



Brief paper

Convex incremental dissipativity analysis of nonlinear systems[☆]Chris Verhoek^{a,*}, Patrick J.W. Koelewijn^a, Sofie Haesaert^a, Roland Tóth^{a,b}^a Control Systems Group, Eindhoven University of Technology, Eindhoven 5600MB, The Netherlands^b Systems and Control Lab, Institute for Computer Science and Control, 1111 Budapest, Hungary

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ABSTRACT

Efficiently computable stability and performance analysis of nonlinear systems becomes increasingly more important in practical applications. Dissipativity can express stability and performance jointly, but existing results are limited to the regions around the equilibrium points of these nonlinear systems. The incremental framework, based on the convergence of the system trajectories, removes this limitation. We investigate how stability and performance characterizations of nonlinear systems in the incremental framework are linked to dissipativity, and how general performance characterization beyond the \mathcal{L}_2 -gain concept can be understood in this framework. This paper presents a matrix inequalities-based convex incremental dissipativity analysis for nonlinear systems via quadratic storage and supply functions. The proposed dissipativity analysis links the notions of incremental, differential, and general dissipativity. We show that through differential dissipativity, incremental and general dissipativity of the nonlinear system can be guaranteed. These results also lead to the incremental extensions of the \mathcal{L}_2 -gain, the generalized \mathcal{H}_2 -norm, the \mathcal{L}_∞ -gain, and passivity of nonlinear systems.

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1. Introduction

The *linear time-invariant* (LTI) framework has been a systematic and easy-to-use approach for modeling, identification and control of physical systems for many years. Its success is driven by powerful theoretical and computational results on stability, performance, and shaping (Skogestad & Postlethwaite, 2005). Growing performance demands in terms of accuracy, response speed and energy efficiency, together with increasing complexity of systems to accommodate such expectations, are pushing beyond the modeling and control capabilities of the LTI framework. Therefore, stability and performance analysis of *nonlinear* systems becomes increasingly more important.

A large variety of stability analysis tools are available for nonlinear systems, including Lyapunov's stability theory (Khalil, 2002), dissipativity theory (Willems, 1972) and contraction theory (Lohmiller & Slotine, 1998). Moreover, techniques such as

backstepping, input-output or feedback linearization (Khalil, 2002) have been introduced to stabilize the behavior and to achieve reference tracking for nonlinear systems. However, these techniques often require cumbersome computations and restrictive assumptions, and – unlike the LTI case – they have not lead to systematic performance analysis and shaping methods. While dissipativity theory in principle allows for analysis of nonlinear systems, current results are not computationally attractive. Furthermore, they only provide local stability and performance guarantees, i.e., only w.r.t. a single point of natural storage (usually the origin), which is undesirable for disturbance rejection and reference tracking. Hence, there is need for a computationally efficient analysis tool for *global* conclusions on the dissipativity property of a nonlinear system.

Several frameworks have been developed to extend computationally efficient LTI tools to nonlinear systems, e.g., using piecewise affine, *linear time-varying* (LTV), Fuzzy, or *linear parameter-varying* (LPV) system representations. The LPV framework specifically aims at providing convex tools to analyze nonlinear systems as a predefined convex set of LTI systems. However, the stability and performance guarantees are still only valid w.r.t. a single equilibrium point (Koelewijn, Sales Mazzoccante, Tóth, & Weiland, 2020). To analyze global stability properties of nonlinear systems, independent of a specific equilibrium point, notions such as incremental stability (Angeli, 2002) were introduced. Incremental stability analyzes stability of a system w.r.t. arbitrary trajectories of the system, instead of w.r.t. a single equilibrium point. Similar stability notions have also been developed, such

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as contraction (Lohmiller & Slotine, 1998; Manchester & Slotine, 2018) and convergence theory (Pavlov, Pogromsky, van de Wouw, & Nijmeijer, 2004) with strong connections to incremental stability theory (Rüffer, van de Wouw, & Mueller, 2013). Similar notions for performance have also been introduced such as incremental \mathcal{L}_2 -gain Fromion, Monaco, and Normand-Cyrot (2001) and passivity (Pavlov & Marconi, 2008). Extensions towards global dissipativity analysis in the literature are differential dissipativity (Forni & Sepulchre, 2013; Forni, Sepulchre, & van der Schaft, 2013; van der Schaft, 2013), incremental dissipativity (Pavlov & Marconi, 2008) and equilibrium independent dissipativity (Simpson-Porco, 2019). However, they do not provide computationally efficient methods to verify these dissipativity notions. Works discussing differential and incremental dissipativity only focus on passivity-based performance and how the various dissipativity notions are linked to general dissipativity is generally not discussed.

To address these shortcomings, the main contributions of this paper are (i) conditions on general quadratic performance analysis using incremental dissipativity, (ii) establishing the missing link between general dissipation theory and incremental analysis of nonlinear systems, and (iii) computationally efficient convex tools to analyze incremental stability and performance of nonlinear systems. This is achieved by developing a general incremental dissipativity framework that connects differential dissipativity, incremental dissipativity and general dissipativity. As a consequence, incremental notions of the \mathcal{L}_2 -gain, the generalized \mathcal{H}_2 -norm, the \mathcal{L}_∞ -gain and passivity are systematically introduced, also recovering some existing results on these concepts. Furthermore, convex analysis tools to compute the resulting conditions for differential and incremental dissipativity are derived using a so-called *differential parameter-varying* (DPV) inclusion of the nonlinear system.

In Section 2, a formal definition of the problem setting is given. Section 3 gives the main results on differential, incremental and general dissipativity and their connection. In Section 4, the incremental extensions of well-known performance measures are derived and the concept of DPV inclusions are discussed, yielding convex computation methods. The introduced concepts and methods are demonstrated on an academic example in Section 5, while the conclusions are provided Section 6.

Notation.

\mathbb{R} is the set of real numbers, while \mathbb{R}_0^+ and \mathbb{R}^+ stand for non-negative reals and positive reals. The convex hull of a set S is $\text{co}\{S\}$. Projection of $\mathbb{D} := \mathbb{A} \times \mathbb{B}$, with elements (a, b) , onto \mathbb{A} is denoted by $\pi_a \mathbb{D}$, meaning $a \in \pi_a \mathbb{D} = \mathbb{A}$. If a mapping $f : \mathbb{R}^p \rightarrow \mathbb{R}^q$ is in \mathcal{C}^n , it is n -times continuously differentiable. \mathcal{L}_2^n is the signal space of real-valued square integrable functions $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$ with associated norm $\|f\|_2 := (\int_0^\infty \|f(t)\|^2 dt)^{\frac{1}{2}}$ where $\|\cdot\|$ is the Euclidean (vector) norm. \mathcal{L}_∞^n is the signal space of functions $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$ with finite amplitude, i.e., bounded $\|f\|_\infty := \sup_{t \geq 0} \|f(t)\|$. We use $(*)$ to denote a symmetric term in a quadratic expression, e.g., $(*)^T Q (a - b) = (a - b)^T Q (a - b)$ for $Q \in \mathbb{R}^{n \times n}$ and $a, b \in \mathbb{R}^n$. The notation $A > 0$ ($A \succ 0$) indicates that A is positive (semi-) definite, while $A < 0$ ($A \preccurlyeq 0$) denotes a negative (semi-) definite A . The zero-matrix and the identity matrix of appropriate dimensions are denoted as 0 and I . Furthermore, $\text{col}(x_1, \dots, x_n)$ denotes the column vector $[x_1^T \dots x_n^T]^T$.

2. Problem definition

In this paper, we consider nonlinear, time-invariant systems of the form

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), u(t)); \\ y(t) = h(x(t), u(t)); \end{cases} \quad (1)$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ is the input, and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^{n_y}$ is the output of the system. The sets \mathcal{X} , \mathcal{U} and \mathcal{Y} are open sets containing the origin, with \mathcal{X} , \mathcal{U} being convex, and the mappings $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^{n_x}$ and $h : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$ are in \mathcal{C}^1 . We only consider solutions of (1) that are forward complete, unique and satisfy (1) in the ordinary sense. The trajectories of (1) are also restricted to have left-compact support, i.e., $\exists t_* \in \mathbb{R}$ such that (x, u, y) is zero outside the left-compact set $[t_*, \infty)$. We define the state-transition map as $\phi_x : \mathbb{R} \times \mathbb{R} \times \mathcal{X} \times \mathcal{U}^{\mathbb{R}} \rightarrow \mathcal{X}$, describing the evolution of the state such that

$$x(t) = \phi_x(t, t_0, x_0, u), \quad (2)$$

with $x_0 = x(t_0)$. The behavior of the system, i.e., the set of all possible solutions, is denoted by

$$\mathfrak{B} := \{(x, u, y) \in (\mathcal{X} \times \mathcal{U} \times \mathcal{Y})^{\mathbb{R}} \mid x \in \mathcal{C}^1 \text{ and } (x, u, y) \text{ satisfies (1) with left-compact support}\}. \quad (3)$$

Note that $\mathfrak{B} \subseteq \mathcal{B}^{\mathbb{R}}$, where $\mathcal{B} = \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$ is called the signal value set.

In this paper, the form presented in (1) will be referred to as the *primal form* of the nonlinear system. For the primal form, an extensive dissipativity theory has been developed over the years, with its roots in Willems (1972). From the notion of dissipativity, many system properties can be derived, such as performance characteristics and stability (Hill & Moylan, 1980; Willems, 1972), as well as a link with the physical interpretation of the system. Therefore, dissipativity is an important fundament in nonlinear system theory, which we will briefly review. We consider Willems' dissipativity notion (Willems, 1972) that allows for simultaneous stability and performance analysis.

Definition 1 (General Dissipativity). The system (1) is dissipative w.r.t. a supply function $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$, if there exists a storage function $\mathcal{V} : \mathcal{X} \rightarrow \mathbb{R}_0^+$ with an $x_* \in \mathcal{X}$, such that $\mathcal{V}(x_*) = 0$ and

$$\mathcal{V}(x(t_1)) - \mathcal{V}(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t)) dt, \quad (4)$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$, and for all $(x, u, y) \in \mathfrak{B}$.

The storage function \mathcal{V} can be interpreted as a representation of the stored 'energy' in the system with a point of neutral storage x_* (energy minimum), while the supply function s can be seen as the total energy flowing in and out of the system. If $\mathcal{V}(x(t))$ is differentiable, the *dissipation inequality* (DI) (4) can be rewritten as the so-called *differentiated dissipation inequality* (DDI), i.e., $\frac{d}{dt}(\mathcal{V}(x(t))) \leq s(u(t), y(t))$. In this paper, dissipativity of the primal form of a system will be referred to as *general dissipativity*. Note that x_* , i.e., the point where \mathcal{V} is considered to be zero, does not need to be at $x_* = 0$. In fact, it can be chosen to be any (forced) equilibrium point of (1). However, if the system is nonlinear, the DDI is different for each considered x_* and unlike in the LTI case, this difference cannot be eliminated by a coordinate transformation. This means that performance and stability analysis through general dissipativity is equilibrium point dependent.

An extension to this concept is *incremental dissipativity*, i.e., analysis of the (dissipated) energy flow between any two system trajectories. We give an extension of the definition of incremental passivity in van der Schaft (2017, Def. 4.7.1):

Definition 2 (Incremental Dissipativity). The system (1) is called incrementally dissipative w.r.t. the supply function $s_\Delta : \mathcal{U} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$, if there exists a storage function $\mathcal{V}_\Delta : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_0^+$, with $\mathcal{V}_\Delta(x, x) = 0$, such that for any two trajectories

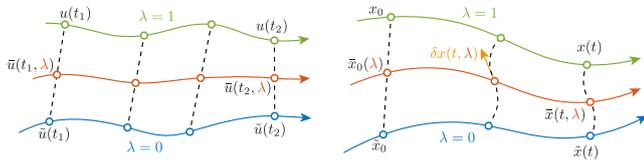


Fig. 1. Collection of trajectories $(\bar{x}(\lambda), \bar{u}(\lambda), \bar{y}(\lambda)) \in \mathfrak{B}$ parametrized in $\lambda \in [0, 1]$ between two arbitrary state and input trajectories, where $\bar{u}(t, \lambda) = \bar{u}(t) + \lambda(u(t) - \bar{u}(t))$ and $\bar{x}_0(\lambda) = \bar{x}_0 + \lambda(x_0 - \bar{x}_0)$. Variation δx of \bar{x} is given in orange.

$(x, u, y), (\bar{x}, \bar{u}, \bar{y}) \in \mathfrak{B}$, it holds that

$$\begin{aligned} \mathcal{V}_\Delta(x(t_1), \bar{x}(t_1)) - \mathcal{V}_\Delta(x(t_0), \bar{x}(t_0)) \\ \leq \int_{t_0}^{t_1} \mathcal{S}_\Delta(u(t), \bar{u}(t), y(t), \bar{y}(t)) dt, \end{aligned} \quad (5)$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.

Besides analyzing the difference between two trajectories, it is also possible to analyze *infinitesimal* variations of trajectories. First, define the set of paths

$$\Gamma_\Phi(\varphi, \bar{\varphi}) := \{\bar{\varphi} \in \Phi^{[0,1]} \mid \bar{\varphi} \in C^1, \bar{\varphi}(0) = \bar{\varphi}, \bar{\varphi}(1) = \varphi\},$$

corresponding to all smooth paths along $\lambda \in [0, 1]$ between points $\varphi \in \Phi \subseteq \mathbb{R}^n$ and $\bar{\varphi} \in \Phi \subseteq \mathbb{R}^n$. Consider any two trajectories of (1): $(x, u, y), (\bar{x}, \bar{u}, \bar{y}) \in \mathfrak{B}$. As \mathcal{X}, \mathcal{U} are considered to be convex, any trajectory *between* these can be parametrized with $\bar{x}_0 \in \Gamma_{\mathcal{X}}(x_0, \bar{x}_0)$ and $\bar{u}(t) \in \Gamma_{\mathcal{U}}(u(t), \bar{u}(t))$, e.g., $\bar{x}_0(\lambda) = \bar{x}_0 + \lambda(x_0 - \bar{x}_0) \in \mathcal{X}$ and $\bar{u}(t, \lambda) = \bar{u}(t) + \lambda(u(t) - \bar{u}(t)) \in \mathcal{U}$, resulting in the state transition map $\bar{x}(t, \lambda) = \phi_{\bar{x}}(t, t_0, \bar{x}_0(\lambda), \bar{u}(\lambda)) \in \mathcal{X}$. Note that this parametrization covers transitions between all possible solutions in \mathfrak{B} . Given a λ , it holds that

$$\dot{\bar{x}}(t, \lambda) = f(\bar{x}(t, \lambda), \bar{u}(t, \lambda)); \quad (6a)$$

$$\bar{y}(t, \lambda) = h(\bar{x}(t, \lambda), \bar{u}(t, \lambda)), \quad (6b)$$

and for $\lambda \in [0, 1]$, the resulting collection of trajectories $(\bar{x}(\lambda), \bar{u}(\lambda), \bar{y}(\lambda)) \in \mathfrak{B}$ is depicted in Fig. 1. As $f, h \in C^1$, taking the derivative of (6) w.r.t. λ , the infinitesimal variation of the original trajectories can be analyzed. A similar concept has been introduced in Crouch and van der Schaft (1987), Forni et al. (2013) and Reyes-Báez (2019) as *variational dynamics*¹. Differentiation of (6) w.r.t. λ yields the variational system

$$\Sigma_\delta : \begin{cases} \delta \dot{x}(t, \lambda) = \bar{A}(t, \lambda) \delta x(t, \lambda) + \bar{B}(t, \lambda) \delta u(t, \lambda); \\ \delta y(t, \lambda) = \bar{C}(t, \lambda) \delta x(t, \lambda) + \bar{D}(t, \lambda) \delta u(t, \lambda); \end{cases} \quad (7)$$

with $\delta x(t, \lambda) = \frac{\partial \bar{x}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_x}$, $\delta u(t, \lambda) = \frac{\partial \bar{u}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_u}$, $\delta y(t, \lambda) = \frac{\partial \bar{y}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_y}$, and, by omitting the dependence on t and λ for brevity, $\bar{A} = A(\bar{x}, \bar{u}), \dots, \bar{D} = D(\bar{x}, \bar{u})$ with

$$\begin{aligned} A(\bar{x}, \bar{u}) &= \frac{\partial f}{\partial x}(\bar{x}, \bar{u}), & B(\bar{x}, \bar{u}) &= \frac{\partial f}{\partial u}(\bar{x}, \bar{u}), \\ C(\bar{x}, \bar{u}) &= \frac{\partial h}{\partial x}(\bar{x}, \bar{u}), & D(\bar{x}, \bar{u}) &= \frac{\partial h}{\partial u}(\bar{x}, \bar{u}), \end{aligned} \quad (8)$$

where $(\bar{x}, \bar{u}) \in \pi_{x,u}\mathfrak{B}$ for all $\lambda \in [0, 1]$. Note that for a trajectory in \mathfrak{B} , λ is fixed for all time. Hence, given a trajectory $\bar{x}(\lambda_*)$, corresponding to λ_* , its variations are captured in $\delta x(\lambda_*)$. In the sequel, we will generally omit λ for brevity, e.g., $\delta x(t) = \delta x(t, \lambda_*)$. Analogous to the primal form, solutions of the variational system (7) are considered in the ordinary sense and are restricted to have left-compact support. In this paper, we will refer to (7) as the *differential form* of the nonlinear system (1).

¹ In fact, we can obtain a variational system for any smooth (x, u) parametrization (see Reyes-Báez (2019) for an alternative approach).

With the differential form of a system defined, we can define the notion of *differential dissipativity*, interpreted as the ‘energy’ dissipation of variations of the system trajectory that are not forced by the input. If the energy of these variations in the system trajectories decreases over time, the trajectory will eventually only be determined by the input of the system. Hence, the primal form of the system will converge to a *steady-state* solution, which is not necessary a forced equilibrium point, e.g., it can be a periodic orbit. We use the definition from Forni and Sepulchre (2013).

Definition 3 (Differential Dissipativity). Consider a system Σ of the form (1) and its differential form (7), Σ_δ . Σ is differentially dissipative w.r.t. a supply function $\mathcal{S}_\delta : \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}$, if there exists a storage function $\mathcal{V}_\delta : \mathcal{X} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_0^+$, with $\mathcal{V}_\delta(\cdot, 0) = 0$, such that

$$\mathcal{V}_\delta(\bar{x}(t_1), \delta x(t_1)) - \mathcal{V}_\delta(\bar{x}(t_0), \delta x(t_0)) \leq \int_{t_0}^{t_1} \mathcal{S}_\delta(\delta u(t), \delta y(t)) dt, \quad (9)$$

for all $(\bar{x}, \bar{u}) \in \pi_{x,u}\mathfrak{B}$ and for all $t_0, t_1 \in \mathbb{R}$, with $t_0 \leq t_1$.

Differential passivity definitions can be found in Forni et al. (2013) and van der Schaft (2013).

Remark 4 (Differentiated dissipation inequalities). Note that when the incremental and differential storage functions \mathcal{V}_Δ and \mathcal{V}_δ are differentiable, we can also define the *differentiated* forms of (5) and (9).

Despite the interest in general dissipativity, incremental dissipativity and differential dissipativity, the underlying connection between these notions have not been explored in the literature yet. We will establish this connection in case of quadratic supply functions in the next section, based on which performance analysis of nonlinear systems is achieved. Furthermore, we will discuss implications of these dissipativity notions on stability as well.

3. Main results

In this section, we present our main results. We first examine differential dissipativity, then we show that this property implies incremental dissipativity and general dissipativity of the nonlinear system.

3.1. Differential dissipativity of a nonlinear system

Consider the differential form (7) of a nonlinear system, which describes the variation of the system over a trajectory $(\bar{x}, \bar{u}, \bar{y}) \in \mathfrak{B}$. Note that this system always exists if the mappings f and h are in C^1 . To formulate our results for differential dissipativity, we consider a quadratic storage function of the form

$$\mathcal{V}_\delta(\bar{x}, \delta x) = \delta x^\top M(\bar{x}) \delta x, \quad (10)$$

where we assume:

- A1 The matrix function $M \in C^1$ is real, symmetric, bounded and positive definite, i.e., there exists $k_1, k_2 \in \mathbb{R}^+$, such that $k_1 I \preceq M(\bar{x}(t)) \preceq k_2 I$ for all $\bar{x}(t) \in \mathcal{X}$.

This storage function represents the energy of the variation along the state trajectory \bar{x} . We consider the following quadratic supply function,

$$\mathcal{S}_\delta(\delta u, \delta y) = (*)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} \delta u \\ \delta y \end{pmatrix}, \quad (11)$$

with real, constant, bounded matrices $R = R^\top, Q = Q^\top$ and S . With (10) and (11), we formulate the following theorem.

Theorem 5 (Differential Dissipativity Condition). *The system in primal form (1) is differentially dissipative w.r.t. the quadratic supply function (11) under a quadratic storage function (10) satisfying A1, if and only if for all $(\bar{x}, \bar{u}) \in \pi_{x,u}\mathfrak{B}$ and $t \in \mathbb{R}$, omitting dependence on time for brevity,*

$$(*)^\top \begin{pmatrix} \dot{M}(\bar{x}) & M(\bar{x}) \\ M(\bar{x}) & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A(\bar{x}, \bar{u}) & B(\bar{x}, \bar{u}) \end{pmatrix} - (**)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C(\bar{x}, \bar{u}) & D(\bar{x}, \bar{u}) \end{pmatrix} \preceq 0, \quad (12)$$

with $\dot{M}(\bar{x}) = \sum_{i=1}^{n_x} \frac{\partial M(\bar{x})}{\partial \bar{x}_i} \dot{\bar{x}}_i$, $\dot{\bar{x}}_i = \frac{d\bar{x}_i}{dt}$, and A, \dots, D as in (8).

Proof. By Definition 3, the primal form (1) is differentially dissipative, if the differential form (7) is dissipative. Hence, it suffices to show that if (12) holds, the differential form is dissipative with storage function (10) and supply function (11). Note that (10) is differentiable. Therefore, we start with substituting (10) and (11) into the differentiated differential dissipation inequality,

$$\frac{d}{dt} (\delta x^\top M(\bar{x}) \delta x) \leq (**)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} \delta u \\ \delta y \end{pmatrix}. \quad (13)$$

By Willems (1972), (13) is satisfied for all possible trajectories of (7) if and only if (13) holds for all values $(\delta x(t), \delta u(t), \delta y(t)) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y}$, and $\bar{x}(t) \in \mathcal{X}$. Writing out (13) yields,

$$2\delta x^\top \dot{M}(\bar{x}) (A(\bar{x}, \bar{u}) \delta x + B(\bar{x}, \bar{u}) \delta u) + \delta x^\top \dot{M}(\bar{x}) \delta x \leq \delta u^\top Q \delta u + 2\delta u^\top S (C(\bar{x}, \bar{u}) \delta x + D(\bar{x}, \bar{u}) \delta u) + (**)^\top R (C(\bar{x}, \bar{u}) \delta x + D(\bar{x}, \bar{u}) \delta u), \quad (14)$$

with A, \dots, D as in (8) and $\dot{M}(\bar{x}) = \sum_{i=1}^{n_x} \frac{\partial M(\bar{x})}{\partial \bar{x}_i} \dot{\bar{x}}_i$. It is trivial to see that (14) is equivalent to the pre- and post multiplication of (12) with $\text{col}(\delta x, \delta u)^\top$ and $\text{col}(\delta x, \delta u)$, respectively. Requiring (14) to hold for all $(\bar{x}, \bar{u}) \in \pi_{x,u}\mathfrak{B}$ and $t \in \mathbb{R}$ is equivalent to require the condition in (12) to hold for all $(\bar{x}, \bar{u}) \in \pi_{x,u}\mathfrak{B}$ and $t \in \mathbb{R}$, which proves the statement. ■

Note that the velocity of \bar{x} is required to verify differential dissipativity. Often this is solved in practice by capturing $\dot{\bar{x}}(t)$ in a set \mathcal{D} , such that $\dot{\bar{x}}(t) \in \mathcal{D}$ for all time.

3.2. Incremental dissipativity of a nonlinear system

First, we show that the property of differential dissipativity under supply function (11) implies the property of incremental dissipativity with supply function

$$\mathfrak{S}_\Delta(u, \tilde{u}, y, \tilde{y}) = (**)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} u - \tilde{u} \\ y - \tilde{y} \end{pmatrix}. \quad (15)$$

Secondly, we give a computable condition to analyze incremental dissipativity. The following result is the core of our contribution.

Theorem 6 (Induced Incremental Dissipativity). *When the system in primal form (1) is differentially dissipative w.r.t. the supply function (11) with $R \preceq 0$ under a storage function \mathfrak{V}_δ , then there exists a storage function \mathfrak{V}_Δ such that the system is incrementally dissipative w.r.t. the supply function (15).*

Proof. Writing out the λ -dependence in (9) for differential dissipativity, allows to integrate it over λ :

$$\int_0^1 \left[\mathfrak{V}_\delta(\bar{x}(t_1, \lambda), \delta x(t_1, \lambda)) - \mathfrak{V}_\delta(\bar{x}(t_0, \lambda), \delta x(t_0, \lambda)) - \int_{t_0}^{t_1} \mathfrak{S}_\delta(\delta u(\tau, \lambda), \delta y(\tau, \lambda)) d\tau \right] d\lambda \leq 0. \quad (16)$$

We compute the integral of the storage terms first. We define the following minimum energy path between x and \bar{x} by

$$\chi_{(x,\bar{x})}(\lambda) := \arg \inf_{\hat{x} \in \Gamma_{\mathcal{X}}(x,\bar{x})} \int_0^1 \mathfrak{V}_\delta \left(\hat{x}(\lambda), \frac{\partial \hat{x}(\lambda)}{\partial \lambda} \right) d\lambda. \quad (17)$$

When $\mathfrak{V}_\delta(\bar{x}, \delta x) = \delta x^\top M(\bar{x}) \delta x$, $\chi_{(x,\bar{x})}$ can be seen as the geodesic connecting x and \bar{x} corresponding to the Riemannian metric $M(\bar{x})$, see also Manchester and Slotine (2018) and Reyes-Báez (2019). Next, we define

$$\mathfrak{V}_\Delta(x, \bar{x}) := \int_0^1 \mathfrak{V}_\delta \left(\chi_{(x,\bar{x})}(\lambda), \frac{\partial \chi_{(x,\bar{x})}(\lambda)}{\partial \lambda} \right) d\lambda, \quad (18)$$

which will be our incremental storage function. Note that $\mathfrak{V}_\Delta \geq 0$ as by definition $\mathfrak{V}_\delta \geq 0$. Furthermore, $\mathfrak{V}_\Delta(x, \bar{x}) = 0$ when $x = \bar{x}$ as $\frac{\partial \chi_{(x,\bar{x})}(\lambda)}{\partial \lambda} = 0$ and by definition $\mathfrak{V}_\delta(\cdot, 0) = 0$. Using this incremental storage function, we have that

$$\mathfrak{V}_\Delta(x(t_1), \bar{x}(t_1)) \leq \int_0^1 \mathfrak{V}_\delta(\bar{x}(t_1, \lambda), \delta x(t_1, \lambda)) d\lambda, \quad (19)$$

for any $\bar{x}(t_1) \in \Gamma_{\mathcal{X}}(x(t_1), \bar{x}(t_1))$ with $x(t_1), \bar{x}(t_1) \in \mathcal{X}$, $t_1 \in \mathbb{R}$, and $\bar{x}(\lambda) \in \pi_{x,u}\mathfrak{B}$ for any $\lambda \in [0, 1]$. Furthermore, we take as parametrization for our initial condition $\bar{x}(t_0, \lambda) = \bar{x}_0(\lambda) = \chi_{(x_0, \bar{x}_0)}(\lambda)$. Hence, we have that

$$- \mathfrak{V}_\Delta(x(t_0), \bar{x}(t_0)) = - \int_0^1 \mathfrak{V}_\delta(\bar{x}(t_0, \lambda), \delta x(t_0, \lambda)) d\lambda. \quad (20)$$

Combining (19) and (20) gives that

$$\mathfrak{V}_\Delta(x(t_1), \bar{x}(t_1)) - \mathfrak{V}_\Delta(x(t_0), \bar{x}(t_0)) \leq \int_0^1 [\mathfrak{V}_\delta(\bar{x}(t_1, \lambda), \delta x(t_1, \lambda)) - \mathfrak{V}_\delta(\bar{x}(t_0, \lambda), \delta x(t_0, \lambda))] d\lambda.$$

This together with (16) implies

$$\mathfrak{V}_\Delta(x(t_1), \bar{x}(t_1)) - \mathfrak{V}_\Delta(x(t_0), \bar{x}(t_0)) \leq \int_0^1 \int_{t_0}^{t_1} \mathfrak{S}_\delta(\delta u(\tau, \lambda), \delta y(\tau, \lambda)) d\tau d\lambda. \quad (21)$$

We now consider the right-hand side of the inequality (21). Changing the order of integration gives

$$\int_{t_0}^{t_1} \int_0^1 (**)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} \delta u(\tau, \lambda) \\ \delta y(\tau, \lambda) \end{pmatrix} d\lambda d\tau. \quad (22)$$

We now solve the individual terms in the inner integral,

$$\int_0^1 (**)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} \delta u(\tau, \lambda) \\ \delta y(\tau, \lambda) \end{pmatrix} d\lambda = \int_0^1 (**)^\top Q \delta u(\tau, \lambda) d\lambda + 2 \int_0^1 \delta u(\tau, \lambda)^\top S \delta y(\tau, \lambda) d\lambda + \int_0^1 (**)^\top R \delta y(\tau, \lambda) d\lambda. \quad (23)$$

Taking $\tilde{u}(t, \lambda) = \tilde{u}(t) + \lambda(u(t) - \tilde{u}(t))$ as a parametrization, we obtain $\delta u(t) = \frac{\partial \tilde{u}(t, \lambda)}{\partial \lambda} = u(t) - \tilde{u}(t)$. Hence, the first term in (23) resolves to $(**)^\top Q(u(\tau) - \tilde{u}(\tau))$, while the second term gives

$$2(u(\tau) - \tilde{u}(\tau))^\top S \int_0^1 \frac{\partial \tilde{y}(\tau, \lambda)}{\partial \lambda} d\lambda = 2(u(\tau) - \tilde{u}(\tau))^\top S(y(\tau) - \tilde{y}(\tau)).$$

For the third term in (23) where $R \preceq 0$, i.e., $-R \succeq 0$, we use Lemma 23 in Appendix to obtain an upper bound:

$$\int_0^1 [(**)^\top R \frac{\partial \tilde{y}(\tau, \lambda)}{\partial \lambda}] d\lambda \leq (**)^\top R \left(\int_0^1 \frac{\partial \tilde{y}(\tau, \lambda)}{\partial \lambda} d\lambda \right) = (**)^\top R (\tilde{y}(\tau, 1) - \tilde{y}(\tau, 0)) = (**)^\top R (y(\tau) - \tilde{y}(\tau)). \quad (24)$$

Combining our results yields

$$\int_{t_0}^{t_1} (**)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} u(\tau) - \tilde{u}(\tau) \\ y(\tau) - \tilde{y}(\tau) \end{pmatrix} d\tau, \quad (25)$$

as an upper bound for (22). Thus, if (9) holds, we know that (16) holds, which in turn implies, considering a supply function (11) with $R \preceq 0$, that

$$\mathcal{V}_\Delta(x(t_1), \tilde{x}(t_1)) - \mathcal{V}_\Delta(x(t_0), \tilde{x}(t_0)) \leq \int_{t_0}^{t_1} (*)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} u(\tau) - \tilde{u}(\tau) \\ y(\tau) - \tilde{y}(\tau) \end{pmatrix} d\tau, \quad (26)$$

via the upper bound (25). Hence, if the system is differentially dissipative w.r.t. the supply function (11) with $R \preceq 0$, then the system is incrementally dissipative w.r.t. the equally parametrized supply function (15). ■

Remark 7 (Restricted R). Restriction $R \preceq 0$ is a technical necessity in the proof of Theorem 6. In case of $R > 0$ or R being indefinite, validity of Theorem 6 is an open question.

Comparing Theorem 6 to existing results in this context, we want to highlight that Waitman, Massioni, Bako, Scorletti, and Fromion (2016a, 2016b) also give some results on incremental dissipativity. However, these works only focus on a specific and restrictive form of the supply function. Moreover, the technical result of Waitman et al. (2016a) refers to a proof in a paper that has never appeared to the authors' knowledge.

From Theorem 6, we have the following (trivial) result:

Corollary 8 (Incremental Dissipativity Condition). *The system in primal form (1) is incrementally dissipative w.r.t. the supply function (15) with $R \preceq 0$, if (12) holds for all $(\bar{x}, \bar{u}) \in \pi_{x,u}B$ with M satisfying A1.*

Corollary 8 gives a sufficient condition to verify incremental dissipativity of a general nonlinear system. Note that by this result, if the matrix inequality (12) holds for all $(\bar{x}, \bar{u}) \in \pi_{x,u}B$, then we know that there exists a valid storage function of the form (18). However, calculating this function in an explicit form might be difficult (see Section 3.3). If no positive definite M can be found to satisfy (12), then it does not necessarily mean that the system is not differentially or incrementally dissipative. Inequality (12) might hold for a non-quadratic \mathcal{V}_δ , or a more complex M .

3.3. Explicit incremental storage function

Even if deriving an explicit form of (18) is challenging in general, under the quadratic form of (10), we can take an extra assumption to give an explicit construction:

A2 $M(\bar{x})$ can be decomposed as $M(\bar{x}) = N^\top(\bar{x})PN(\bar{x})$, $P > 0$, and $\exists \nu : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ s.t. $\frac{\partial \nu(\bar{x})}{\partial \bar{x}} = N(\bar{x})$.

While this decomposition of $M(\bar{x})$ is always possible if it satisfies A1, see Verhoek, Koelewijn, Tóth, and Haesaert (2022), existence of ν such that $\frac{\partial \nu(\bar{x})}{\partial \bar{x}} = N(\bar{x})$ is not guaranteed for any $M(\bar{x})$. This illustrates well the challenges for obtaining an explicit construction of \mathcal{V}_Δ . For the sake of simplicity, we assume in the remainder of this subsection that $\mathcal{X} = \mathbb{R}^{n_x}$.

Lemma 9 (Induced Incremental Storage Function). *If the system in primal form (1) is differentially dissipative with a storage function*

$$\mathcal{V}_\delta(\bar{x}, \delta x) = \delta x^\top M(\bar{x})\delta x, \quad (27)$$

where M satisfies A1 and A2, then the incremental storage function \mathcal{V}_Δ in Theorem 6 is given by

$$\mathcal{V}_\Delta(x, \tilde{x}) = (\nu(x) - \nu(\tilde{x}))^\top P(\nu(x) - \nu(\tilde{x})).$$

Additionally, if $M(\bar{x}) = M$ for all $\bar{x} \in \mathcal{X}$, then, the incremental storage function simplifies to

$$\mathcal{V}_\Delta(x, \tilde{x}) = (x - \tilde{x})^\top M(x - \tilde{x}). \quad (28)$$

Proof. Based on (16), we need to compute the terms

$$\int_0^1 \mathcal{V}_\delta(\bar{x}(t_1, \lambda), \delta x(t_1, \lambda))d\lambda \quad (29a)$$

and

$$- \int_0^1 \mathcal{V}_\delta(\bar{x}(t_0, \lambda), \delta x(t_0, \lambda))d\lambda. \quad (29b)$$

Based on A2, we can decompose $M(\bar{x})$ into

$$M(\bar{x}) = N^\top(\bar{x})PN(\bar{x}), \quad (30)$$

where $P > 0$ with $P \in \mathbb{R}^{n_x \times n_x}$ and, because of A1, $N(\bar{x}(t, \lambda)) \in \mathbb{R}^{n_x \times n_x}$ is invertible on \mathcal{X} , i.e., $\det N(\bar{x}) \neq 0$, $\forall \bar{x} \in \mathcal{X}$. Furthermore, by A2, there exists a diffeomorphism $\nu : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ such that $\frac{d\nu}{d\bar{x}}(\bar{x}) = N(\bar{x})$, $\forall \bar{x} \in \mathcal{X}$. Next, define $\bar{z}(t, \lambda) := \nu(\bar{x}(t, \lambda))$, which satisfies that

$$\delta z(t, \lambda) = \frac{\partial}{\partial \bar{x}} \bar{z}(t, \lambda) = N(\bar{x}(t, \lambda))\delta x(t, \lambda). \quad (31)$$

This allows to rewrite (10) as

$$\delta x^\top M(\bar{x})\delta x = \delta x^\top N^\top(\bar{x})PN(\bar{x})\delta x = \delta z^\top P\delta z. \quad (32)$$

Using this relation, the first term (29a) can be written as

$$\int_0^1 \delta z^\top(t_1, \lambda)P\delta z(t_1, \lambda)d\lambda. \quad (33)$$

Applying Lemma 23, see Appendix, to (33) results in

$$(*)^\top P \left(\int_0^1 \delta z(t_1, \lambda)d\lambda \right) \leq \int_0^1 \delta z^\top(t_1, \lambda)P\delta z(t_1, \lambda)d\lambda.$$

Hence,

$$(*)^\top P(\nu(x(t_1)) - \nu(\tilde{x}(t_1))) = (*)^\top P(\bar{z}(t_1, 1) - \bar{z}(t_1, 0)) \leq \int_0^1 (\delta z^\top(t_1, \lambda)P\delta z(t_1, \lambda))d\lambda = \int_0^1 (\delta x^\top(t_1, \lambda)M(\bar{x}(t_1, \lambda))\delta x(t_1, \lambda))d\lambda. \quad (34)$$

Before looking at the second term, i.e., (29b), let us recall some definitions. As aforementioned, the parametrized initial condition $\bar{x}(t_0, \lambda) = \bar{x}_0(\lambda)$ can be taken as any smooth parametrization $\bar{x}_0 \in \Gamma_{\mathcal{X}}(x_0, \tilde{x}_0)$. Recall that ν is a diffeomorphism, implying that $\nu^{-1} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ exists and $\nu, \nu^{-1} \in \mathcal{C}_1$. Hence, w.l.o.g. we take

$$\bar{x}_0(\lambda) = \nu^{-1}(\bar{z}(t_0, \lambda)), \quad (35)$$

where $\bar{z}(t_0, \lambda) = \nu(\tilde{x}_0) + \lambda(\nu(x_0) - \nu(\tilde{x}_0))$. Note that this choice of $\bar{x}_0(\lambda)$ satisfies the aforementioned conditions. Consequently, we have that

$$\delta z(t_0, \lambda) = \frac{\partial}{\partial \bar{x}} \bar{z}(t_0, \lambda) = \nu(x_0) - \nu(\tilde{x}_0). \quad (36)$$

Using this result and (32), the second term (29b) gives

$$\int_0^1 (\nu(x_0) - \nu(\tilde{x}_0))^\top P(\nu(x_0) - \nu(\tilde{x}_0))d\lambda = (\nu(x_0) - \nu(\tilde{x}_0))^\top P(\nu(x_0) - \nu(\tilde{x}_0)). \quad (37)$$

Combining the results of (34) and (37), it holds that

$$\int_0^1 [\mathcal{V}_\delta(\bar{x}(t_1, \lambda), \delta x(t_1, \lambda)) - \mathcal{V}_\delta(\bar{x}(t_0, \lambda), \delta x(t_0, \lambda))]d\lambda \geq (*)^\top P(\nu(x(t_1)) - \nu(\tilde{x}(t_1))) - (*)^\top P(\nu(x(t_0)) - \nu(\tilde{x}(t_0))). \quad (38)$$

Combining this result with (25) gives

$$\mathcal{V}_\Delta(x(t_1), \tilde{x}(t_1)) - \mathcal{V}_\Delta(x(t_0), \tilde{x}(t_0)) \leq \int_{t_0}^{t_1} (*)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} u(\tau) - \tilde{u}(\tau) \\ y(\tau) - \tilde{y}(\tau) \end{pmatrix} d\tau, \quad (39)$$

where \mathcal{V}_Δ is according to (27). Hence, (27) qualifies as an incremental storage function for (1).

In case $M(\bar{x}) = M$ for all $\bar{x} \in \mathcal{X}$, the decomposition in (30) simplifies to $N = I$ and $P = M$, hence, $\nu(x) = x$ and we obtain (28). Note that the same result is obtained when solving (17) and (18) directly for $V_\delta(\bar{x}, \delta x) = \delta x^\top M \delta x$, as in that case $\chi(x, \bar{x}) = \bar{x} + \lambda(x - \bar{x})$ and hence V_Δ is given by (28). ■

In case \mathcal{X} is a bounded convex set, Lemma 9 can be also shown to hold true, if either beyond A2 it holds that $\nu(\mathcal{X})$ is also convex, or if M is a constant matrix.

3.4. General dissipativity analysis of a nonlinear system

We now show that incremental dissipativity implies that the considered system is globally dissipative, i.e., dissipative w.r.t. any forced equilibrium point in \mathcal{B} .

Theorem 10 (Induced General Dissipativity). *Given a nonlinear system in its primal form (1). Suppose that $(x_e, u_e, y_e) \in \mathcal{B}$ is a (forced) equilibrium point of the system, i.e., $(\check{x}(t), \check{u}(t), \check{y}(t)) = (x_e, u_e, y_e)$ and $(\check{x}(t), \check{u}(t), \check{y}(t))$ satisfies (1) for all $t \in \mathbb{R}$. If the system is incrementally dissipative under the supply function (15), then for every equilibrium (x_e, u_e, y_e) , the system is dissipative w.r.t. an equally parametrized supply function.*

Proof. If the system is incrementally dissipative w.r.t. the supply function (15) under the storage function \mathcal{V}_Δ , then it holds that

$$\mathcal{V}_\Delta(x(t_1), \tilde{x}(t_1)) - \mathcal{V}_\Delta(x(t_0), \tilde{x}(t_0)) \leq \int_{t_0}^{t_1} (*)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} u(t) - \tilde{u}(t) \\ y(t) - \tilde{y}(t) \end{pmatrix} dt,$$

for all $t_0, t_1 \in \mathbb{R}$, with $t_0 \leq t_1$. Let the trajectory $(\tilde{x}, \tilde{u}, \tilde{y})$ be equal to the equilibrium trajectory $(\check{x}, \check{u}, \check{y})$, i.e., the equilibrium point (x_e, u_e, y_e) . Hence, for all $t_0, t_1 \in \mathbb{R}$, with $t_0 \leq t_1$

$$\mathcal{V}_\Delta(x(t_1), x_e) - \mathcal{V}_\Delta(x(t_0), x_e) \leq \int_{t_0}^{t_1} (*)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} u(t) - u_e \\ y(t) - y_e \end{pmatrix} dt,$$

Next, introduce the coordinate shift

$$q = x - x_e, \quad w = u - u_e, \quad z = y - y_e,$$

and define

$$\mathcal{V}(q) := \mathcal{V}_\Delta(q + x_e, x_e), \tag{40}$$

which is non-negative and satisfies that $\mathcal{V}(0) = 0$. Substituting this in the inequality gives that

$$\mathcal{V}(q(t_1)) - \mathcal{V}(q(t_0)) \leq \int_{t_0}^{t_1} (*)^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt,$$

holds for all $t_0, t_1 \in \mathbb{R}$, with $t_0 \leq t_1$, which is the general dissipation inequality (4) with \mathcal{V} as defined in (40) being the corresponding storage function. Hence, (1) is dissipative w.r.t. any arbitrary forced equilibrium point if it is incrementally dissipative. ■

By this last result, we have obtained a *chain of implications*, which connect the notions of dissipativity. Moreover, we gave a condition (matrix inequality (12)) that allows to examine differential, incremental and general dissipativity and thus examine global stability and performance of a nonlinear system. This chain of implications is summarized in Fig. 2. A result similar to Theorem 10 is given in Liu, Hill, and Zhao (2014) for single-input-single-output networked nonlinear systems. However, note that Theorem 10 is more general, as it holds for general nonlinear multi-input-multi-output systems of the form (1).

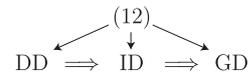


Fig. 2. Chain of implications with the dissipativity notions: differential dissipativity (DD) implies incremental dissipativity (ID), and incremental dissipativity implies general dissipativity (GD). Condition (12) can be used to analyze the various dissipativity notions.

Remark 11 (Implied Stability). If the supply function satisfies

$$s(0, y) \leq 0 \quad \forall y \in \pi_y \mathfrak{B}, \tag{41}$$

then it is well-known that dissipativity implies Lyapunov stability of (1) (Angeli, 2002; van der Schaft, 2017). Under a similar condition on \mathcal{S}_Δ , incremental dissipativity implies incremental stability, which means that there exists a function $\beta \in \mathcal{KL}$, such that, for all $u \in \pi_u \mathfrak{B}$, all $x_0, \tilde{x}_0 \in \mathcal{X}$ and all $t \geq 0$,

$$\|\phi_x(t, 0, x_0, u) - \phi_x(t, 0, \tilde{x}_0, u)\| \leq \beta(\|x_0 - \tilde{x}_0\|, t).$$

See Angeli (2002) for more details. Similarly, we have that differential dissipativity implies stability of (7) when $\forall \delta y \in (\mathbb{R}^{n_y})^{\mathbb{R}} : \mathcal{S}_\delta(0, \delta y) \leq 0$. As $R \preceq 0$, these conditions are trivially satisfied by our considered supply functions and through Theorems 6 and 10, the same chain of implications hold between these stability notions as in Fig. 2. Hence, by showing differential dissipativity with the considered supply functions, we also show incremental and Lyapunov stability of (1). If the above conditions on \mathcal{S} hold in the strict sense, then the implications hold in terms of asymptotic forms of stability.

4. Performance analysis via convex tests

We now use the dissipativity results of Section 3 to recover incremental notions of well-known performance indicators (\mathcal{L}_2 -gain, \mathcal{L}_∞ -gain, passivity and the generalized \mathcal{H}_2 -norm) and propose a method that allows for global, convex performance analysis of nonlinear systems. This contribution can also serve as a stepping stone for the formulation of incremental controller synthesis methods. We want to highlight that the results in this section resemble to conditions of respective performance indicators² of LPV systems. The LPV conditions differ from these results as we use the differential form of a nonlinear system. Hence, the relations follow from a completely different analysis that allows for global nonlinear performance analysis.

We will introduce the incremental performance notions for storage functions of the form of (28). It is trivial to extend these results to the case when a matrix function $M(\bar{x})$ is considered.

4.1. Incremental \mathcal{L}_2 -gain

A system has finite \mathcal{L}_2 -gain $\gamma < \infty$ if the system is dissipative w.r.t. to the supply rate $s(u, y) = \gamma^2 \|u\|_2^2 - \|y\|_2^2$ (Scherer & Weiland, 2015), i.e., u must be in $\mathcal{L}_2^{n_u}$. Let \mathfrak{B}_2 be defined as $\mathfrak{B}_2 := \{(x, u, y) \in \mathfrak{B} \mid u \in \mathcal{L}_2^{n_u}\}$. There are several definitions in the literature that extend the classical \mathcal{L}_2 -gain definition towards the incremental setting (Fromion, Scorletti, & Ferreres, 1999; Koelewijn, Tóth, Nijmeijer, & Weiland, 2021; van der Schaft, 2017). The following definition fits with the incremental dissipativity notion discussed in this paper.

² Therefore, the proofs in this section are omitted. For the interested reader, the proofs are included in the extended version of this paper Verhoek, Koelewijn, Haesaert, and Tóth (2022).

Definition 12 (Incremental \mathcal{L}_2 -Gain). The incremental \mathcal{L}_2 -gain, i.e., \mathcal{L}_{i2} -gain, of the system Σ of the form (1) is

$$\|\Sigma\|_{\mathcal{L}_{i2}} := \sup_{0 < \|u - \tilde{u}\|_2 < \infty} \frac{\|y - \tilde{y}\|_2}{\|u - \tilde{u}\|_2}, \quad (42)$$

where $(x, u, y), (\tilde{x}, \tilde{u}, \tilde{y}) \in \mathfrak{B}_2$ are any two arbitrary trajectories of Σ for which $x(0) = \tilde{x}(0)$.

Remark 13 (\mathcal{L}_{i2} -Gain in the LTI Case). The \mathcal{L}_2 -gain and the \mathcal{L}_{i2} -gain are equivalent for LTI systems (Koelewijn & Tóth, 2019). Hence, the \mathcal{L}_2 -gain of a differential LTI system is equal to the \mathcal{L}_2 -gain of a primal LTI system.

The results in Fromion et al. (1999), Koelewijn et al. (2021) and van der Schaft (2017), together with Corollary 8 lead to the following result:

Corollary 14 (\mathcal{L}_{i2} -Gain Bound). Consider Σ as the system (1) and let $\gamma \in \mathbb{R}^+$. If there exists an $M = M^T > 0$ s.t. $\forall(\tilde{x}, \tilde{u}) \in \pi_{x,u}\mathcal{B}$,

$$\begin{pmatrix} A(\bar{\eta})^T M + MA(\bar{\eta}) & MB(\bar{\eta}) & C(\bar{\eta})^T \\ B(\bar{\eta})^T M & -\gamma^2 I & D(\bar{\eta})^T \\ C(\bar{\eta}) & D(\bar{\eta}) & -I \end{pmatrix} \preceq 0, \quad (43)$$

where $\bar{\eta} = \text{col}(\tilde{x}, \tilde{u})$, then $\|\Sigma\|_{\mathcal{L}_{i2}} \leq \gamma$.

Proof. The proof follows by combining Corollary 8 with the results in van der Schaft (2017). See also Verhoek, Koelewijn, Haesaert, and Tóth (2022). ■

In Fromion and Scorletti (2003), it is shown that $\|\Sigma_\delta\|_{\mathcal{L}_2} < \gamma \Leftrightarrow \|\Sigma\|_{\mathcal{L}_{i2}} < \gamma$. It is an interesting (open) question how necessity can also be established via Theorem 6 in this case. Additionally, note that (43) is linear, i.e., convex, in M and γ^2 , but it is an infinite semi-definite problem. We will discuss in Section 4.5 how to turn it into a finite number of linear matrix inequalities (LMIs)-based optimization problem.

4.2. Incremental \mathcal{L}_∞ -gain

The well-known \mathcal{L}_1 -norm is defined for stable LTI systems that map inputs with bounded amplitude to outputs with bounded amplitude. For LTI systems, the \mathcal{L}_1 -norm is equivalent with the induced \mathcal{L}_∞ -norm, i.e., the peak-to-peak gain of a system. We extend the notion of the \mathcal{L}_∞ -gain to the incremental setting, which characterizes the peak-to-peak gain between two arbitrary trajectories of a system. Let \mathfrak{B}_∞ be defined as $\mathfrak{B}_\infty := \{(x, u, y) \in \mathfrak{B} \mid u \in \mathcal{L}_\infty^{n_u}\}$.

Definition 15 (Incremental \mathcal{L}_∞ -Gain). The incremental \mathcal{L}_∞ -gain, i.e., $\mathcal{L}_{i\infty}$ -gain, of the system Σ of the form (1) is

$$\|\Sigma\|_{\mathcal{L}_{i\infty}} := \sup_{0 < \|u - \tilde{u}\|_\infty < \infty} \frac{\|y - \tilde{y}\|_\infty}{\|u - \tilde{u}\|_\infty}, \quad (44)$$

where $(x, u, y), (\tilde{x}, \tilde{u}, \tilde{y}) \in \mathfrak{B}_\infty$ are any two arbitrary trajectories of Σ for which $x(0) = \tilde{x}(0)$.

As an extension of Scherer (2000, Sec. 10.3) and Scherer and Weiland (2015, Sec. 3.3.5), the following result gives a sufficient condition for an upper bound γ of the $\mathcal{L}_{i\infty}$ -gain of a nonlinear system.

Corollary 16 ($\mathcal{L}_{i\infty}$ -Gain Bound). Consider Σ as the system (1) and let $\gamma \in \mathbb{R}^+$. If there exist an $M^T = M > 0$, $\kappa > 0$ and $\mu > 0$ such that $\forall(\tilde{x}, \tilde{u}) \in \pi_{x,u}\mathcal{B}$,

$$\begin{pmatrix} A(\bar{\eta})^T M + MA(\bar{\eta}) + \kappa M & MB(\bar{\eta}) \\ B(\bar{\eta})^T M & -\mu I \end{pmatrix} \prec 0, \quad (45a)$$

$$\begin{pmatrix} \kappa M & 0 & C(\bar{\eta})^T \\ 0 & (\gamma - \mu)I & D(\bar{\eta})^T \\ C(\bar{\eta}) & D(\bar{\eta}) & \gamma I \end{pmatrix} \succ 0, \quad (45b)$$

where $\bar{\eta} = \text{col}(\tilde{x}, \tilde{u})$, then $\|\Sigma\|_{\mathcal{L}_{i\infty}} < \gamma$.

Proof. The proof follows by combining Corollary 8 with the results in Scherer (2000) and Scherer and Weiland (2015). See also Verhoek, Koelewijn, Haesaert, and Tóth (2022). ■

Despite of the fact that (45a) is not convex in κ and M due to their multiplicative relation, by fixing κ and performing a line-search over it, (45a) again corresponds to an infinite Semi-Definite Program (SDP).

4.3. Incremental passivity

Passivity is a widely studied system property and it has been recently extended towards the incremental setting (Pavlov & Marconi, 2008; van der Schaft, 2017) and the differential setting (Forni & Sepulchre, 2013; Forni et al., 2013; van der Schaft, 2013). In Kawano, Kosaraju, and Scherpen (2020), the connection between differential and incremental passivity has been established for a storage function (10) with constant M . That work might serve as a parallel proof for Theorem 6, when focusing only on passivity.

A system is said to be passive if it is dissipative w.r.t. to the supply rate $\mathfrak{s}(u, y) = u^T y + y^T u$. Based on van der Schaft (2017), the definition of incremental passivity is as follows:

Definition 17 (Incremental Passivity). A system of the form (1) is incrementally passive, if for the supply

$$\mathfrak{s}_\Delta(u, \tilde{u}, y, \tilde{y}) = (u - \tilde{u})^T (y - \tilde{y}) + (y - \tilde{y})^T (u - \tilde{u}), \quad (46)$$

there exists a storage function $\mathcal{V}_\Delta : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_0^+$ s.t.

$$\mathcal{V}_\Delta(x(t_1), \tilde{x}(t_1)) - \mathcal{V}_\Delta(x(t_0), \tilde{x}(t_0)) \leq \int_{t_0}^{t_1} \mathfrak{s}_\Delta(u, \tilde{u}, y, \tilde{y}) dt,$$

for all $(x, u, y), (\tilde{x}, \tilde{u}, \tilde{y}) \in \mathfrak{B}$.

Based on Corollary 8, the following result holds:

Corollary 18 (Incremental Passivity Condition). The system (1) with $n_y = n_u$ is incrementally passive if there exists an $M^T = M > 0$ such that $\forall(\tilde{x}, \tilde{u}) \in \pi_{x,u}\mathcal{B}$

$$\begin{pmatrix} A(\bar{\eta})^T M + MA(\bar{\eta}) & MB(\bar{\eta}) - C(\bar{\eta})^T \\ B(\bar{\eta})^T M - C(\bar{\eta}) & -D(\bar{\eta}) - D(\bar{\eta})^T \end{pmatrix} \preceq 0, \quad (47)$$

where $\bar{\eta} = \text{col}(\tilde{x}, \tilde{u})$.

Proof. The proof follows by direct application of Corollary 8 with $Q = R = 0$ and $S = I$. ■

Comparing Corollary 18 to Kawano et al. (2020) and van der Schaft (2013), these papers give results on differential passivity for a combined primal and differential system formulation (a prolonged system Crouch & van der Schaft, 1987) using a specific form of storage function. The result depends on equality constraints, which serve as a decoupling condition between the differential storage and the primal storage, while in this paper the differential storage and the primal storage have the same structure (quadratic form with the same M), not requiring such equality constraints.

4.4. Generalized incremental \mathcal{H}_2 -norm

There are several extensions of the \mathcal{H}_2 -norm for nonlinear systems embedded as LPV systems (Bouali, Yagoubi, & Chevrel, 2008; de Souza, Trofino, & de Oliveira, 2003; Xie, 2005). In this paper, we extend the notion of the generalized \mathcal{H}_2 -norm to the incremental setting:

Definition 19 (Incremental Generalized \mathcal{H}_2 -Norm). Consider Σ as the system (1) with $\frac{\partial h}{\partial u} = 0$. The generalized incremental \mathcal{H}_2 -norm, i.e., \mathcal{H}_{i2}^g -norm, of Σ is

$$\|\Sigma\|_{\mathcal{H}_{i2}^g} := \sup_{0 < \|u - \tilde{u}\|_2 < \infty} \frac{\|y - \tilde{y}\|_\infty}{\|u - \tilde{u}\|_2}, \quad (48)$$

where $(x, u, y), (\tilde{x}, \tilde{u}, \tilde{y}) \in \mathfrak{B}_2$ are any two arbitrary trajectories of Σ for which $x(0) = \tilde{x}(0)$.

Note that if assumption $\frac{\partial h}{\partial u} = 0$ does not hold, then the \mathcal{H}_{i2}^g -norm is trivially unbounded. As an extension of Scherer and Weiland (2015, Sec. 3.3.4), the following result characterizes an upper bound γ on the \mathcal{H}_{i2}^g -norm.

Corollary 20 (\mathcal{H}_{i2}^g -Gain Bound). Consider Σ as the system (1) with $\frac{\partial h}{\partial u} = 0$ and let $\gamma \in \mathbb{R}^+$. If there exists an $M^T = M \succ 0$ such that $\forall(\bar{x}, \bar{u}) \in \pi_{x,u}\mathcal{B}$,

$$\begin{pmatrix} A(\bar{\eta})^T M + MA(\bar{\eta}) & MB(\bar{\eta}) \\ B(\bar{\eta})^T M & -\gamma I \end{pmatrix} \prec 0, \quad \begin{pmatrix} M & C(\bar{\eta})^T \\ C(\bar{\eta}) & \gamma I \end{pmatrix} \succ 0,$$

where $\bar{\eta} = \text{col}(\bar{x}, \bar{u})$, then $\|\Sigma\|_{\mathcal{H}_{i2}^g} < \gamma$.

Proof. The proof follows by combining Corollary 8 with the results in Scherer and Weiland (2015). See also Verhoek, Koelewijn, Haesaert, and Tóth (2022). ■

4.5. Convex computation with DPV inclusions

So far, the obtained results have yielded matrix inequalities that correspond to infinite dimensional SDPs. This section presents a convexification of the constraint variation to recast these problems as regular SDPs by embedding of the differential form of the system in a DPV inclusion. Inspired by Tóth (2010) and Wang, Tóth, and Manchester (2020), we define the DPV inclusion of (1) as follows.

Definition 21 (DPV Inclusion). The DPV inclusion of (1), given by

$$\Sigma_{\text{DPV}} : \begin{cases} \delta \dot{x}(t) = A(p(t))\delta x(t) + B(p(t))\delta u(t), \\ \delta y(t) = C(p(t))\delta x(t) + D(p(t))\delta u(t), \end{cases} \quad (49)$$

with $p(t) \in \mathcal{P}$ being the scheduling variable, is an embedding of the differential form of (1) on the compact convex region $\mathcal{P} \subset \mathbb{R}^{n_p}$, if there exists a function $\psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_p}$, the so-called scheduling map, such that $\forall(\bar{x}(t), \bar{u}(t)) \in \mathcal{X} \times \mathcal{U}$:

$$\begin{aligned} A(\psi(\bar{x}, \bar{u})) &= \frac{\partial f}{\partial x}(\bar{x}, \bar{u}), & B(\psi(\bar{x}, \bar{u})) &= \frac{\partial f}{\partial u}(\bar{x}, \bar{u}), \\ C(\psi(\bar{x}, \bar{u})) &= \frac{\partial h}{\partial x}(\bar{x}, \bar{u}), & D(\psi(\bar{x}, \bar{u})) &= \frac{\partial h}{\partial u}(\bar{x}, \bar{u}), \end{aligned}$$

where A, \dots, D belong to a given function class (affine, polynomial, etc.), implying that $p(t) = \psi(\bar{x}(t), \bar{u}(t))$, and $\mathcal{P} \supseteq \psi(\mathcal{X}, \mathcal{U})$.

The convex set \mathcal{P} is usually a superset of the ψ -projected values of possible state and input trajectories (even if \mathcal{X}, \mathcal{U} are convex), hence the DPV embedding of a nonlinear system introduces conservatism. However, this is considered to be the trade-off for efficiently computable stability and performance analysis of nonlinear systems. To reduce the conservatism of the DPV embedding (49) for a given preferred dependency class of

A, B, C, D (e.g., affine, polynomial, rational), we can optimize ψ (with minimal n_p) such that $\text{co}\{\psi(\mathcal{X}, \mathcal{U})\} \setminus \psi(\mathcal{X}, \mathcal{U})$ has minimal volume (Sadeghzadeh & Tóth, 2020; Tóth, 2010). Note that the DPV embedding serves as an important tool to convexify the variation of the matrix inequalities in the analysis. In turn, that allows to solve the derived infinite set of LMIs using a finite set of LMIs using SDP, e.g., via polytopic or multiplier based methods (Hoffmann & Werner, 2014).

In case that $\mathcal{X} \times \mathcal{U}$ is unbounded, the DPV embedding is often realized on a convex subset $\mathfrak{X} \times \mathfrak{U}$ of $\mathcal{X} \times \mathcal{U}$, such that there exists a compact and convex $\mathcal{P} \supseteq \psi(\mathfrak{X}, \mathfrak{U})$. In this case, one either requires to add an extra condition of invariance of the system on $\mathfrak{X} \times \mathfrak{U}$ or assume it, which may introduce conservatism in the analysis, as not the full behavior of the original primal system is considered. Note that existence of a compact and convex \mathcal{P} , in case of unbounded $\mathcal{X} \times \mathcal{U}$, follows when $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial u}$ are bounded matrix functions, e.g., if $\frac{\partial f}{\partial x} = \sin(x)$, with $x \in \mathbb{R}$, we can take $p = \psi(x) = \sin(x) \in [-1, 1]$.

5. Example

This section demonstrates the developed notions of incremental dissipativity theory³ and the analysis tools on an example. The extended version (Verhoek, Koelewijn, Haesaert, & Tóth, 2022) contains an additional example on the relation between incremental and general dissipativity.

Example 22. Consider a second-order Duffing oscillator given in a state-space form by

$$\begin{cases} \dot{x}_1(t) = x_2(t); \\ \dot{x}_2(t) = -ax_2(t) - (b + cx_1^2(t))x_1(t) + u(t); \\ y(t) = x_1(t), \end{cases} \quad (50)$$

where a and b represent the linear damping and stiffness, respectively, and c represents the nonlinear stiffness component. The differential form of (50) is given by

$$\begin{cases} \delta \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -b - 3cx_1^2(t) & -a \end{pmatrix} \delta x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta u(t); \\ \delta y(t) = (1 \quad 0) \delta x(t). \end{cases} \quad (51)$$

Moreover, we assume for this system that $(x_1, x_2) \in \mathcal{X}$, with $\mathcal{X} = \mathcal{X}^{\mathbb{R}}$, where $\mathcal{X} = [-\sqrt{2}, \sqrt{2}] \times \mathbb{R}$, and

$$u \in \mathcal{U} := \mathcal{L}_2 \cap \left\{ \mathbb{R}^{\mathbb{R}} \mid (50) \text{ holds and } (x_1, x_2) \in \mathcal{X} \right\}.$$

By choosing $a = 3.3, b = 7.9, c = 1$, (50) yields a system with finite \mathcal{L}_{i2} -gain. In this example, we determine the \mathcal{L}_{i2} -gain of the system, using Corollary 14. Note that the nonlinearity $x_1^2(t)$ in (51) can be captured by using a DPV inclusion $p(t) = \psi(x_1(t)) = x_1^2(t) \in [0, 2]$. By this substitution, (43) becomes a matrix inequality linear in p , which can be reduced to a finite number of LMI constraints at the vertices, due to convexity of $[0, 2]$. Solving the resulting SDP (constrained minimization of γ) yields $M = \begin{pmatrix} 0.592 & 0.0896 \\ 0.0896 & 0.0543 \end{pmatrix} \succ 0$ and $\gamma = 0.155$. Hence, within less than a second, we know that the nonlinear system is differentially, incrementally and generally dissipative on \mathcal{X} w.r.t. the supply function (11) with $Q = 0.155^2, R = -1$ and $S = 0$, and that it has an \mathcal{L}_{i2} -gain less than 0.155. The system is simulated with two different input signals, given in (52), for which we know they are in \mathcal{U} .

$$u_1(t) = 3e^{-0.2t} \cos(\pi t) \mathbb{1}(t), \quad (52a)$$

³ There are some works under review that apply the developed theory in Section 3 on a practical example.

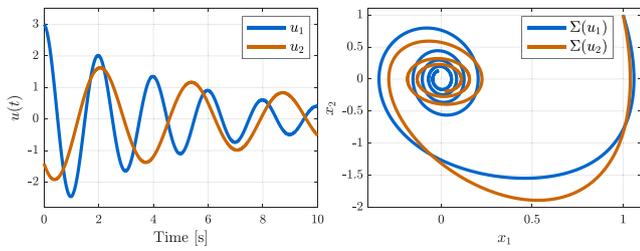
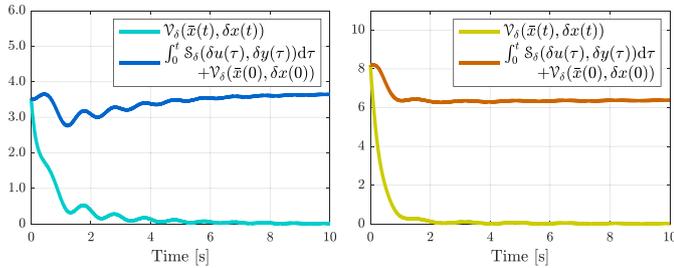
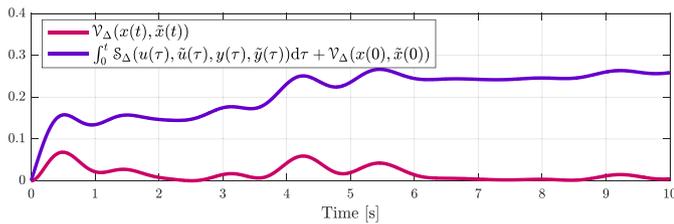


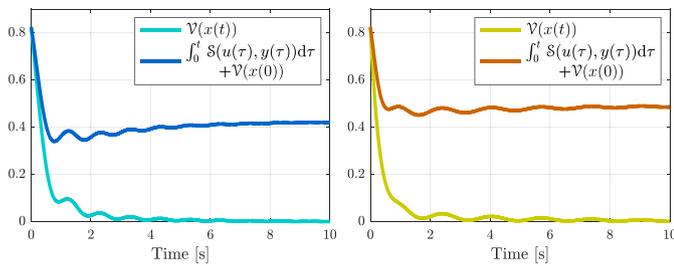
Fig. 3. The applied inputs to the system (left) and the resulting state trajectories (right). Both trajectories start at $(x_1(0), x_2(0)) = (1, 1)$.



(a) Differential dissipativity of the system trajectories with $u_1(t)$ as input (left) and $u_2(t)$ as input (right).



(b) Incremental dissipativity based on the system trajectories with $u_1(t)$ and $u_2(t)$ as input.



(c) General dissipativity of the system trajectories with $u_1(t)$ as input (left) and $u_2(t)$ as input (right).

Fig. 4. Simulation results for the different notions of dissipativity for a Duffing oscillator w.r.t. the supply function $s(u, y) = \gamma^2 \|u\|^2 - \|y\|^2$, corresponding to the (incremental) \mathcal{L}_2 -gain.

$$u_2(t) = -2e^{-0.1t} \sin\left(0.6\pi t + \frac{\pi}{4}\right) \mathbb{1}(t), \quad (52b)$$

where $\mathbb{1}(t)$ is the unit step-function. The inputs and the state trajectories are shown in Fig. 3, which shows that the states stay within the defined state-space \mathcal{X} .

To verify whether the system is differentially dissipative, considering these specific trajectories, the signals of (51) are substituted in the DI for the differential form (9). The left- and right-hand side of the DI (9) are plotted in Fig. 4(a) corresponding to the system trajectories of Fig. 3. As can be seen in Fig. 4(a), the stored energy in the system is always less than the supplied energy plus the initial stored energy, hence the system is differentially dissipative w.r.t. the considered \mathcal{L}_2 -gain supply.

Since the system is differentially dissipative it is also incrementally dissipative. Fig. 4(b) shows the incremental dissipation inequality, i.e., the stored energy and the supplied energy between the two trajectories in Fig. 3. As can be observed in Fig. 4(b), the stored energy between two trajectories is always less than the supplied energy between two trajectories. Hence, considering these trajectories, the system is incrementally dissipative. Therefore, we can state (based on these two trajectories) that these results correspond to the developed theory.

Moreover, by Theorem 10, incremental dissipativity implies general dissipativity of the original system (50). Fig. 4(c) gives the storage and supply function evolution over time for the two considered trajectories, showing that the original system is dissipative, since the stored energy is always less than the supplied energy. ◀

6. Conclusions

In this paper, we established the link between general dissipation theory, incremental dissipativity analysis and differential dissipativity analysis for nonlinear systems. Moreover, we have given results on general quadratic incremental performance notions and parameter-varying inclusion based computation tools to analyze the different notions of dissipativity in a convex setting by SDPs. The established link gives us a generic framework to analyze stability and performance of a nonlinear system from a global perspective. Finally, the presented computation tools allow to efficiently analyze global stability and performance of a rather general class of nonlinear systems. These results open up the possibility to establish controller synthesis based on PV inclusions of the differential form, such that we can synthesize (nonlinear) controllers for nonlinear systems with incremental stability and performance guarantees of the closed-loop behavior. For future work, we aim to extend the developed theory for discrete-time and time-varying nonlinear systems.

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Appendix. Norm integral inequality

From Koelewijn and Tóth (2021, Lem. 16), we have the following useful lemma.

Lemma 23. For $0 < M = M^T \in \mathbb{R}^{n \times n}$ and a continuous function $\phi : [0, 1] \rightarrow \mathbb{R}^n$, it holds that

$$\left(\int_0^1 \phi(t) dt\right)^T M \left(\int_0^1 \phi(t) dt\right) \leq \int_0^1 \phi^T(t) M \phi(t) dt. \quad (A.1)$$

References

Angeli, D. (2002). A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*.

Bouali, A., Yagoubi, M., & Chevrel, P. (2008). \mathcal{H}_2 gain scheduling control for rational LPV systems using the descriptor framework. In *Proc. of the 47th IEEE conference on decision and control*.

Crouch, P. E., & van der Schaft, A. J. (1987). *Variational and Hamiltonian Control Systems*. Berlin: Springer-Verlag.

Forni, F., & Sepulchre, R. (2013). On differentially dissipative dynamical systems. In *Proc. of the 9th IFAC symposium on nonlinear control systems*.

Forni, F., Sepulchre, R., & van der Schaft, A. J. (2013). On differential passivity of physical systems. In *Proc. of the 52nd IEEE conference on decision and control*.

Fromion, V., Monaco, S., & Normand-Cyrot, D. (2001). The weighted incremental norm approach: From linear to nonlinear \mathcal{H}_∞ control. *Automatica*.

Fromion, V., & Scorletti, G. (2003). A theoretical framework for gain scheduling. *International Journal of Robust and Nonlinear Control*.

Fromion, V., Scorletti, G., & Ferreres, G. (1999). Nonlinear performance of a PI controlled missile: An explanation. *International Journal of Robust and Nonlinear Control*.

Hill, D. J., & Moylan, P. J. (1980). Dissipative dynamical systems: Basic input-output and state properties. *Journal of the Franklin Institute*.

Hoffmann, C., & Werner, H. (2014). A survey of linear parameter-varying control applications validated by experiments or high-fidelity simulations. *IEEE Transactions on Control Systems Technology*.

Kawano, Y., Kosaraju, K. C., & Scherpen, J. M. A. (2020). Krasovskii and shifted passivity based control. *IEEE Transactions on Automatic Control*.

Khalil, H. K. (2002). *Nonlinear Systems* (3rd ed.). Upper Saddle River, NJ, USA: Prentice Hall.

Koelewijn, P. J. W., Sales Mazzoccante, G., Tóth, R., & Weiland, S. (2020). Pitfalls of guaranteeing asymptotic stability in LPV control of nonlinear systems. In *Proc. of the European control conference Saint Petersburg*.

Koelewijn, P. J. W., & Tóth, R. (2019). Incremental gain of LTI systems. *Technical Report TUE CS*, Eindhoven University of Technology.

Koelewijn, P. J. W., & Tóth, R. (2021). Incremental stability and performance analysis of discrete-time nonlinear systems using the LPV framework. In *Proc. of the 4th IFAC workshop on linear parameter-varying systems*.

Koelewijn, P. J. W., Tóth, R., Nijmeijer, H., & Weiland, S. (2021). Nonlinear tracking and rejection using linear parameter-varying control. arXiv preprint arXiv:2104.09938.

Liu, T., Hill, D. J., & Zhao, J. (2014). Incremental-dissipativity-based output synchronization of dynamical networks with switching topology. In *Proc. of the 53rd IEEE conference on decision and control*. IEEE.

Lohmiller, W., & Slotine, J.-J. E. (1998). On contraction analysis for non-linear systems. *Automatica*.

Manchester, I. R., & Slotine, J.-J. E. (2018). Robust control contraction metrics: A convex approach to nonlinear state-feedback H^∞ control. *IEEE Control Systems Letters*.

Pavlov, A., & Marconi, L. (2008). Incremental passivity and output regulation. *Systems & Control Letters*.

Pavlov, A., Pogromsky, A., van de Wouw, N., & Nijmeijer, H. (2004). Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Systems & Control Letters*.

Reyes-Báez, R. (2019). *Virtual contraction and passivity based control of nonlinear mechanical systems* (Ph.D. thesis), Groningen, The Netherlands: University of Groningen.

Rüffer, B. S., van de Wouw, N., & Mueller, M. (2013). Convergent systems vs. incremental stability. *Systems & Control Letters*.

Sadeghzadeh, A., & Tóth, R. (2020). Linear parameter-varying embedding of nonlinear models with reduced conservativeness. In *Proc. of the 21st IFAC world congress*. Berlin.

van der Schaft, A. J. (2013). On differential passivity. In *Proc. of the 9th IFAC symposium on nonlinear control systems*.

van der Schaft, A. J. (2017). *\mathcal{L}_2 -gain and passivity techniques in nonlinear control* (3rd ed.). Cham, Switzerland: Springer International Publishing AG.

Scherer, C. W. (2000). Robust mixed control and linear parameter-varying control with full block scalings. In *Advances in linear matrix inequality methods in control* (pp. 187–207). SIAM.

Scherer, C. W., & Weiland, S. (2015). *Linear matrix inequalities in control*. <https://www.imng.uni-stuttgart.de/mst/files/LectureNotes.pdf>.

Simpson-Porco, J. W. (2019). Equilibrium-independent dissipativity with quadratic supply rates. *IEEE Transactions on Automatic Control*.

Skogestad, S., & Postlethwaite, I. (2005). *Multivariable feedback control* (2nd ed.). John Wiley & Sons Ltd.

de Souza, C. E., Trofino, A., & de Oliveira, J. (2003). Parametric Lyapunov function approach to \mathcal{H}_2 analysis and control of linear parameter-dependent systems. *IEE Proceedings D (Control Theory and Applications)*.

Tóth, R. (2010). *Modeling and identification of linear parameter-varying systems* (1st ed.). Springer-Verlag.

Verhoek, C., Koelewijn, P. J. W., Haesaert, S., & Tóth, R. (2022). Convex incremental dissipativity analysis of nonlinear systems (extended version). arXiv preprint arXiv:2006.14201.

Verhoek, C., Koelewijn, P. J. W., Tóth, R., & Haesaert, S. (2022). Decomposition of a positive definite matrix function that is continuously differentiable. *Technical Report TUE CS*, Eindhoven University of Technology.

Waitman, S., Massioni, P., Bako, L., Scorletti, G., & Fromion, V. (2016a). Incremental analysis of nonlinear systems with efficient methods for piecewise-affine systems. arXiv preprint arXiv:1611.08322.

Waitman, S., Massioni, P., Bako, L., Scorletti, G., & Fromion, V. (2016b). Incremental \mathcal{L}_2 -gain analysis of piecewise-affine systems using piecewise quadratic storage functions. In *Proc. of the 55th conference on decision and control*.

Wang, R., Tóth, R., & Manchester, I. R. (2020). Virtual Control contraction metrics: Convex nonlinear feedback design via behavioral embedding. arXiv preprint arXiv:2003.08513.

Willems, J. C. (1972). Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis*.

Xie, W. (2005). \mathcal{H}_2 gain scheduled state feedback for LPV system with new LMI formulation. *IEE Proceedings D (Control Theory and Applications)*.



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