs $\mathcal{H}_1, \ldots, \mathcal{H}_m$ of $\mathcal{H}$ whose union is $\mathcal{H}$. If each $\mathcal{H}_i$ is isomorphic to a fixed hypergraph $\mathcal{F}$, then $\{\mathcal{H}_1, \ldots, \mathcal{H}_m\}$ is called an $\mathcal{F}$*-decomposition* of $\mathcal{H}$. Obviously, if $\mathcal{H}$ admits an $\mathcal{F}$-decomposition, then the number of edges in $\mathcal{H}$ is a multiple of the number of edges in $\mathcal{F}$. Here we study decompositions into 3-uniform cycles of lengths 6 and 9, and prove that this obvious necessary condition is also sufficient for the decomposability of every nearly complete hypergraph $K_v^{(3)} - I$ obtained from the complete 3-uniform hypergraph $K_v^{(3)}$ of order v by the deletion of a 1-factor, i.e. omitting $v/3$ mutually disjoint edges.

Edge decompositions of complete graphs K_v, and complete graphs minus a 1-factor, $K_v - I$, of order v have a long history for over a century. Concerning the existence of decompositions into cycles of a fixed length it was proved by Alspach and Gavlas [2] and Šajna [23], with a substantially different proof by Buratti [8] for odd cycle lengths, that the standard necessary arithmetic conditions—i.e., the degree $v-1$ or $v-2$ must be even, and the number $\binom{v}{2}$ or $\frac{v(v-2)}{2}$ of edges must be a multiple of the cycle length—are also sufficient. Also the existence of decompositions into Hamiltonian cycles requires just the proper parity of v; this well-known fact dates back to the 19th century.

1.1. Berge cycles in hypergraphs

The situation becomes more complicated when larger edge size is considered. This is so already for the complete 3-uniform hypergraphs $K_v^{(3)}$ of order v. In hypergraphs there are several ways to define cycles, and the stricter one is taken, the harder the question of decomposability becomes.

* Corresponding author.
E-mail addresses: keszler@sztaki.hu (A. Keszler), tuza.zsolt@mik.uni-pannon.hu (Zs. Tuza).

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The weakest form of cycle in a hypergraph is *Berge k-cycle*, defined as an alternating cyclic sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k$ of k mutually distinct vertices and edges such that $v_i \in e_i \cap e_{i-1}$ for all $1 < i \le k$ and $v_1 \in e_1 \cap e_k$. The particular case where k is the number v of vertices means Hamiltonian Berge cycle. Here the necessary condition for $K_v^{(3)}$ means $v \mid \binom{v}{3}$, and it turns out to be sufficient. Namely, the existence of decompositions of $K_v^{(3)}$ into Hamiltonian Berge cycles was proved by Bermond [4] for $v \equiv 2, 4, 5 \pmod 6$, and by Verall [25] for $v \equiv 1 \pmod 6$. Kühn and Osthus [18] generalized this result, proving the sufficiency of $v \mid \binom{v}{r}$ for Hamiltonian Berge-cycle decompositions of complete r-uniform hypergraphs $K_v^{(r)}$ $(r \ge 4)$, with the slight restriction $v \ge 30$ if $r = 4$ and $v \ge 20$ if $r \ge 5$.

Assume next that the number v of vertices is a multiple of the edge size r. If $r = 3$, then in the hypergraph $K_v^{(3)} - I$ the number $\frac{v^2(v-3)}{6}$ of edges is a multiple of v, and $K_v^{(3)} - I$ admits a decomposition into Hamiltonian Berge cycles [25]. Similarly, if $r \ge 4$ and v is not too small, then every r-uniform hypergraph with v vertices and $\binom{v}{r} - (\binom{v}{r} \bmod r)$ edges is decomposable into Hamiltonian Berge cycles [18].

Concerning cycles of any fixed length the decomposability problem on $K_v^{(r)}$ (and even on complete multi-hypergraphs) was solved by Javadi, Khodadadpour and Omidi [12] for all $v \ge 108$. For the particular cycle lengths $k = 4$ and $k = 6$ with edge size $r = 3$ it is also known that the lower bounds on v can be omitted; see [14] and [19], respectively.

1.2. Tight cycles in uniform hypergraphs

A stricter cycle definition is *r-uniform tight k-cycle* that means a cyclic sequence $v_1, v_2, \ldots, v_k$ of k vertices, together with k edges formed by the r-tuples of consecutive vertices $v_i, v_{i+1}, \ldots, v_{i+r-1}$ $(i = 1, 2, \ldots, k$, subscript addition taken modulo $k)$.

From now on, by k-cycle we mean 3-uniform tight k-cycle. The decomposition problem into such cycles seems much harder than the one above on Berge cycles, already on $K_v^{(3)}$ and $K_v^{(3)} - I$. Decomposability of $K_v^{(3)}$ into Hamiltonian cycles has the simple necessary condition $3 \mid (v-1)(v-2)$. But its sufficiency has been studied only within a limited range ($v \le 16$ by Bailey and Stevens [3], $v \le 32$ by Meszka and Rosa [21], $v \le 46$ by Huo et al. [11].)

For fixed cycle length k concerning $K_v^{(3)}$, only three cases are solved: the very famous class of Steiner Quadruple Systems $(k = 4)$ by the classical theorem of Hanani [10], and the very recent works by Akin et al. [1] for $k = 6$ and by Bunge et al. [7] for $k = 9$. There are many constructions for $k = 5$ and $k = 7$, but no complete solution is available on them; for partial results and further references we cite [15] and [20].

In this paper we initiate the study of k-cycle decompositions of $K_v^{(3)} - I$. Hence, let us assume $3 \mid v$, and consider $k = 4$ first. Note that the edges of a 3-uniform tight 4-cycle are exactly the 3-element subsets of a 4-element set, therefore edge-disjoint collections of 4-cycles are in one-to-one correspondence with partial Steiner Quadruple Systems. In this way, in the particular case $k = 4$, estimates due to Johnson [13] and Schönheim [24] yield that the maximum number of edge-disjoint 4-cycles on v vertices does not exceed

$$\left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \left\lfloor \frac{v-2}{2} \right\rfloor \right\rfloor \right\rfloor.$$

For $v \equiv 3 \pmod 6$ this means $v(v-1)(v-3)/24$. Moreover, as discussed by Brouwer in [6], another upper bound due to Johnson yields $v(v^2 - 3v - 6)/24$ if $v \equiv 0 \pmod 6$. As a consequence, in either case the number of edges covered by any collection of edge-disjoint 4-cycles is smaller than $v^2(v-3)/6$. In our context this fact has the following important consequence.

Corollary 1. *No $K_v^{(3)} - I$ can admit a decomposition into tight 4-cycles.*

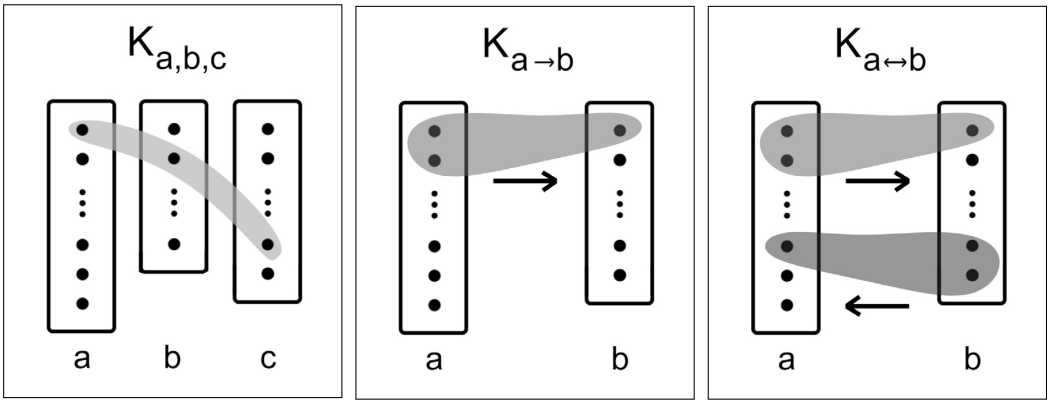
The main goal of the present note is to prove that in the other two cases that are solved for $K_v^{(3)}$, namely $k = 6$ and $k = 9$, the k-cycle decompositions of $K_v^{(3)}$ have their natural analogues for $K_v^{(3)} - I$. In this way we solve the spectrum problem for the decomposability of $K_v^{(3)} - I$ for the cases of 6-cycles and 9-cycles. More explicitly, we prove the following two results.

Theorem 2. *The hypergraph $K_v^{(3)} - I$ admits a decomposition into 6-cycles if and only if $v \ge 6$ and $v \equiv 0, 3, 6 \pmod{12}$.*

Theorem 3. *The hypergraph $K_v^{(3)} - I$ admits a decomposition into 9-cycles if and only if $v \ge 9$ and v is a multiple of 3.*

Since $K_v^{(3)} - I$ has $v^2(v-3)/6$ edges, and a k-cycle has k edges, the general necessary conditions now mean $6k \mid v^2(v-3)$. That is, $3 \mid v$ for both $k = 6$ and $k = 9$, moreover the residue class $v \equiv 9 \pmod{12}$ is excluded if $k = 6$.

Remark 4. Since $K_v^{(3)} - I$ has $v^2(v-3)/6$ edges, and every k-cycle has exactly k edges, for the existence of a decomposition into k-cycles we must have $6k \mid v^2(v-3)$, therefore the conditions given in Theorems 2 and 3 are necessary.



(a) $K^{(3)}_{a,b,c}$ (b) $K^{(3)}_{a\to b}$ (c) $K^{(3)}_{a\leftrightarrow b}$

Fig. 1. Illustrations for $K^{(3)}_{a,b,c}$, $K^{(3)}_{a\to b}$ and $K^{(3)}_{a\leftrightarrow b}$.

Sufficiency of the conditions in Theorems 2 and 3 will be proved in Section 2 and Section 3, respectively. Further systems with additional properties are constructed in Sections 4 and 5. One of those properties specifies "2-split systems" [9] that have been used frequently in recursive constructions for other cycle lengths. The other considered type is "cyclic systems" having a rotational symmetry.

1.3. Notation

We write $\mathcal{C}^*(3,k,v)$ to denote any decomposition of $K^{(3)}_v - I$ into tight 3-uniform k-cycles. Moreover, for some particular types of 3-uniform hypergraphs we use the following notation (see Fig. 1 for illustrations):

- $K^{(3)}_{a,b,c}$ – complete 3-partite hypergraph whose vertex set is partitioned into three sets A,B,C with $|A|=a$, $|B|=b$, $|C|=c$, and a 3-element set $T\subset A\cup B\cup C$ is an edge if and only if $|T\cap A|=|T\cap B|=|T\cap C|=1$.
- $K^{(3)}_{a\to b}$ – 3-uniform hypergraph whose vertex set is partitioned into two sets A and B with $|A|=a$ and $|B|=b$, and a 3-element set $T\subset A\cup B$ is an edge if and only if $|T\cap A|=2$ and $|T\cap B|=1$.
- $K^{(3)}_{a\leftrightarrow b}$ – a hypergraph of "crossing triplets": 3-uniform hypergraph whose vertex set is partitioned into two sets A and B with $|A|=a$ and $|B|=b$, and a 3-element set $T\subset A\cup B$ is an edge if and only if it meets both A and B.

2. The spectrum of $\mathcal{C}^*(3,6,v)$ systems

In this section we prove Theorem 2. Let $v=12m+3s$, where $m\ge 1$ is any integer and $s=0,1,2$. We denote by $\mathcal{H}_2$ the 3-uniform hypergraph with four vertices a,b,c,d and two edges abc, abd. We shall apply the following well-known result.

Theorem 5 *(Bermond, Germa, Sotteau [5]). If $n\equiv 0,1,2 \pmod 4$, then $K^{(3)}_n$ has a $\mathcal{H}_2$-decomposition.*

Moreover, two building blocks will be used. The first one is derived from the cycle double cover of the edge set of the complete bipartite graph $K_{3,3}$ with the three cycles $x_1y_1x_2y_2x_3y_3$, $y_1x_2y_3x_1y_2x_3$, $x_2y_3x_3y_1x_1y_2$, where the two vertex classes are $\{x_1,x_2,x_3\}$ and $\{y_1,y_2,y_3\}$.

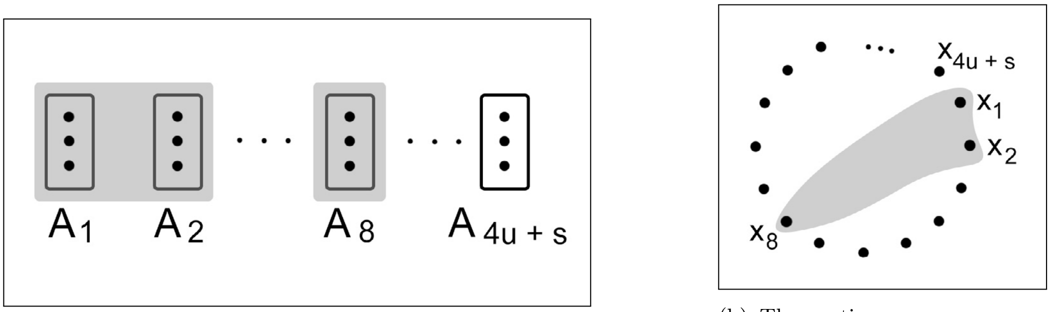
Lemma 6. *The hypergraph $K^{(3)}_6 - I$ obtained from $K^{(3)}_6$ by omitting the 1-factor $I=\{a_1a_2a_3, a_4a_5a_6\}$ has a decomposition into the three 6-cycles*

$$a_1a_4a_2a_5a_3a_6\,,\quad a_4a_2a_6a_1a_5a_3\,,\quad a_2a_6a_3a_4a_1a_5\,.$$

The second small construction is derived by combining the two cyclic P_4-decompositions $x_{1+i}y_{1+i}x_{2+i}y_{3+i}$ and $y_{1+i}x_{1+i}y_{3+i}x_{2+i}$ ($i=0,1,2$, subscript addition taken modulo 3) of the same $K_{3,3}$.

Lemma 7. *The complete 3-partite hypergraph $K^{(3)}_{3,3,2}$ with its three vertex classes $\{a_1,a_2,a_3\}$, $\{b_1,b_2,b_3\}$, $\{c_1,c_2\}$ has a decomposition into the three 6-cycles*

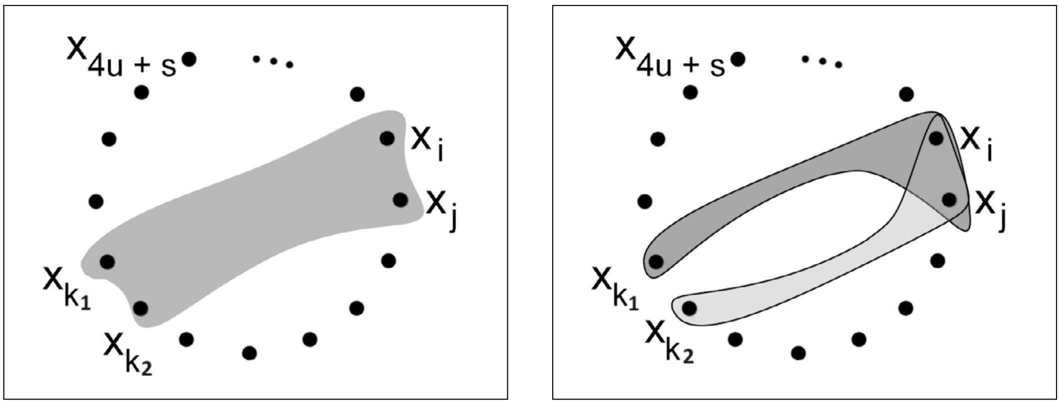
$$a_{1+i}\,b_{1+i}\,c_1\,a_{2+i}\,b_{3+i}\,c_2 \quad (i=0,1,2)$$



(a) The $4u+s$ triplets $A_1,\dots,A_{4u+s}$, with the example triplets A_1,A_2,A_8 highlighted.

(b) The vertices $x_1,\dots,x_{4u+s}$ of $K^{(3)}_{4u+s}$. $A_1,A_2,A_8 \to x_1,x_2,x_8$

Fig. 2. We view $K_v^{(3)} - I$ as the result of substituting $4u+s$ triplets $A_1,\dots,A_{4u+s}$ into the vertices $x_1,\dots,x_{4u+s}$ of $K^{(3)}_{4u+s}$. An example with triplets A_1,A_2,A_8 and the corresponding three vertices x_1,x_2,x_8 is highlighted.



(a) The vertices x_i,x_j,x_{k_1},x_{k_2} that belong to a copy of $\mathcal{H}_2$.

(b) The triplets (hyperedges) $x_ix_jx_{k_1}$, $x_ix_jx_{k_2}$ forming a copy of $\mathcal{H}_2$.

Fig. 3. Example $\mathcal{H}_2$ formed by the triplets $x_ix_jx_{k_1}$, $x_ix_jx_{k_2}$ in the decomposition.

where subscript addition is taken modulo 3.

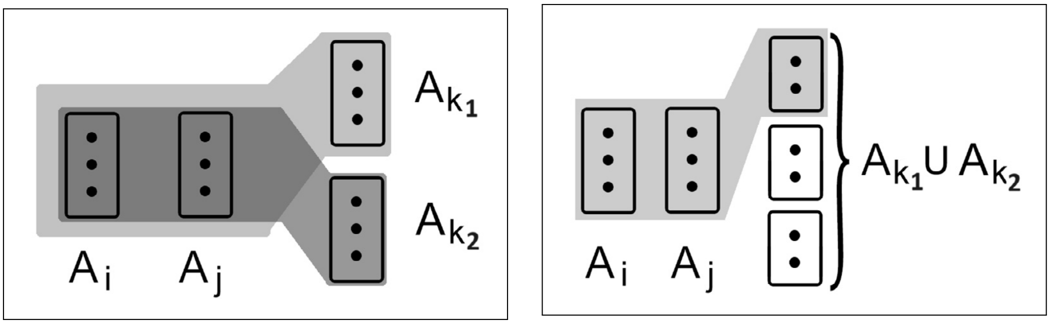
Alternative constructions for Lemmas 6 and 7, which have a less symmetric structure, can be found in Examples 2 and 5 of [1].

Proof of Theorem 2. The case of $v=6$ is settled in Lemma 6. Let now $v=12u+3s$, where $u \ge 1$ and $s=0,1,2$. We view $K_v^{(3)} - I$ as the result of substituting $4u+s$ triplets $A_1,\dots,A_{4u+s}$ into the vertices $x_1,\dots,x_{4u+s}$ of $K^{(3)}_{4u+s}$ (see Fig. 2). The sets A_i will play the role of edges in the 1-factor whose omission from the edge set of $K_v^{(3)}$ yields $K_v^{(3)} - I$. Then each edge of $K_v^{(3)} - I$ meets two or three of the sets A_i. Those two types of edges will be covered with 6-cycles separately.

(a) For the edges meeting two sets from $A_1,\dots,A_{4u+s}$ we consider all pairs i,j with $1 \le i < j \le 4u+s$, and apply Lemma 6 to cover all triplets but A_i and A_j inside $A_i \cup A_j$. This step creates three 6-cycles for each pair i,j.

(b) For the edges meeting three of the A_i we take an $\mathcal{H}_2$-decomposition of $K^{(3)}_{4u+s}$ as guaranteed by Theorem 5. Suppose that the triplets $x_ix_jx_{k_1}$, $x_ix_jx_{k_2}$ form a copy of $\mathcal{H}_2$ in this decomposition (see Fig. 3). They represent the complete 3-partite hypergraph $K^{(3)}_{3,3,6}$ whose vertex classes are A_i, A_j, and $A_{k_1} \cup A_{k_2}$. We split $A_{k_1} \cup A_{k_2}$ into three pairs $C_{k_0}, C_{k_1}, C_{k_2}$, in this way decomposing $K^{(3)}_{3,3,6}$ into three copies of $K^{(3)}_{3,3,2}$ (see Fig. 4). Now Lemma 7 can be applied to find 6-cycle decompositions of the complete 3-partite hypergraphs whose vertex classes are A_i, A_j, and C_{k_ℓ} for $\ell = 0,1,2$. This step creates nine 6-cycles for each copy of $\mathcal{H}_2$.

These collections of 6-cycles decompose $K_v^{(3)} - I$. $\square$



(a) A_i, A_j, A_{k_1} and A_{k_2} are the corresponding sets of the $\mathcal{H}_2$, forming a $K^{(3)}_{3,3,6}$.

(b) The $K^{(3)}_{3,3,6}$ can be decomposed into three copies of $K^{(3)}_{3,3,2}$.

Fig. 4. A copy of $\mathcal{H}_2$ represents the complete 3-partite hypergraph $K^{(3)}_{3,3,6}$. Continuing the example of Fig. 3, the vertex classes of the $K^{(3)}_{3,3,6}$ are A_i, A_j and $A_{k_1} \cup A_{k_2}$. We split $A_{k_1} \cup A_{k_2}$ into three pairs, decomposing $K^{(3)}_{3,3,6}$ into three copies of $K^{(3)}_{3,3,2}$.

3. The spectrum of $\mathcal{C}^*(3, 9, v)$ systems

In this section we prove Theorem 3. Although its form is simpler than Theorem 2, it requires more types of building blocks than the construction for 6-cycles, moreover a distinction between the three residue classes will also be needed. Before the main part of the proof we give several constructions of 9-cycle decompositions of hypergraphs on a small number of vertices.

Lemma 8. *There exists a decomposition of $K^{(3)}_9 - I$ into nine 9-cycles.*

Proof. We construct a system with cyclic symmetry, composed from the following 9-cycles, subscript addition taken modulo 9:

$$a_{1+i}\, a_{2+i}\, a_{3+i}\, a_{8+i}\, a_{7+i}\, a_{5+i}\, a_{9+i}\, a_{6+i}\, a_{4+i} \quad (i = 0, 1, \ldots, 8)\,.$$

These cycles cover all vertex triplets but $a_1a_4a_7$, $a_2a_5a_8$, and $a_3a_6a_9$. □

The method of the following construction works in a more general way also, for an infinite sequence of cycle lengths as shown in [17]; but here we only need its particular case yielding 9-cycles.

Lemma 9. *There exists a decomposition of $K^{(3)}_{9\to 3}$ into twelve 9-cycles.*

Proof. Let $A = \{a_1, \ldots, a_9\}$ and $B = \{b_1, b_2, b_3\}$ be the two vertex classes of $K^{(3)}_{9\to 3}$. As mentioned in the introduction, cycle systems on ordinary complete graphs K_v exist whenever v is odd and the number of edges is divisible by the given cycle length. In particular, K_9 with 9 vertices and 36 edges can be decomposed into six 6-cycle subgraphs. We take this auxiliary decomposition over the vertex set A, and construct two 9-cycles in $K^{(3)}_{9\to 3}$ for each of its graph 6-cycles. Say, $a_1a_2a_3a_4a_5a_6$ is one of the cycles in K_9. We define the two 9-cycles

$$a_1\, b_1\, a_2\, a_3\, b_2\, a_4\, a_5\, b_3\, a_6\,, \quad b_2\, a_1\, a_2\, b_3\, a_3\, a_4\, b_1\, a_5\, a_6\,.$$

The 18 edges of size 3 in these two 9-cycles are precisely those triplets that consist of two consecutive vertices along the graph 6-cycle and one vertex from B. Hence, taking the same for all the six cycles in the decomposition of K_9, a required collection of 9-cycles is obtained in $K^{(3)}_{9\to 3}$. □

A similarly symmetric construction cannot be expected for $K^{(3)}_{6\to 3}$, nevertheless a 9-cycle decomposition still exists.

Lemma 10. *The five 9-cycles*

$$\begin{array}{c} a_1\, a_2\, b_1\, a_3\, a_4\, b_2\, a_5\, a_6\, b_3 \\ a_1\, a_5\, b_1\, a_6\, a_4\, b_2\, a_2\, a_3\, b_3 \\ a_2\, a_4\, b_1\, a_1\, a_6\, b_2\, a_3\, a_5\, b_3 \\ a_3\, a_6\, b_1\, a_2\, a_5\, b_2\, a_1\, a_4\, b_3 \\ a_4\, a_5\, b_1\, a_3\, a_1\, b_2\, a_2\, a_6\, b_3 \end{array}$$

decompose $K^{(3)}_{6\to 3}$ whose two vertex classes are $A = \{a_1, a_2, \ldots, a_6\}$ and $B = \{b_1, b_2, b_3\}$.

We also recall the following construction, which is derived from three Hamiltonian cycles of the complete graph K_9.

Lemma 11 *(Bunge et al. [7], Example 5). The three* 9*-cycles*

$$a_1\, a_{1+i}\, a_{1+2i} \ldots a_{1+8i} \quad (i = 1, 2, 4)$$

with subscript addition modulo 9 *decompose* $K^{(3)}_{3,3,3}$ *whose vertex classes are* $\{a_1, a_4, a_7\}$, $\{a_2, a_5, a_8\}$, *and* $\{a_3, a_6, a_9\}$.

It will be convenient to put these small structures together in some larger hypergraphs as follows.

Lemma 12. *All of the following types of hypergraphs admit decompositions into* 9*-cycles:*

(i) $K^{(3)}_{3p,3q,3r}$ *for all* $p, q, r \geq 1$,
(ii) $K^{(3)}_{3p\to 3q}$ *for all* $p \geq 2$ *and* $q \geq 1$,
(iii) $K^{(3)}_{3p\leftrightarrow 3q}$ *for all* $p, q \geq 2$,

where p, q, r *denote integers.*

Proof. Concerning (i), the following decomposition chain is easily seen:

$$K^{(3)}_{3p,3q,3r} \longrightarrow r \times K^{(3)}_{3p,3q,3} \longrightarrow qr \times K^{(3)}_{3p,3,3} \longrightarrow pqr \times K^{(3)}_{3,3,3}$$

and for $K^{(3)}_{3,3,3}$ we have a decomposition into three 9-cycles by Lemma 11. Similarly in (ii) we can do the step $K^{(3)}_{3p\to 3q} \longrightarrow q \times K^{(3)}_{3p\to 3}$. Now we write p in the form $p = 2a + 3b$, which is possible whenever $p \geq 2$.

(In fact $b = 0$ or $b = 1$ can always be ensured.) Splitting the $3p$ vertices in the first class of $K^{(3)}_{3p\to 3}$ into a sets of size 6 and b sets of size 9 we can proceed with the step

$$K^{(3)}_{3p\to 3} \longrightarrow \left(a \times K^{(3)}_{6\to 3}\right) \cup \left(b \times K^{(3)}_{9\to 3}\right) \cup \left(ab \times K^{(3)}_{3,6,9}\right) \cup \left(\binom{a}{2} \times K^{(3)}_{3,6,6}\right) \cup \left(\binom{b}{2} \times K^{(3)}_{3,9,9}\right)$$

completing the decomposition by Lemmas 9 and 10, also using (i). Finally, (iii) is implied by $K^{(3)}_{3p\leftrightarrow 3q} \longrightarrow \left(K^{(3)}_{3p\to 3q}\right) \cup \left(K^{(3)}_{3q\to 3p}\right)$ that can be done due to (ii). □

We shall need two further small cases for the general proof of Theorem 3. The first one is $v = 12$.

Lemma 13. *For* $v = 12$ *the two* 9*-cycles*

$$a_1\, a_2\, a_3\, a_5\, a_6\, a_9\, a_{11}\, a_4\, a_7\,, \quad a_1\, a_2\, a_8\, a_{12}\, a_3\, a_{10}\, a_{11}\, a_7\, a_9$$

and their rotations modulo 12 *form a decomposition of* $K^{(3)}_{12} - I$, *with uncovered (omitted) triplets* $a_1a_5a_9$, $a_2a_6a_{10}$, $a_3a_7a_{11}$, $a_4a_8a_{12}$.

As a final auxiliary step, we also need to handle the case of $v = 15$ separately.

Lemma 14. *The hypergraph* $K^{(3)}_{15} - I$ *admits a* 9*-cycle decomposition into* 50 *cycles.*

Proof. Let the vertex set be $A \cup B \cup C$, with $|A| = |B| = 6$ and $|C| = 3$. Inside $A \cup C$ and also inside $B \cup C$ we take a copy of the 9-cycle decomposition of $K^{(3)}_9 - I$, where C is an uncovered vertex triplet. These 18 cycles cover all triplets inside A and inside B—with the exception of two disjoint triplets in each—and also the A–C and B–C crossing triplets. The A–B crossing triplets can be covered by a decomposition of $K^{(3)}_{6\leftrightarrow 6}$ (20 cycles); and the triplets meeting all the three parts A, B, C decompose as $K^{(3)}_{6,6,3}$ (12 cycles), applying Lemma 12 for both cases. □

Now we are in a position to prove Theorem 3.

Proof of Theorem 3. The three feasible residue classes 0, 3, 6 modulo 9 will be treated separately.

Case 1: $v \equiv 0 \pmod 9$.

For $v = 9u$ let the vertex set be $A_1 \cup \cdots \cup A_u$ with $|A_i| = 9$ for all $1 \le i \le u$. Inside each A_i we take a copy of the 9-cycle decomposition of $K_9^{(3)} - I$ given in Lemma 8. The family of triplets meeting an A_i in two vertices and another A_j in one vertex can be covered by the 9-cycle decomposition of $K_{9\leftrightarrow 9}^{(3)}$, as in Lemma 12. Finally, the triplets meeting three distinct parts A_i, A_j, A_k form a copy of $K_{9,9,9}^{(3)}$ for any $1 \le i < j < k \le u$, hence this type is also decomposable into 9-cycles by Lemma 12.

Case 2: $v \equiv 3$ (mod 9).
If $v = 9u + 3$, beside the sets $A_1, \dots, A_u$ with $|A_i| = 9$ we also take an A_0 with $|A_0| = 3$. Now inside each $A_0 \cup A_i$ we insert a copy of the $K_{12}^{(3)} - I$ decomposition as in Lemma 13, in such a way that A_0 is one of the uncovered triplets. Hence the copies of $K_{12}^{(3)} - I$ for distinct i are independent of each other. Inside $A_1 \cup \cdots \cup A_u$ we cover the triplets meeting more than one A_i in the same way as we did in the case of $v = 9u$. Hence only those triplets remain to be covered that meet A_0 and two further A_i, A_j ($1 \le i < j \le u$). For any fixed pair i, j those triplets form a copy of $K_{9,9,3}^{(3)}$, thus they are decomposable into 9-cycles by Lemma 12.

Case 3: $v \equiv 6$ (mod 9).
If $v = 9u + 6$, beside the sets $A_1, \dots, A_u$ with $|A_i| = 9$ we take an A_0 with $|A_0| = 6$. In this case A_1 will be treated differently from the other parts A_i, $i \ge 2$. Then we take:

- $K_{15}^{(3)} - I$ inside $A_0 \cup A_1$, see Lemma 14;
- $K_9^{(3)} - I$ inside each A_i for $2 \le i \le u$, see Lemma 8;
- $K_{6\leftrightarrow 9}^{(3)}$ between A_0 and each A_i for $2 \le i \le u$, see Lemma 12 (iii);
- $K_{6,9,9}^{(3)}$ with vertex classes A_0, A_i, A_j for all $1 \le i < j \le u$, see Lemma 12 (i);
- $K_{9\leftrightarrow 9}^{(3)}$ with vertex classes A_i, A_j for all $1 \le i < j \le u$, see Lemma 12 (iii);
- $K_{9,9,9}^{(3)}$ with vertex classes A_i, A_j, A_k for all $1 \le i < j < k \le u$, see Lemma 12 (i).

The decompositions of these parts can be done according to the lemmas above, and they together decompose $K_{9u+6}^{(3)} - I$ into 9-cycles. □

4. 2-split systems

The notion of 2-split systems was introduced and applied in [9], with the intention to build cycle systems of double size from available constructions. In our present context a *2-split system* of order v consists of a decomposition of two vertex-disjoint copies of $K_{v/2}^{(3)} - I$, together with a decomposition of $K_{v/2\leftrightarrow v/2}^{(3)}$ for the set of edges that meet both of the two disjoint parts.

In this section we prove that this can be done for all feasible residue classes; i.e., 2-split $\mathcal{C}^*(3, 6, v)$ and $\mathcal{C}^*(3, 9, v)$ systems exist for all v that admit $\mathcal{C}^*(3, 6, v/2)$ and $\mathcal{C}^*(3, 9, v/2)$ systems, respectively. In fact, the existence of 2-split $\mathcal{C}^*(3, 9, v)$ systems already follows from our previous lemmas.

Theorem 15. *There exists a 2-split 9-cycle decomposition of $K_v^{(3)} - I$ if and only if $v \ge 18$ and $v \equiv 0$ (mod 6).*

Proof. A $\mathcal{C}^*(3, 9, v/2)$ system with $v/2 = 3p$ exists for every $p \ge 3$ by Theorem 3, and a $K_{3p\leftrightarrow 3p}^{(3)}$ system exists by part (*iii*) of Lemma 12. On the other hand, the condition $3 \mid \frac{v}{2}$ is clearly necessary. □

The construction of 2-split 6-cycle systems requires more work. Along the way we shall also need a fundamental result on Kirkman triple systems. Recall from the literature that a *Kirkman triple system* of order v is a collection of 3-element sets (blocks) such that each pair of elements belongs to exactly one block, and the set of blocks can be partitioned into $(v-1)/2$ so-called parallel classes, each of those classes consisting of $v/3$ mutually disjoint blocks.

Theorem 16 *(Ray-Chaudhuri, Wilson [22]). For every $v \equiv 3$ (mod 6) there exists a Kirkman triple system of order v.*

We shall also use the following small construction.

Lemma 17 *(Akin et al. [7], Example 3). There is a decomposition of $K_{6\leftrightarrow 6}^{(3)}$ into 6-cycles.*

Proof. The construction in [7] takes $\mathbb{Z}_{12}$ as vertex set, and defines 30 cycles derived from two cycles $(0, 5, 10, 8, 11, 2)$ and $(0, 1, 9, 4, 3, 7)$ turning them into 12 positions via the mappings $j \mapsto j + i$ (mod 12) for $i \in \mathbb{Z}_{12}$, and from a third cycle $(0,1,2,6,7,8)$ turned into 6 positions via $j \mapsto j+i \pmod{12}$ for $i = 0,1,\dots,5$. Here the two vertex classes are $A = \{0,2,4,6,8,10\}$ and $B = \{1,3,5,7,9,11\}$. □

The spectrum of 2-split $\mathcal{C}^*(3,6,v)$ systems can now be determined.

Theorem 18. *There exists a 2-split 6-cycle decomposition of $K_v^{(3)} - I$ if and only if $v \ge 12$ and $v \equiv 0, 6, 12 \pmod{24}$.*

Proof. According to Theorem 2, the arithmetic condition $v/2 \equiv 0,3,6 \pmod{12}$ is necessary. To prove sufficiency, consider first the cases where v is of the form $v = 12p$. We know from Theorem 2 that a $\mathcal{C}^*(3,6,6p)$ system exists for every $p \ge 1$. Let now $A_1 \cup \cdots \cup A_p$ and $B_1 \cup \cdots \cup B_p$ be the vertex sets of two such systems, where the A_i and B_i are mutually disjoint 6-element sets. For all $1 \le i, j \le p$ we take decompositions of $K_{6\leftrightarrow 6}^{(3)}$ as given in Lemma 17, for the crossing triplets in $A_i \cup B_j$. It remains to cover the triplets that meet two subsets on one side of $\mathcal{C}^*(3,6,6p)$ and one subset on the other side. For any three of those 6-element sets we can apply the decomposition chain

$$K_{6,6,6}^{(3)} \longrightarrow 2 \times K_{3,6,6}^{(3)} \longrightarrow 4 \times K_{3,3,6}^{(3)} \longrightarrow 12 \times K_{3,3,2}^{(3)}$$

and find a decomposition according to Lemma 7.

For the third residue class $v = 24p + 6$, the construction starts with two copies of a $\mathcal{C}^*(3,6,12p+3)$ system, say over the disjoint sets A' and A'', guaranteed by Theorem 2. Apply Theorem 16 to find Kirkman triple systems over A' and A'', to be used as auxiliary tools. We denote by $F'_1, \dots, F'_{6p+1}$ and $F''_1, \dots, F''_{6p+1}$ the corresponding parallel classes. Then for $i = 1, \dots, 6p+1$, for all $T' \in F'_i$ and $T'' \in F''_i$ we take a decomposition of $K_6^{(3)} - I$ whose missing edges are T' and T'' (three 6-cycles for each pair T', T'') as given in Lemma 6. The collection of those cycles covers all crossing triplets exactly once. Indeed, a triplet T meeting both A' and A'' has two vertices on one side; those two vertices are contained in a unique block of the Kirkman triple system on that side; and the block uniquely determines the index i of $F'_i \cup F''_i$ in which the block occurs; hence T is contained in a single well-defined set $T' \cup T''$ and appears in just one of its 6-cycles. Consequently a $\mathcal{C}^*(3,6,24p+6)$ system is obtained. □

5. Concluding remarks

In this note we determined the spectrum of 6-cycle decompositions and 9-cycle decompositions of nearly complete 3-uniform hypergraphs $K_v^{(3)} - I$. This problem is now completely solved. On the other hand it would be of interest to find decompositions satisfying some further structural requirements. Besides the 2-split systems discussed in Section 4 we would like to mention cyclic systems as well. In a cyclic system, with the vertex set $\mathbb{Z}_v$, the mapping $i \mapsto i+1 \pmod{v}$ is an automorphism of the decomposition. The missing triplets are $(i, i+v/3, i+2v/3)$, because $i \mapsto i+1$ has to be an automorphism of the complement of $K_v^{(3)} - I$, too. Concerning cyclic systems we formulate the following open problems.

Conjecture 1.

(*i*) *For every $v \equiv 0,3,6 \pmod{12}$, $v \ge 6$, there exists a cyclic 6-cycle decomposition of $K_v^{(3)} - I$.*
(*ii*) *For every $v \equiv 0,6,12 \pmod{24}$, $v \ge 12$, there exists a cyclic 2-split 6-cycle decomposition of $K_v^{(3)} - I$.*
(*iii*) *For every $v \equiv 0 \pmod 3$, $v \ge 9$, there exists a cyclic 9-cycle decomposition of $K_v^{(3)} - I$.*
(*iv*) *For every $v \equiv 0 \pmod 6$, $v \ge 18$, there exists a cyclic 2-split 9-cycle decomposition of $K_v^{(3)} - I$.*

Examples of cyclic 9-cycle decompositions have already been given in Lemmas 8, 11, and 13 above. In further support of Conjecture 1, in the two subsections below we list the base cycles generating cyclic systems $\mathcal{C}^*(3,6,v)$ and $\mathcal{C}^*(3,9,v)$ for all feasible values of $v \le 30$. Those systems have been found via a combination of intuitively pre-defined base cycles and partial computer search. In order to facilitate checking that those are decompositions indeed, detailed tables are presented in the arXiv version [16] of this paper.

Remark 19. Under the mapping $i \mapsto i+1$ the number of orbits of triplets other than $(i, i+v/3, i+2v/3)$ is $\frac{1}{6}v(v-3)$. This is not divisible by 6 if $v = 12p+6$, and not divisible by 9 if $v = 9p+6$. In those residue classes, one “exceptional” base cycle has an automorphism $i \mapsto i + v/2$ for 6-cycles and $i \mapsto i + v/3$ for 9-cycles.

5.1. Base cycles of cyclic 6-cycle decompositions

The cyclic system $\mathcal{C}^*(3,6,6)$ has 1 base cycle:

(0,1,2,5,4,3) (exceptional).[1]

The cyclic system $\mathcal{C}^*(3,6,12)$ has 3 base cycles:
(0,1,2,4,5,8), (0,1,5,8,3,6), (0,1,9,3,5,7).

The cyclic system $\mathcal{C}^*(3,6,15)$ has 5 base cycles:
(0,1,2,4,5,8), (0,1,5,3,8,6), (0,1,7,3,6,10), (0,1,9,14,7,11),
(0,3,12,6,2,8).

The cyclic system $\mathcal{C}^*(3,6,18)$ has 8 base cycles:
(0,1,2,9,10,11) (exceptional),
(0,1,3,4,7,9), (0,1,5,3,8,10), (0,1,6,5,2,12), (0,1,7,3,9,13),
(0,2,7,13,3,8), (0,3,7,17,4,11), (0,3,12,8,15,6).

The cyclic system $\mathcal{C}^*(3,6,24)$ has 14 base cycles:
(0,1,2,4,5,8), (0,1,5,3,8,6), (0,1,7,3,6,9), (0,1,10,3,5,11),
(0,1,12,3,6,13), (0,1,14,3,6,15), (0,1,16,3,6,17), (0,1,18,3,6,20),
(0,2,10,5,1,19), (0,2,12,7,16,20), (0,2,14,7,3,16), (0,4,12,17,3,7),
(0,4,17,23,7,16), (0,5,13,19,7,14).

The cyclic system $\mathcal{C}^*(3,6,27)$ has 18 base cycles:
(0,1,3,5,24,26), (0,3,4,8,23,24), (0,4,6,1,21,23), (0,5,6,12,21,22),
(0,6,9,1,18,21), (0,7,3,8,24,20), (0,8,1,10,26,19), (0,10,2,11,25,17),
(0,11,1,8,26,16), (0,12,2,16,25,15), (0,13,1,4,26,14), (0,1,9,4,13,16),
(0,2,13,5,18,22), (0,2,16,23,5,18), (0,3,16,6,21,17), (0,4,12,19,7,21),
(0,6,17,23,12,19), (0,6,18,4,23,7).

The cyclic system $\mathcal{C}^*(3,6,30)$ has 23 base cycles:
(0,1,2,15,16,17) (exceptional),
(0,2,3,15,27,28), (0,3,4,15,26,27), (0,4,6,15,24,26), (0,5,1,15,29,25),
(0,6,7,15,23,24), (0,7,2,15,28,23), (0,1,6,3,10,15), (0,1,8,10,2,20),
(0,2,10,5,14,17), (0,2,13,5,19,14), (0,3,9,16,1,10), (0,4,10,29,13,23),
(0,4,12,25,7,21), (0,10,21,6,24,13), (0,5,12,9,14,15), (0,18,10,12,19,20),
(0,3,12,7,15,17), (0,25,9,1,12,14), (0,9,24,1,7,10), (0,10,24,13,19,23),
(0,14,26,9,17,21), (0,19,7,22,3,13).

5.2. Base cycles of cyclic 9-cycle decompositions

The cyclic system $\mathcal{C}^*(3,9,9)$ has 1 base cycle:
(0,1,2,7,6,4,8,5,3). (Cf. Lemma 8.)

The cyclic system $\mathcal{C}^*(3,9,12)$ has 2 base cycles:
(0,1,2,4,5,8,10,3,6), (0,1,7,11,2,9,10,6,8).

The cyclic system $\mathcal{C}^*(3,9,15)$ has 4 base cycles:
(0,1,2,5,6,7,10,11,12) (exceptional),
(0,1,3,4,8,2,5,7,9), (0,1,6,3,10,2,12,4,7), (0,1,10,14,6,12,2,5,11).

The cyclic system $\mathcal{C}^*(3,9,18)$ has 5 base cycles:
(0,1,2,4,5,8,3,6,7), (0,1,5,3,9,4,2,10,8), (0,1,9,4,8,10,3,6,13),
(0,1,10,3,7,14,17,2,11), (0,3,13,17,5,10,14,4,9).

The cyclic system $\mathcal{C}^*(3,9,21)$ has 7 base cycles:
(0,1,2,4,5,8,3,6,7), (0,1,5,3,9,4,2,10,8), (0,1,9,4,8,10,3,6,12),
(0,1,10,3,7,11,2,6,13), (0,1,11,3,8,13,2,4,14), (0,3,6,18,4,13,1,5,14),
(0,3,13,7,12,20,6,17,11).

The cyclic system $\mathcal{C}^*(3,9,24)$ has 10 base cycles:
(0,1,2,8,9,10,16,17,18) (exceptional),
(0,1,3,4,7,2,5,6,10), (0,1,6,5,13,2,3,11,12), (0,1,14,3,5,7,11,2,15),
(0,1,17,3,5,10,2,6,19), (0,2,8,5,1,7,9,16,12), (0,2,10,7,1,12,5,14,19),
(0,3,9,15,1,4,14,20,10), (0,3,12,15,4,22,13,5,17), (0,4,9,16,1,11,18,5,13).

[1] Interestingly enough, here the mapping $i \mapsto i + v/2$ reverses the vertex order along the base cycle, but nevertheless it is an automorphism. The point is that the "reflection through the 1–4 line," $i \mapsto 2 - i \pmod 6$, is a further automorphism of this cycle.

The cyclic system $\mathcal{C}^*(3,9,27)$ has 12 base cycles:
(0,1,3,6,10,17,21,24,26), (0,2,6,1,7,20,26,21,25), (0,3,4,10,1,26,17,23,24),
(0,4,9,1,8,19,26,18,23), (0,5,7,13,2,25,14,20,22), (0,6,10,18,1,26,9,17,21),
(0,7,9,19,5,22,8,18,20), (0,8,9,18,4,23,9,18,19), (0,10,3,15,1,26,12,24,17),
(0,11,1,22,8,19,5,26,16), (0,12,9,1,16,11,26,18,15), (0,13,4,19,20,7,8,23,14).

The cyclic system $\mathcal{C}^*(3,9,30)$ has 15 base cycles:
(0,1,3,6,7,23,24,27,29), (0,2,6,1,7,23,29,24,28), (0,3,7,12,1,29,18,23,27),
(0,4,10,1,8,22,29,20,26), (0,5,7,13,6,24,17,23,25), (0,6,13,1,4,26,29,17,24),
(0,7,8,16,1,29,14,22,23), (0,8,10,19,2,28,11,20,22), (0,9,10,20,2,28,10,20,21),
(0,11,3,16,28,2,14,27,19), (0,12,1,5,20,10,25,29,18), (0,13,1,11,8,22,19,29,17),
(0,14,2,17,3,27,13,28,16), (0,3,13,22,8,18,21,5,16), (0,6,15,25,8,18,4,13,19).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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