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Inverses of Rational Functions*

Tamas Dozsa[a]

Abstract

We consider the numerical construction of inverses for a class of rational functions. We propose two inverse algorithms, which can be used to simultaneously identify every zero of a rational function or polynomial. In the first case, we propose a generalization of an inverse algorithm based on our previous work and specify a class of rational functions, for which this generalized algorithm is applicable. In the second case, we provide a method to construct Blaschke-products, whose roots match the roots of a polynomial or a rational function. We also consider different iterative methods to numerically calculate the inverse points and discuss their properties.

Keywords: rational functions, Blaschke-products, fixed point iterations, winding numbers

1 Introduction

Rational functions play a crucial role in many theoretical and engineering applications. Rational orthogonal systems, such as the Malmquist-Takenaka system were proven to be well suited for several biological signal processing tasks [8, 13]. The transfer functions of linear systems are also rational, making the study of rational functions essential in system identification [12, 14]. Special types of rational functions, such as Blaschke-products also form the basis of many theoretical applications such as the Riesz-Nevanlinna factorization of Hardy-spaces [13], hyperbolic wavelet construction [12] and the construction of bi-orthogonal systems [6].

Our objective in this paper is to describe and numerically produce all solutions of the implicit equation

$$f(\phi) = \Gamma \subset \mathbb{C}, \tag{1}$$

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[a]Department of Numerical Analysis, Eötvös Loránd University, Budapest, Hungary and Systems and Control Laboratory, SZTAKI, Budapest, Hungary, E-mail: dotuaai@inf.elte.hu, dozsatamas@sztaki.hu, ORCID: 0000-0003-0919-4385

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where $f \in \mathcal{R}$ belongs to a class of rational functions and Γ is a simple smooth curve. The proposed methods aim to generalize our previous results for Blaschke-products described in [4, 5]. In [5], we provided sufficient conditions on Γ for the distinct, continuous solutions $\phi_k,\ k = 1, \ldots, n$ to uniquely exist if f is an n-factor Blaschke-product. Furthermore, we proposed an inverse algorithm, which can be used to find all solutions of (1).

The rest of this paper is organized as follows. In Section 2 we provide sufficient conditions on Γ for the existence of distinct continuous solutions to (1), if f is rational. In Section 3, we specify the class of considered rational functions $\mathcal{R}$, and propose a generalization of the inverse algorithm in [5] to produce the inverses of any $f \in \mathcal{R}$. Alternatively, one can construct a Blaschke-product, whose zeros match the zeros of the function in question and apply the algorithm proposed in [5] as-is. This approach and its properties are discussed in Section 3.5. In Section 4 we consider different numerical iterative methods and highlight their advantages for use with the proposed, generalized inverse algorithm. Finally, we summarize our results in Section 5.

2 Inverses of Analytic Functions along a Curve

In this section we will discuss the inverses of functions along a curve. Let f be an analytic function on the region $\Omega \subset \mathbb{C}$ and denote by $\Omega' := f(\Omega)$ its range. Furthermore, let $K := \{z \in \Omega : f'(z) = 0\}$ be the set of critical points and $K' = f(K)$ their image with respect to f. We note that, if f is a polynomial every point in K falls into the convex hull of the roots of f [11], while if f happens to be a Blaschke-product, all of its critical points fall into the hyperbolic convex hull of its zeros [11].

The analytic function f can be locally inverted in any $z_0 \in \Omega \setminus K$ [7, 11]. In other words, for any $W_0 \subset \Omega'$ neighborhood of the point $w_0 = f(z_0)$, we can find an $U_0 \subset \Omega$ neighborhood of z_0, such that $f : U_0 \to W_0$ is injective (one-to-one function). Our proposed algorithms rely on a generalization of this statement to curves. Let

$$\Gamma := \{\gamma(s) : s \in J = [\alpha, \beta]\} \subset \Omega' \tag{2}$$

be a simple smooth curve with γ parameterization. That is, $\gamma : J \to \Gamma$ is a continuously differentiable bijection, for which $\gamma'(s) \neq 0$ $(s \in J)$. We say that the smooth function $\phi : J \to f^{-1}(\Gamma)$ is the inverse function of f along the curve Γ in notation

$$f(\phi) = \Gamma, \tag{3}$$

if, $f(\phi(s)) = \gamma(s)$ $(s \in J)$ holds. We will use the following theorem regarding the solutions of (3).

Theorem 1. *Suppose $\Gamma \cap K' = \varnothing$. Then,*

1. *For any $z_0 \in f^{-1}(\Gamma)$, equation (3) has a solution that passes through z_0: $\exists \phi: \ \phi(s_0) = z_0, \ f(\phi(s)) = \gamma(s) \ (s \in J)$.*

2. *If any two solutions of (3) have a common point, then these solutions coincide: $\phi_1(s_0) = \phi_2(s_0) \implies \phi_1(s) = \phi_2(s) \ (s \in J)$.*

A proof of Theorem 1 can be found in [5]. This relies on the existence and uniqueness theorem for differential equations. An alternative way to prove the theorem can be found in [11] (2.1, pp. 25).

If f is a rational function, then it has a finite number of critical points. Furthermore, supposing $\Gamma \cap K' = \varnothing$, the equation $f(\phi) = \Gamma$ has a finite number of $\phi_1, \ldots, \phi_m, \ m \in \mathbb{N}$ solutions. Based on Theorem 1, the ranges $\Gamma^j := \phi_j(J) \subset f^{-1}(\Gamma) \ (j = 1, \ldots, m)$ of these solutions, are distinct smooth curves.

3 An inverse algorithm for rational functions

In this section we discuss how to find the inverse curves $\Gamma^j \ (j = 1, \ldots, m)$. We propose a generalization of the inverse algorithm introduced in [5], where f was assumed to be a finite Blaschke-product:

$$B(z) := \varepsilon \prod_{k=1}^{m} \frac{z - a_k}{1 - \overline{a_k} z} \quad (z \in \overline{\mathbb{D}}, \ a_k \in \mathbb{D}, \ k = 1, \ldots, m, \ m \in \mathbb{N}, \varepsilon \in \mathbb{T}). \tag{4}$$

These special rational functions, as defined in (4), have many applications such as the construction of rational orthogonal systems [6]. Our algorithm proposed in [5] had two main ideas. First, we showed that if f is an m-factor Blaschke-product and we choose a point $w \in \mathbb{T}$, then every solution $z_i \in \mathbb{T}$ of

$$f(z_i) = w \ (i = 1, \ldots, m) \tag{5}$$

can easily be identified. In this work, we introduce a class of rational functions $\mathcal{R}$ and generalize this idea for $f \in \mathcal{R}$ in Section 3.3. The second idea of the algorithm in [5] was that given the initial solutions in (5), a successive application of Newton's iteration can be used to produce every inverse of the Blaschke-product f along the curve $\Gamma \subset \mathbb{D} \cup \mathbb{T}$.

The main contribution of this paper therefore is a generalization of the inverse algorithm introduced in [5] for a wide class of rational functions. We begin by comparing the proposed algorithm's properties to well-established root finding methods in Section 3.1. We will discuss a generalization of this iterative method for arbitrary analytic functions in Section 3.2. Furthermore, we are going to propose a method to identify every zero of the rational function $f \in \mathcal{R}$ in Section 3.4. Finally, we will investigate an alternative root finding algorithm involving the construction of Blaschke-products in Section 3.5.

3.1 Comparison with existing approaches

If f is a rational function, then for any $w \in \mathbb{C}$, the implicit equation

$$f(z) = w \tag{6}$$

can be rewritten as the polynomial root finding problem

$$P(z) = H(z) - w \cdot Q(z) = 0, \tag{7}$$

where H and Q are polynomials such that $f(z) = \frac{H(z)}{Q(z)}$. Many well-established numerical algorithms exist for solving such problems. In this section we will compare the proposed method to the well-known algorithms [1, 3]. Comparison with these methods makes sense, because both the proposed inverse algorithm for rational functions and [1, 3] were created to produce *every solution* of (6) and (7) simultaneously.

The Graeffe-Dandelin-Lobachesky method detailed in [3] introduces an iteration which squares the zeros of a polynomial in each step. This separates the roots by magnitude, then the Vieta-relations can be exploited to get good estimates on the absolute values of the roots. These estimates can either serve as a starting point for some other root finding algorithm, or one of numerous strategies can be applied to estimate the angles of the zeros as well.

Another well-known and popular algorithm for finding every zero of a polynomial is Aberth's method [1]. This algorithm is cubically convergent for simple zeros and can be interpreted as an improvement of the Durand-Kerner method [10]. Aberth's method updates an initial estimate of the roots in each step of the iteration. The iteration can encounter problems in the case when both the zeros of the polynomial and the initial approximations are distributed in a symmetrical fashion.

The advantages of the rational inverse algorithm proposed in this manuscript over the above mentioned well-known polynomial root finding methods are twofold. First, in order to acquire the form (7) from (6), one assumes that the values of the polynomials H and Q can be accessed separately. If the value of f is available in a sufficient number of points, one could apply interpolation to achieve this, however at the cost of possibly introducing numerical errors (especially in real life applications in the presence of noise). The second advantage of the proposed method is that it makes no assumptions on the order of f. The root finding algorithms [1, 3] require us to have apriori information about the order of the polynomial whose roots we are trying to identify. In contrast, the algorithm presented here can produce every solution to (6), regardless of the number of solutions, provided that f belongs to a certain class of rational functions. For some applications however this condition is naturally satisfied. For example our algorithm could presumably be applied to identify the zeros (and thus poles) of the transfer function of an all-pass filter [2] without knowing the order of the transfer function.

Finally, we would like to mention that our approach in considering rational functions for inverse problems instead of polynomials is not without precedent. In fact, the classical Bernoulli-method [10] constructs a special rational function and identifies its so-called dominant pole in order to determine a zero of a polynomial.

3.2 Finding the inverses given an initial solution

Henceforth, denote by $D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ and $\overline{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq r\}$ the open and closed neighborhoods of z_0 and let $\Omega = D_R(0)$. Suppose the function f is analytic on $\overline{\Omega} = \overline{D}_R(0)$. Furthermore, let

$$M_j := \max_{z \in \overline{\Omega}} |f^{(j)}(z)|. \tag{8}$$

In order to produce the inverse curves $\Gamma^j \subset \Omega$ $(j = 1, \ldots, m)$ introduced in Section 2, we need to find neighborhoods which separate the curve $\Gamma \subset \Omega' = f(\Omega)$ from K' and the Γ^j curves from each other. Let

$$\rho(H, L) := \inf\{|z - w| : z \in H, w \in L\} \tag{9}$$

denote the distance between sets $H, L \subset \mathbb{C}$ and

$$\Gamma_r := \{w \in \Omega' : \rho(w, \Gamma) < r\} \tag{10}$$

denote the neighborhood of the curve Γ with a radius of r. In addition, let $K_r^c = \overline{\Omega} \setminus \cup_{\kappa \in K} D_r(\kappa)$ be the complement of the r radius neighborhood of the critical points. If $\Gamma \cap K' = \varnothing$, then Γ can be separated from K' in the following sense. There exists a number $r_1 > 0$ such that

$$\rho(\Gamma_{r_1}, K') > r_1. \tag{11}$$

By (11),

$$\rho(L, K) \geq \sqrt{r_1/M_2} =: r_2, \tag{12}$$

where $L := f^{-1}(\Gamma_{r_1})$. Indeed, if $\kappa \in K,\ w = f(z) \in \Gamma_{r_1}$, then

$$|f(z) - f(\kappa)| = |f(z) - f(\kappa) - f'(\kappa)(z - \kappa)| \leq M_2 |z - \kappa|^2.$$

From here, (12) is a consequence of

$$\rho(\Gamma_{r_1}, K') \leq M_2 \rho^2(K, L).$$

Since by Theorem 1, the inverse curves Γ^j are pairwise distinct, there exists r_0 for which

$$\Gamma^j_{r_0} \subset L,\ \Gamma^j_{r_0} \cap \Gamma^k_{r_0} = \varnothing, \quad j \neq k,\ 1 \leq j, k \leq m,\ L := f^{-1}(\Gamma_{r_1}). \tag{13}$$

Furthermore let

$$m_1 := \max_{z \in K_r^c} |1/f'(z)| \geq \max_{z \in L} |1/f'(z)|. \tag{14}$$

We note, that the constants m_1 and M_j only depend on Γ and f.

In order to solve the equation $f(\phi) = \Gamma$, suppose we already acquired for some $w_0 = \gamma(s_0) \in \Gamma$ point the solutions $z_{0,j} = \phi_j(s_0) \in \Gamma^j$ $(j = 1, \ldots, m)$. We are going to discuss iterative methods, with which we can determine the inverses $z \in \Gamma^j$ of $w \in \Gamma$, provided w is close enough to w_0. The solution $z \in \Gamma^j$ can be found on the disk $\overline{D}_r(z_{0,j})$, as the fixed point of an iteration generated by the function

$$h(v) = v - (f(z_0 + v) - w) \cdot g(z_0 + v) \ (|v| < r). \tag{15}$$

Indeed, if g does not vanish, then

$$h(v) = v \iff f(z) = w \ (z := z_0 + v), \tag{16}$$

and since $|z - z_0| = |v| < r$, based on (13), z falls on Γ^j, provided $r < r_0$.

In order to find v which satisfies (16), we are going to show for some functions h, that they are contraction mappings. That is, for any $|v_k| < r$ $(k = 1, 2)$,

$$|h(v_1) - h(v_2)| < q \cdot |v_1 - v_2| \tag{17}$$

for some constant $q \in [0, 1)$. In Section 4, we provide specific examples of h and show that there exist $0 < r \leq r_0$ and $0 < \overline{r} < r_1$, such that

$$z_0 \in \Gamma^j, \ f(z_0) = w_0, \ w \in \Gamma, \ |w - w_0| \leq \overline{r} \implies h : \overline{D}_r \to \overline{D}_r \tag{18}$$

and h also possesses the property described in (17). Such mappings h satisfy the conditions of the Fixed-point theorem and therefore iterations of the type $v_{k+1} := h(v_k)$ will converge to the solution (16). Using these iterations, we can invert the function f in the $w_k := \gamma(s_k) \in \Gamma$ points, where s_k belongs to the partitioning $s_0 = \alpha < s_1 < \ldots < s_N = \beta \quad (J = [\alpha, \beta])$. If the partitioning is dense enough, beginning from some initial solution $z_0 \in \Gamma^j$ satisfying $f(z_0) = w_0 \in \Gamma$, we can find the rest of the solutions $z_k \in \Gamma^j$ for which $f(z_k) = w_k, \ (k = 1, \ldots, N)$ recursively. These z_k solutions are the limits of fixed point iterations.

3.3 Finding an initial solution

In this section we introduce an algorithm to produce every initial solution $z_{0,j} = \phi(s_0) \in \Gamma^j, \ (j = 1, \ldots, m)$. The proposed algorithm is a generalization of the method introduced in [5], where similar ideas were used to produce these solutions if f is an m-factor Blaschke-product (4).

We begin by specifying the class of rational functions $\mathcal{R}$, for which the discussed ideas are applicable. For a rational function f, let Z_f and P_f denote the set of its zeros and poles respectively. Let $\mathcal{R}$ be the class of rational functions, for which

$$R^* := \max\{|\xi| : \xi \in Z_f\} < R_* := \min\{|\zeta| : \zeta \in P_f\}. \tag{19}$$

Polynomials and Blaschke-products obviously belong to $\mathcal{R}$. We will make use of the notion of the Nyquist-plot, which for a function f belonging to $\mathcal{R}$ can be defined by (20).

$$f_{T_R} := f(T_R)$$
$$(T_R := \{z = R \cdot e^{it}, t \in \mathbb{I} = [-\pi, \pi)\},\ R^* < R < R_*,\ f \in \mathcal{R}). \tag{20}$$

Our reason for considering the class of functions $\mathcal{R}$ is summarized by the next theorem.

Theorem 2. *If $f \in \mathcal{R}$, then the Nyquist-plot f_{T_R} can be written in the form*

$$f(Re^{it}) = A(t)e^{i\theta(t)}\ (t \in \mathbb{R}),$$

where A is a positive continuous function and $\theta : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. Furthermore θ satisfies $\theta(t + 2\pi) = \theta(t) + 2m\pi$ $(t \in \mathbb{R})$, where m denotes the number of f's zeros with multiplicities.

Proof. The winding number

$$\mathrm{Ind}(u, f_{T_R}) = \frac{1}{2\pi i}\int_{f_{T_R}} \frac{1}{z-u} dz\ (u \in \mathbb{C})$$

specifies the integer number of times the Nyquist-plot travels around the point u in a counter clockwise manner [7, 9, 11]. Cauchy's argument principle [7, 9, 11], makes a connection between the poles and zeros of f and the winding number of the Nyquist-plot at $u = 0$:

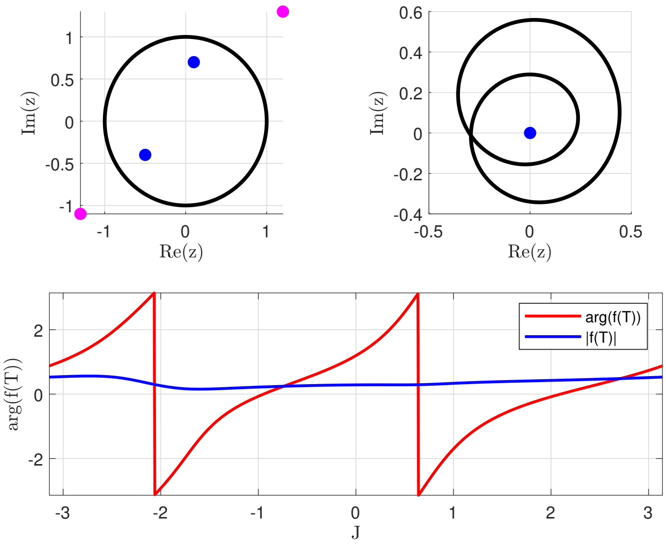
$$\mathrm{Ind}(0, f_{T_R}) = Z_{f,T_R} - P_{f,T_R},$$

where Z_{f,T_R} and P_{f,T_R} denote the number of zeros and poles that fall inside T_R. From this and the above mentioned interpretation of the winding number, choosing $f \in \mathcal{R}$ guarantees that in the Nyquist-plot

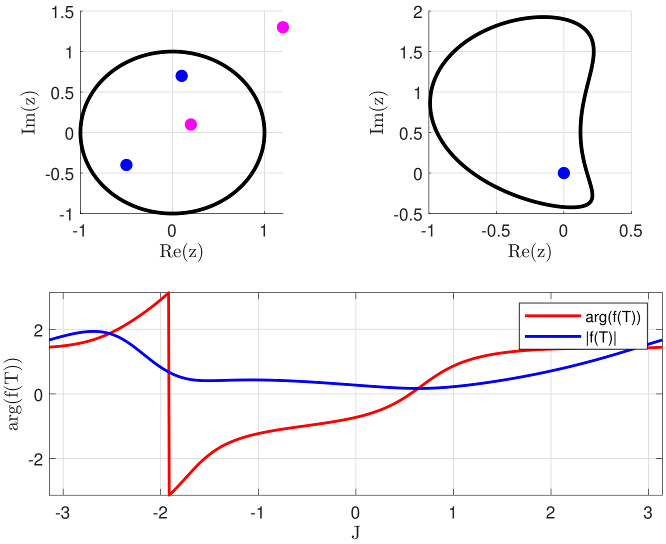
$$f(R \cdot e^{it}) = A(t)e^{i\theta(t)}\ (t \in \mathbb{R})$$

the argument function $\theta : \mathbb{R} \to \mathbb{R}$ is strictly increasing and satisfies $\theta(t + 2\pi) = \theta(t) + 2m\pi$ $(t \in \mathbb{R})$. □

We note that for Blaschke-products (4) $A(t) = 1$ $(t \in \mathbb{R})$. Figure 1 illustrates the Nyquist-plots of some examples of rational functions.



(a) The curve T_R with $R = 1$ (top left), its image $f(T_R)$ with respect to $f \in \mathcal{R}$ (top right), $\theta(t) \mod 2\pi$ and $A(t)$ (bottom).



(b) The curve T_R with $R = 1$ (top left), its image $f(T_R)$ with respect to $f \notin \mathcal{R}$ (top right), $\theta(t) \mod 2\pi$ and $A(t)$ (bottom).

Figure 1: 1a: Nyquist-plot of a rational function belonging to $\mathcal{R}$. The argument function θ is made up of 2 strictly increasing parts. 1b: Nyquist-plot of a rational function not in $\mathcal{R}$. Now the winding number is 1 and the Nyquist-plot makes a single revolution around 0. Blue points denote the zeros (and their images) of the functions, magenta points denote the poles.

From Theorem 2 it follows, that if $f \in \mathcal{R}$, each set

$$\mathbb{I}_\tau := \{t = \theta^{-1}(\theta(\tau) + 2j\pi) : t \in [-\pi, \pi),\ j \in \mathbb{Z}\} \quad (\tau \in [-\pi, \pi)) \tag{21}$$

has exactly m members. Furthermore, for any fixed τ we can easily produce the set $\mathbb{I}_\tau$ numerically (i.g. by interval halving). Then, we can identify m initial solutions by

$$f(Re^{it_j}) = f(z_{0,j}) = A(t_j)e^{i\theta(\tau)} = w_{0,j}\ (t_j \in \mathbb{I}_\tau,\ j = 1, \ldots, m). \tag{22}$$

3.4 Identifying every zero

In this section we discuss an application of the proposed inverse algorithm to find every zero of $f \in \mathcal{R}$. Then, we can give a parametric representation of the boundary of the star-like domain $f(D_R)$ as

$$F_R = \{A^*(\tau)e^{i\theta(\tau)} : \tau \in [-\pi, \pi)\}, \tag{23}$$

where $A^*(\tau) := \max_{t \in \mathbb{I}_\tau} A(t)$. The point $w \in f(D_R)$ is said to be an internal self intersecting point of the diagram $f(T_R)$, if there exist $t_1, t_2 \in \mathbb{I},\ t_1 \neq t_2$ that satisfy $f(Re^{it_1}) = f(Re^{it_2}) = w$. If $f \in \mathcal{R}$, then the S set of internal self intersecting points is finite. In order to find the zeros of f, we are going to produce the inverses along the line segments

$$\Gamma := [0, w_{0,j}] = \{\gamma(s) := (1-s)w_{0,j} : 0 \leq s \leq 1\}, \tag{24}$$

that connect 0 with the initial points $w_{0,j}$ $(j = 1, \ldots, m)$. We only consider the inverses along the line segments for which

$$[0, w_{0,j}] \cap (K' \cup S) = \varnothing \tag{25}$$

holds. Let F_R^* denote the set of possible $w_{0,j}$ endpoints, with which the segment $[0, w_{0,j}]$ satisfies (25). The set $F_R \setminus F_R^*$ is a finite set. We bring attention to the fact, that if the initial inverse points were determined according to Section 3.3, then the points $w_{0,j}$ all fall on the same line segment $(j = 1, \ldots, m)$. Henceforth we assume that the elements of $\mathbb{I}_\tau$ are indexed in a way so that the points $w_{0,j} = A(t_j)e^{i\theta(\tau)}$ satisfy $|w_{0,1}| < |w_{0,2}| < \ldots < |w_{0,m}|$ and therefore

$$[0, w_{0,1}] \subset [0, w_{0,2}] \subset \ldots \subset [0, w_{0,m}]. \tag{26}$$

Suppose that f has only simple roots. Consider the functions $\phi_j : [0, A(t_j)] \to f^{-1}(\Gamma)$ starting from the origin going backwards. That is, as a first step we define the inverse images of the segment $[0, w_{0,1}]$, which start from the m zeros of f. Now $\phi(A(t_1)) = z_{0,1} = Re^{it_1}$. Taking the inverse images of the segment $[w_{0,1}, w_{0,2}]$ starting from the point $\phi_j(t_1)$ $(j = 2, \ldots, m)$, we get $m-1$ smooth curves, furthermore $\phi_1(A(t_2)) = z_{0,2} = Re^{it_2}$. Continuing this method finally brings us to consider the inverse of $[w_{0,m-1}, w_{0,m}]$ starting from the point $\phi_m(t_{m-1})$, which gives us a smooth curve ending in $\phi_1(A(t_m)) = z_{0,m} = Re^{it_m}$. Thus, we showed that the functions ϕ_j considered over the intervals $[0, A(t_j)]$ are smooth solutions of the equation $f(\phi) = [0, w_{0,j}]$. Furthermore, these solutions connect the $z_{0,j}$ points on the boundary with the zeros of f. Our numerical experiments show, that if a zero of f has a multiplicity greater than 1, then the number of ϕ_j solution trajectories ending in this root matches the multiplicity. Figures 2 and 3 illustrate the above described root finding algorithm.

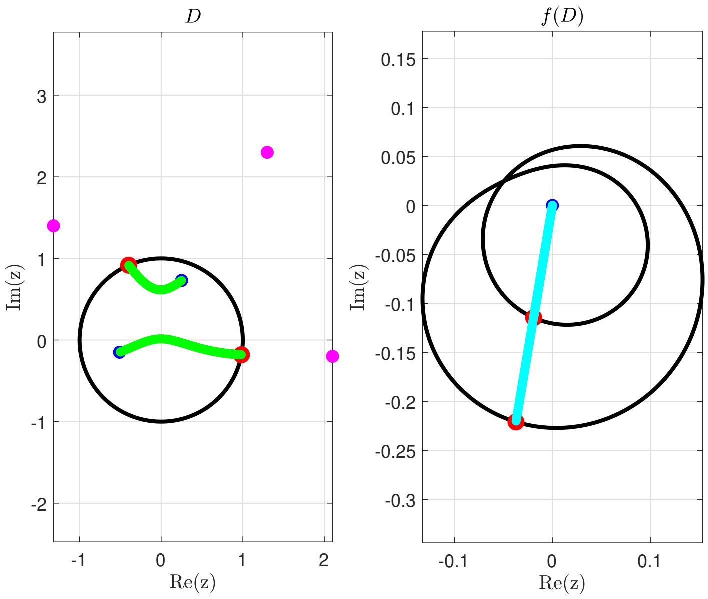
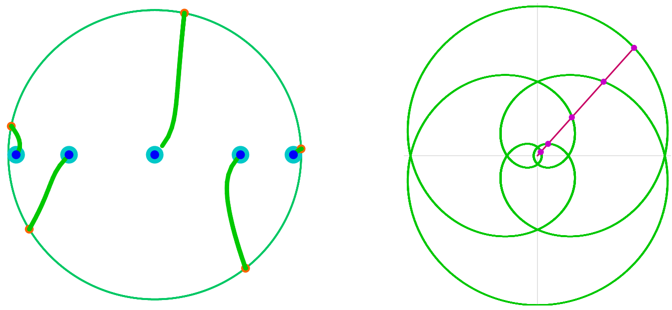


Figure 2: LEFT: The domain D_R (bordered by black circle), the zeros (blue points) and the poles (purple points) of $f \in \mathcal{R}$, initial inverse points (red points on the circle), and the inverse curves Γ^1 and Γ^2 (green curves). RIGHT: The range $f(D_R)$ bordered by the Nyquist-plot (black curve), $w_{0,1}$ and $w_{0,2}$ (red points), the inverted line segments $[0, w_{0,1}]$ and $[0, w_{0,2}]$ (light blue segments)

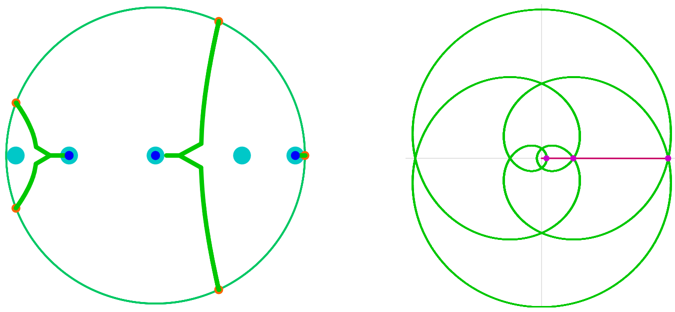
3.5 Construction of equivalent Blaschke-products

We now detail an alternative approach to identify the zeros of polynomials. Namely, we will construct Blaschke-products (4), whose zeros match the zeros the polynomial in question, then apply the inverse algorithm introduced in [5] to identify these. Suppose first that P is a polynomial of degree m. We can then consider the reciprocate polynomial P_r:

$$P_r(z) := z^m \overline{P}(1/\overline{z}) \ (z \in \mathbb{C}). \tag{27}$$



(a) LEFT: The complex unit circle, containing the roots (blue points) of a Chebyshev-polynomial. The inverse curves found by the proposed inverse algorithm are colored green. RIGHT: The Nyquist-plot $f(T_R)$ and the line segments $[0, w_{0,j}]$ $(j = 1, \dots, 5)$ to be inverted. Here, $[0, w_{0,j}] \cap S = \varnothing$.



(b) LEFT: The complex unit circle, containing the roots (blue points) of a Chebyshev-polynomial. The inverse curves found by the proposed inverse algorithm are colored green. RIGHT: The Nyquist-plot $f(T_R)$ and the line segments $[0, w_{0,j}]$ $(j = 1, \dots, 5)$ to be inverted. Here, $[, w_{0,j}] \cap S \neq \varnothing$.

Figure 3: Internal self intersecting points S

Using (27), we can construct the m-factor Blaschke-product B:

$$B(z) := \frac{P(z)}{P_r(z)} = \prod_{i=k}^{m} \frac{z - a_k}{1 - z\overline{a}_k}, \tag{28}$$

where a_k, $(k = 1, \dots, m)$ are the zeros of P including multiplicities. Then, the algorithm described in [5] can be applied to find the zeros a_k, $(k = 1, \dots, m)$.

4 Fixed point iterations

In this section we are going to give some concrete examples for the contraction mappings (15) and consider their properties. More precisely, we suppose that for a rational function $f \in \mathcal{R}$, we already have an initial inverse point z_0 satisfying $f(z_0) = w_0$. We are going to show that the proposed iterations satisfy (18) and (16), hence they produce the inverse at a point $w \in \Gamma$ as explained in 3.2, provided w is close enough to w_0.

4.1 A linearly convergent iteration

Our first example is a linearly convergent iterative method. Let

$$h(v) := v - \frac{f(z_0+v)-w}{f'(z_0)} \quad (|v| < r), \tag{29}$$

where $0 < r$ is assumed to satisfy $r < r_0$, in accordance with (18) and (13). We are going to show that there exists $r_1 > \overline{r} > 0$ such that if $|w - w_0| < \overline{r}$ and r is sufficiently small, then $h : \overline{D}_r \to \overline{D}_r$ is a contraction mapping. Then, according to Section 3.2, for the limit

$$v^* = \lim_{k\to\infty} v_k,\ v_{k+1} := h(v_k),\ v_0 = 0 \tag{30}$$

$f(v^* + z_0) = w$ will hold. Notice, that h has the following properties:

1. $h'(0) = 0$,
2. $|h''(v)| \leq M_2 \cdot m_1 \quad (|v| \leq r)$,

where M_2 and m_1 only depend on f and Γ as defined in (14) and (8). Choosing $v_1, v_2 \in \overline{D}_r$, we get:

$$|h(v_1) - h(v_2)| \leq \max_{s\in[v_1,v_2]} |h'(s)| \cdot |v_1 - v_2| \leq M_2 \cdot r \cdot |v_1 - v_2|. \tag{31}$$

Furthermore, if $v \in \overline{D}_r$

$$|h(v)| \leq |h(v) - h(0)| + |h(0)| \leq M_2 \cdot r \cdot |v| + |w - w_0| \cdot m_1. \tag{32}$$

From (31) and (32), choosing $r := \min\{r_0, 1/(2M_2)\}$ and w such that $|w - w_0| = \min\{\frac{r}{2m_1}, r_1\}$ hold guarantees that $h : \overline{D}_r \to \overline{D}_r$ is a contraction:

$$|h(v_1) - h(v_2)| \leq \frac{1}{2}|v_1 - v_2|,\ |h(v)| \leq r \quad (v, v_1, v_2 \in \overline{D}_r).$$

Convergence of (30) then follows from the Fixed-point theorem and the inverse property $h(v) = v \iff f(z_0 + v) = w$ is guaranteed by the considerations in 3.2. We can also apply the Fixed-point theorem to get the error estimate

$$|v_n - v^*| \leq 2^{-n+1}\ (n \in \mathbb{N}). \tag{33}$$

We note that a slight modification of h yields the iteration

$$\tilde{h}(v) := v - \frac{f(z_0+v)-w}{f'(z_0+v)} \quad (|v| \leq r \leq r_0) \tag{34}$$

which shows locally quadratic convergence. The iterative method generated by (34) can be interpreted as a Newton-iteration aimed at finding a zero of the function $g(v) := f(z_0 + v) - w_0$.

4.2 Identifying the zeros of Blaschke-products without access to derivatives

Suppose we are trying to identify the zeros of $f \in \mathcal{R}$, where f is an m-factor Blaschke-product (4). Suppose furthermore that every solution of $f(z_{0,k}) = w_0$ ($k = 1,\dots,m$) has already been acquired for some w_0. In this section we are going to construct a polynomial P based on the initial solutions $z_{0,k}$, whose zeros match the zeros of f. We can then apply the generalized inverse algorithm proposed in this paper to identify the zeros of P, thus identifying the zeros and poles of f. In addition, we are going to show, that when solving the implicit problems $P(z) = w_j$ ($j > 0$), we can express the derivative $P'(z_0)$ in (29) using the solutions from previous steps of the algorithm. This in turn means that when f is an m-factor Blaschke-product, one can find all of its zeros using the proposed inverse algorithm with a variation of the iteration (29), where we can express the needed derivative values from previous solutions.

In Section 3.3 we saw that if we choose the right side of $f(z) = w_0$ carefully, then every solution can be found with a simple numerical method (i.e. by interval halving). If, for example the disk D_R contains every zero of a polynomial H, then for any $w_0 = H(Re^{it_0})$ value, every zero of $Q(z) := H(z) - w_0$ can be easily identified. These could be used as the initial solutions for the proposed algorithm in 3.2. We can, however have other uses for the $z_1,\dots,z_m$ (pairwise different) zeros of Q as well. Namely, since $H' = Q'$, we can use the $Q(z) = q_m \cdot \prod_{k=1}^{m}(z - z_k)$ form of Q to calculate the derivatives of H. Here q_m denotes the leading coefficient of Q. Provided we have access to q_m, we can easily construct the derivative values needed for the linearly convergent iteration (29) using the initial solutions.

We are going to extend this idea to m-factor Blaschke-products. Namely, we are going to construct an m degree polynomial P with a leading coefficient $p_m = 1$, whose zeros match the zeros of the Blaschke-product. Then, the proposed inverse algorithm and the above idea can be used to identify these zeros. Suppose f is a Blaschke-product and for some $w_0 \in \mathbb{T}$, all m solutions to $f(z) = w_0$ have already been found. Since

$$f(z) = \prod_{k=1}^{m} \frac{z - a_k}{1 - \overline{a}_k z} \ (a_k \in \mathbb{D}, z \in \mathbb{D} \cup \mathbb{T}),$$

the solutions $z_1,\dots z_m$ coincide with the roots of the m degree polynomial

$$P_{w_0}(z) := \prod_{k=1}^{m}(z - a_k) - w_0 \cdot \prod_{k=1}^{m}(1 - \overline{a}_k z) = \sum_{k=0}^{m} p_{w_0,k} \cdot z^k. \tag{35}$$

From (35), the leading coefficient is $p_{w_0,m} = 1 - w_0 \cdot (-1)^m \prod_{k=1}^{m} \overline{a}_k$. Notice, that since f was a Blaschke-product, the leading coefficient can also be written as $p_{w_0,m} = 1 - w_0 \cdot \overline{f(0)}$. This means, we can write the polynomial P_{w_0} using the solutions $z_1,\dots,z_m$ to $f(z) = w_0$ as

$$P_{w_0}(z) = \left(1 - w_0 \cdot \overline{f(0)}\right) \prod_{k=1}^{m} (z - z_k). \tag{36}$$

Now consider the equation $f(u) = -w_0$. If $w_0 \in \mathbb{T}$, then the ideas discussed in 3.3 can be used to identify all m solutions to this. These $u_1, \ldots, u_m$ solutions are also the zeros of the polynomial

$$P_{-w_0}(u) = \prod_{k=1}^{m} (u - a_k) + w_0 \cdot \prod_{k=1}^{m} (1 - \overline{a}_k u), \tag{37}$$

which by the above can be written using the solutions to $f(u) = -w_0$ as

$$P_{-w_0}(z) = \left(1 + w_0 \cdot \overline{f(0)}\right) \prod_{k=1}^{m} (z - u_k). \tag{38}$$

By equations (36) and (38) we can query the values of P_{w_0} and P_{-w_0} using the solutions and $f(0)$, while by equations (35) and (37)

$$P(z) = \frac{1}{2}\left(P_{w_0}(z) + P_{-w_0}(z)\right) = \prod_{k=1}^{m} (z - a_k). \tag{39}$$

In (39), $P(z)$ is an m degree polynomial with a leading coefficient $p_m = 1$, whose zeros a_k $(k = 1, \ldots, m)$ match the zeros of the original Blaschke-product f.

4.3 Secant method

We now discuss an alternative to (29), where we replace the derivatives in (29) with divided differences. For $v \in \overline{D}_r$, let (15) take the form

$$h(v) := v - \frac{f(z_0 + v) - w}{f[z_0 + v, z_0]} = v - \frac{v \cdot (f(z_0 + v) - w)}{f(z_0 + v) - f(z_0)}, \ \lim_{v \to 0} h(v) = \frac{w - w_0}{f'(z_0)}, \tag{40}$$

where $f \in \mathcal{R}$, $f(z_0) = w_0 \in \Gamma$, $|w - w_0| < \overline{r}$ and $r < r_0$ in accordance with (18). We are going to show, that such r and $\overline{r}$ exist for (40). Then, by the ideas in 3.2 the limit

$$v^* = \lim_{k \to \infty} v_k, \ v_{k+1} := h(v_k), \ 0 < |v_0| < r$$

satsifies $f(v^* + z_0) = w$.

Consider the Taylor-series of f around z_0:

$$f(z_0 + v) = f(z_0) + f'(z_0) \cdot v + \sum_{k=2}^{\infty} \frac{f^{(k)}(z_0)}{k!} \cdot v^k. \tag{41}$$

Equation (41) gives us

$$f[z_0 + v, z_0] = f'(z_0) + \varepsilon(v) = f'(z_0) + v \cdot \varepsilon_1(v),$$

where $\varepsilon_1(v) = \sum_{k=2}^{\infty} \frac{f^{(k)}(z_0)}{k!} v^{k-2}$. Notice that

$$\varepsilon(0) = 0, \quad \varepsilon'(0) = \frac{f''(z_0)}{2} \tag{42}$$

hold.

Since

$$h(v) := v - \frac{f(z_0 + v) - w}{f'(z_0) + \varepsilon(v)} = v - f_1(v) \cdot g(v), \tag{43}$$

where $f_1(v) = f(z_0 + v) - w$ and $g(v) = 1/(f'(z_0) + \varepsilon(v))$, we can write the second derivative function of h as

$$h'' = f_1''g + 2f_1'g' + f_1g'' = f_1''g - 2f_1'\varepsilon'g^2 + f_1(-\varepsilon''g^2 + 2\varepsilon'^2g^3).$$

The derivatives $f_1^{(j)}$, $\varepsilon^{(j)}$ $(j \le 2)$ are bounded on $\overline{\Omega}$. We are going to show, that for sufficiently small r, the function $1/g$ is bounded from below on Γ^j. Indeed, for $z_0 \in \Gamma^j$,

$$1/|g(v)| \ge |f'(z_0)| - |v||\varepsilon_1(v)| \ge 1/m_1 - |v|m_2, \tag{44}$$

where $m_2 := \max_{|v|<r} |\varepsilon_1(v)|$ and m_1 is defined in (14). From this, if $|v| \le r := \frac{1}{2m_2m_1}$, then $|g(v)| \le 2m_1$. It follows that h'' is bounded from above:

$$|h''(v)| \le m_3 \ (|v| \le r). \tag{45}$$

In order to show that h is a contraction mapping, we introduce the function

$$h_1(v) = h(v) - v \cdot h'(0). \tag{46}$$

It is clear, that for h_1,

$$h_1'(0) = 0 \tag{47}$$

holds. Using (47) and the mean value theorem, we get that for any $v_1, v_2 \in \overline{D}_r$:

$$|h_1(v_1) - h_1(v_2)| \le \max_{v \in [v_1, v_2]} |h_1'(v)||v_1 - v_2| \le m_3 \cdot r \cdot |v_1 - v_2|. \tag{48}$$

Now we can use (48) to show h is a contraction mapping. Let $v_1, v_2 \in \overline{D}_r$, then

$$\begin{aligned} &|h(v_1) - h(v_2)| = |(h_1(v_1) + h'(0)v_1) - (h_1(v_2) + h'(0)v_2)| \le \\ &m_3 \cdot r \cdot |v_1 - v_2| + |h'(0)||v_1 - v_2| = (m_3 \cdot r + |h'(0)|)|v_1 - v_2| \end{aligned} \tag{49}$$

and

$$|h(v)| \le |h(v) - h(0)| + |h(0)| \le (m_3r + |h'(0)|)|v| + |h(0)|. \tag{50}$$

If we now choose

$$
\begin{aligned}
&r \le \min\{r_0, 1/(4m_3)\}\\
&|w - w_0| \le \overline{r} \le \min\{r_1, 1/(2m_1^2 M_2), r/(2m_1)\},
\end{aligned}
$$

then by (40) and (43)

$$|h(0)| \le |w - w_0| m_1 \le r/2, \quad |h'(0)| \le |w - w_0| M_2 m_1^2/2 \le r/4 \tag{51}$$

and consequently

$$|h(v_1) - h(v_2)| \le |v_1 - v_2|/2, \quad |h(v)| \le |v|/2 + r/2 \le r \ (|v| \le r). \tag{52}$$

Equation (52) shows that $h : \overline{D}_r \to \overline{D}_r$ is a contraction mapping. Thus, by the ideas in 3.2 and the fixed point theorem, the iteration generated by (40) can be used to find the inverse of f at a suitable point w.

5 Conclusion

In this study, we examined the numerical construction of the inverseses of rational functions along a curve. We considered the existence of continous solution curves in Section 2. We then provided an iterative algorithm to produce these solutions numerically in Section 3.2, given some initial solution points. We also proposed a class of rational functions, for which we can easily identify the needed initial solutions in Section 3.3. Furthemore, we proposed an algorithm with which the inverses can be used to identify the zeros of rational functions in Section 3.4. We gave an alternative algorithm for root finding in the case, when f is a polynomial, whose main feature was the construction of special Blaschke-products in Section 3.5. Finally, we investigated fixed point iterations to be used with our iterative algorithm and proved their convergence properties in Section 4.

The investigated algorithms give rise to a number of interesting applications, such as the identification of transfer functions for SISO (single input, single output) systems. We plan to explore these applications in future works.

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