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# Lyapunov function computation for autonomous systems with complex dynamic behavior

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ABSTRACT

A computational approach is presented in this paper to construct local Lyapunov functions for autonomous dynamical systems with multiple isolated locally asymptotically stable (such as point-like, periodic, or strange) attractors. We consider systems of nonlinear ODEs, where the right-hand-side of the dynamic equations is given in the form of rational functions (i.e., fraction of polynomials). The Lyapunov function is searched in a parameterized quadratic form of rational terms of the state variables. The quadratic decomposition of the rational state-dependent inequalities is performed using the linear fractional transformation (LFT) and further algebraic/numeric simplification steps. Unlike the sum of squares (SOS) approach, the sufficient linear matrix inequality (LMI) conditions for the Lyapunov function are formulated only locally on a compact polytopic subset of the state space, which allows indefinite matrix solutions for the quadratic decomposition. The local solution is enforced using affine annihilators with matrix Lagrange multipliers. Alongside the typical Lyapunov conditions, further boundary LMI constraints are prescribed using Finsler's lemma to ensure the required geometric properties of the Lyapunov function. The results are illustrated on four planar benchmark models having either multiple locally stable equlibria or a limit cycle, and on the Lorenz system.



## 1. Introduction

Finding or at least estimating the domain of attraction (DOA) of an attractive equilibrium solution of an autonomous model is still a relevant topic in the field of nonlinear systems [60]. The stability analysis of a dynamical model often involves determining an appropriate Lyapunov function [8], which gives qualitative information (e.g., bounds [15]) on the dynamic behavior of the system. Then, a forward invariant domain for a local attractor can be given as an appropriate level set of the Lyapunov function.

In the literature, numerous *DOA approximation* results are available, which are based on converse Lyapunov theorems [24]. An iterative method is presented in Rozgonyi et al. [50], [61] to approximate a so-called maximal (rational) Lyapunov function, which is closely related to Zubov's approach [63]. Another group of results [2–5,12,19] are based on the numerical approximation of Massera's construction [32], where the Lyapunov function is searched as an integral of an initially given function of the system's solution. These approaches generally involve discretization of both the state space (spatial) and the dynamics (temporal).

The optimization-based approaches usually consider an initially structured parameterized Lyapunov function candidate and formulate convex (sufficient) Lyapunov-type certificates as linear matrix inequalities (LMIs) to estimate the robust DOA of uncertain polynomial/rational systems [37,39,44,46,47,58–60]. Therefore, these approaches result in a Lyapunov function in a closed algebraic form.

However, the research on computing Lyapunov functions generally focuses on dynamical systems with a single locally stable equilibrium point at the origin. Therefore, the nonzero isolated equilibrium points are generally translated to the origin to perform DOA computation. Often, the dynamic equations in the new coordinates have a symbolically more complicated form, which result in a numerically less tractable model. Therefore, it is motivating to compute Lyapunov functions in the original system of coordinates. In [2,4], important numerical approximation results are available on the Lyapunov function construction for dynamical models with

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multiple and not necessarily point-like local attractors, e.g., limit cycles.

It is worth remarking that the stability of a limit cycle is often analysed through the linearized model of the transverse dynamics [20,30,31,53,54,56]. In [31,54], e.g., the authors proposed an SOS solution to estimate the DOA of fixed points of discrete PoincarĘ maps.

The theory of Lyapunov functions for systems with multiple (not necessarily point-like) local attractors is well-founded in Khalil [25, Chap. 4], in book [16], or in Björnsson et al. [2], Björnsson et al. [3], Björnsson and Hafstein [4]. The authors of Björnsson et al. [2], introduced relaxed Lyapunov criteria to prove local stability and compute a forward invariant set for a local attractor (possibly a limit cycle).

Finally, a class of important and closely related results can be found in Fantuzzi et al. [15], Goluskin [17,18], Jones and Peet [23], Tobasco et al. [57]. In these publications, closed-form functions are computed to find minimal attracting sets for autonomous systems with different, possibly multiple attractors. The computed functions determine lower or upper bounds on given state-dependent quantities such as polynomials of the state variables. These approaches are based on the SOS methodology, and are similar to the Lyapunov techniques [39,47,58], but the computed functions do not necessarily have the properties that define a Lyapunov function, such as positive definiteness, polynomial growth, or a negative Lie derivative in the whole state-space.

From a methodological point of view, the SOS solutions rely on the Positivstellensatz [55] and a *polynomial* SOS decomposition with a positive semidefinite matrix, called the Gramian [27,38]. Whereas, the polytopic (also called the *slack-variable* [13]) approaches [41,43,44,46,59] consider *rational* decompositions with a matrix that is not required to be positive semidefinite, but satisfy an affine state-dependent LMI over a bounded polytope. The authors of Chesi [7], De Madeira and Adamy [11], Sato and Peaucelle [51,52], Trofino and Dezuo [59] observed that the polytopic constraint augmented with slack variables generates a less conservative solution set compared to the corresponding SOS constraint. In [41,43,44,46], the rational quadratic decomposition is automated using the LFT [14] and further dimension reduction steps.

In this paper, we follow the polytopic slack-variable LMI solutions, and compute local Lyapunov functions to prove stability and determine invariant domains for systems with (possibly) multiple locally asymptotically stable attractors. Unlike the majority of the Lyapunov function computation results [37,39,44,46,47,58,59], we consider the generalized Lyapunov stability theory of Björnsson et al. [2], which provides strong stability properties, but only outside of a close neighborhood of the attractor.

The paper is organized as follows. After a brief subsection on the notations, we introduce the known definitions and results in Section 2. Then, we summarize the computational framework developed in Polcz [41], Polcz et al. [43,44,46]. The main contributions of this paper are presented in Sections 4 and 5. In Section 6, we demonstrate the operations through four planar models and the Lorenz system.

### 1.1. Notations, abbreviations

Let $\mathbb{R}_+ = [0, \infty)$. We use $\|x\|$ to refer to the Euclidean norm of vector $x \in \mathbb{R}^n$. The $m \times m$ identity matrix is denoted by $I_m$, whereas, $0_{n\times m}$ is the $n \times m$ zero matrix, and

$$\mathbf{1}_m = \begin{pmatrix} 1 & 0_{1\times(m-1)} \\ 0_{(m-1)\times 1} & 0_{(m-1)\times(m-1)} \end{pmatrix}. \tag{1}$$

We say that matrix $A \in \mathbb{R}^{n\times m}$ is *"narrow"* if $n > m$, and *"fat"* if $n < m$. Let $\mathrm{He}\{A\}$ denote $A^\top + A$, where $A$ is a real-valued square matrix and $A^\top$ is its transpose. Relations $P \succeq 0$ and $P \preceq 0$ indicate that symmetric matrix $P$ is positive and negative semidefinite, respectively.

Sets $\mathcal{X}^\circ$ and $\overline{\mathcal{X}}$ denote the interior and the closure of a set $\mathcal{X} \subset \mathbb{R}^n$, respectively. The boundary of a set $\mathcal{X}$ is referred to as $\partial\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{X}^\circ$. Let $\mathfrak{N}(M)$ denote the set of all compact subsets $\mathcal{D} \subset \mathbb{R}^n$ such that $\mathcal{D}^\circ$ is a connected open neighborhood of the compact set $M \subset \mathbb{R}^n$. We use $\mathbf{Ve}(\mathcal{X})$ to refer to the set of vertices of a compact polytope $\mathcal{X}$, whereas, $\mathbf{Co}(A)$ denotes the convex hull of a set $A$.

Let $L_f V$ denote the Lie derivative of function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to (w.r.t) function $f : \mathbb{R}^n \to \mathbb{R}^n$, namely, $L_f V(x) = \frac{\partial V}{x}(x) f(x)$, where $\frac{\partial V}{x} : \mathbb{R}^n \to \mathbb{R}^{1\times n}$ is the gradient *row* function of $V$. In a similar way, let $L_f \pi(x)$ denote $\frac{\partial \pi}{x}(x) f(x)$, where $\frac{\partial \pi}{x} : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is the Jacobian of $\pi : \mathbb{R}^n \to \mathbb{R}^m$. The gradient *column* function of $V$ is denoted by $\nabla V : \mathbb{R}^n \to \mathbb{R}^n$. Let $\Omega_c = \{x \in \mathcal{X} \,|\, V(x) \le c\}$ denote the $c$-level set of function $V : \mathcal{X} \to \mathbb{R}$, and let $\Omega_{c,M}$ denote the connected subset (if exists) of $\Omega_c$ that contains set $M \subset \mathcal{X}$.

## 2. Background

We consider nonlinear (rational) autonomous dynamical systems of the form

$$\dot{x} = f(x), \qquad x_0 \in \mathcal{X} \tag{2}$$

where $x : [0, \infty) \to \mathbb{R}^n$ is the state, $x_0$ is the initial condition, and $\mathcal{X} \subset \mathbb{R}^n$ is a compact polytope given a priori. Function $f : \mathcal{X} \to \mathbb{R}^n$ is a fraction of polynomials (i.e., rational) in $x$, and it can be given as follows:

$$f(x) = f_0 + \textstyle\sum_{j=1}^{J} \frac{q_{1j}(x)}{q_{2j}(x)} f_j, \tag{3}$$

where $f_0, f_j \in \mathbb{R}^n$ are constant vectors and $q_{1j}, q_{2j} : \mathbb{R}^n \to \mathbb{R}$ are polynomials.

**Assumption 1.** We assume that $q_{2j}(x) \ge \varepsilon$ for all $x \in \mathcal{X}$, all $j = 1, \dots, J$, and some $\varepsilon > 0$.

Note that function $f$ in (3) with Assumption 1 is Lipschitz continuous in $\mathcal{X}$ and differentiable in $\mathcal{X}^\circ$.

According to El Ghaoui and Scorletti [14, Lemma 2.1], any rational matrix function $G : x \mapsto G(x)$ with no singularities at the origin admits the following (lower) *linear fractional representation* (LFR) form:

$$\begin{aligned} G(x) &= \mathcal{F}_l\left\{ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \Delta(x) = \begin{pmatrix} I_{r_1} x_1 & & \\ & \dots & \\ & & I_{r_n} x_n \end{pmatrix} \right\} \\ &= M_{11} + M_{12}(I - \Delta(x) M_{22})^{-1} \Delta(x) M_{21}, \end{aligned} \tag{4}$$

where $M_{ij}$ are constant matrices, and $r = r_1 + \ldots + r_n$ is called the order of the LFR. Moreover, $G$ admits a so-called *well-posed* LFR in $\mathcal{X}$ (i.e., $I - \Delta(x) M_{22}$ is invertible for all $x \in \mathcal{X}$) if $G$ is well-defined in $\mathcal{X}$ [29]. The factorized form (4) of function $G$ is computed by the LFT.

In the followings, we introduce standard definitions and theorems on stability of nonlinear systems based on the results of Goebel et al. [16],[25],[30]. Let $t \mapsto \Phi(t, x_0)$ denote the solution of system (2) if the initial value at time 0 is $x_0$. Note that $\Phi(0, x_0) = x_0$.

**Definition 1** ([25]). A set $M$ is said to be *invariant* w.r.t (2) if $\xi \in M$ implies $\Phi(t, \xi) \in M$ for all $t \in \mathbb{R}$. A set $\mathcal{D}$ is said to be *forward invariant* w.r.t (2) if $x_0 \in \mathcal{D}$ implies $\Phi(t, x_0) \in \mathcal{D}$ for all $t \ge 0$.

In the following definition, we use the notion of the distance of $x_0 \in \mathbb{R}^n$ to a set $M$ in the usual way [16,30], namely:

$$\mathrm{dist}(x_0, M) = \inf_{x \in M} \|x_0 - x\|. \tag{5}$$

**Definition 2** ([16,30])**.** Let $M$ be a forward invariant set w.r.t (2). We say that $M$ is locally stable if for all $\varepsilon \in (0, \bar{\varepsilon})$, there exists $\delta > 0$ such that, for all $x_0$ satisfying

$$\operatorname{dist}(x_0, M) \le \delta, \tag{6}$$

the solution $t \mapsto \Phi(t, x_0)$ for $t \ge 0$ exists, is unique, and

$$\operatorname{dist}(\Phi(t, x_0), M) < \varepsilon \text{ for all } t \ge 0. \tag{7}$$

We say that $M$ is locally attractive, i.e., it is a local attractor, if there exists a connected neighborhood $\mathcal{D} \in \mathfrak{N}(M)$ such that, for all $x_0 \in \mathcal{D}$, the solution $t \mapsto \Phi(t, x_0)$ for $t \ge 0$ exists, is unique, and

$$\lim_{t\to\infty} \operatorname{dist}(\Phi(t, x_0), M) = 0 \tag{8}$$

We say that $M$ is locally asymptotically stable if it is locally stable and locally attractive.

**Definition 3.** The set $B(M)$ of all initial conditions $x_0$ satisfying (8), is called the *domain of attraction* (DOA) of attractor $M$.

The following theorem formulates classical stability properties for autonomous nonlinear systems.

**Theorem 4** (LaSalle's theorem Khalil [25, Thm. 4.4])**.** *Let $\Omega \subset \mathcal{X}$ be a compact set that is forward invariant w.r.t (2). Let $V : \mathcal{X} \to \mathbb{R}$ be a continuously differentiable function such that $L_f V(x) \le 0$ in $\Omega$. Let $E$ be the set of all points $x \in \Omega$ where $L_f V(x) = 0$. Let $M$ be the largest invariant set in $E$. Then, every solution starting in $\Omega$ approaches $M$ as $t \to \infty$.*

**Remark 1** (relations between the used notions)**.** When $M$ is an invariant set w.r.t (2), any solution starting from $M$ remains in $M$ even if we go either *forward* or *backward* in time. When $M$ is a local *attractor*, it admits a domain of attraction $B(M)$, satisfying $M \subset B(M)^\circ$, which is *not* necessarily *invariant* but *forward invariant* w.r.t (2). An attractor $M$ is not necessarily invariant either, but *forward invariant* as the limit of the solution in (8) should approach $M$ when the time is going *forward*. An attractor is not necessarily locally stable, because it may happen that a solution starting arbitrarily close to $M$ moves away from $M$ and then returns and approaches $M$ as $t \to \infty$. According to Definition 2, an attractor is a *locally attractive forward invariant* set w.r.t (2).

**Definition 5.** We call $M$ a *stable attractor* if $M$ is locally asymptotically stable w.r.t (2). We call $M$ an *invariant attractor* if attractor $M$ is invariant w.r.t. (2). We call $M$ a *stable invariant attractor* if it is both stable and invariant.

**Remark 2** (examples)**.** A set of equilibria $M = \{\}x_*^{(1)}, x_*^{(2)}, \ldots,$ or a periodic orbit $M = \{\Phi(t, \xi) = \Phi(t + T, \xi) \,|\, t \ge 0\}$ of period $T$ are both *invariant* w.r.t (2). We call a locally attractive periodic orbit $M$ a *limit cycle*.

### *2.1. Recent results on the stability of nonlinear autonomous systems*

In this section, we present a few additional recent results in the Lyapunov theory for nonlinear systems with complex dynamic behavior. These results will be useful when we compute a Lyapunov function for a limit cycle or a strange attractor.

**Definition 6** (Björnsson et al. [2, Def. 2.2])**.** Let $M$ be a compact set, $\mathcal{X}, \mathcal{Y} \in \mathfrak{N}(M)$, and $\mathcal{Y} \subset \mathcal{X}^\circ$. Let $V : \mathcal{X} \setminus \mathcal{Y}^\circ \to \mathbb{R}_+$ be a continuously differentiable function and let $c > 0$ be a constant. Define the set

$$\Psi_c = \mathcal{Y} \cup \{x \in \mathcal{X} \setminus \mathcal{Y}^\circ \,|\, V(x) < c\} \subset \mathcal{X}. \tag{9}$$

Denote by $\Psi_{c,M}$ the connected subset of $\Psi_c$ satisfying $M \subset \Psi_{c,M} \subset \Psi_c$. Let

$$\mathcal{L}_{c,M} = \begin{cases} \Psi_{c,M} & \text{if } \mathcal{Y} \subset \Psi_{c,M}^\circ \subset \overline{\Psi_{c,M}} \subset \mathcal{X}^\circ, \\ \emptyset & \text{if no such } \Psi_{c,M} \text{ exists.} \end{cases} \tag{10}$$

Furthermore, we define

$$\mathcal{L}_M^{\inf} = \bigcap_{\substack{c>0 \\ \mathcal{L}_{c,M} \neq \emptyset}} \mathcal{L}_{c,M} \quad \text{and} \quad \mathcal{L}_M^{\sup} = \bigcup_{c>0} \mathcal{L}_{c,M}. \tag{11}$$

**Definition 7** (Björnsson et al. [2, Def. 2.3])**.** Let $\mathcal{X}, \mathcal{Y} \in \mathfrak{N}(M)$ satisfy $\mathcal{Y} \subset \mathcal{X}^\circ$. Let $\mathcal{G} \subset \mathbb{R}^n$ satisfy $\mathcal{X} \setminus \mathcal{Y}^\circ \subseteq \mathcal{G}$. A continuously differentiable function $V : \mathcal{G} \to \mathbb{R}_+$ is called a Lyapunov function for $M$ on $\mathcal{X} \setminus \mathcal{Y}^\circ$ for (2) if there exists a constant $\alpha > 0$ such that

(L1) $V(x) > 0$ for all $x \in \mathcal{X} \setminus \mathcal{Y}^\circ$,
(L2) $L_f V(x) \le -\alpha$ for all $x \in \mathcal{X}^\circ \setminus \mathcal{Y}$,
(L3) $\mathcal{L}_M^{\inf} \neq \emptyset$.

Note that Definition 7 prescribes the Lyapunov conditions (L1) and (L2) only on $\mathcal{X} \setminus \mathcal{Y}^\circ$, where $\mathcal{Y}$ is ideally a tight outer estimation of the invariant set $M$. This relaxation allows the Lyapunov function to vary along the invariant set $M$. This flexibility is necessary when, e.g., $M$ is a limit cycle, but a closed form of the limit cycle is not known or it does not exist at all. In this case, we cannot expect a *closed-form* Lyapunov function to take the same value along the cycle.

The following theorem formulates strong stability properties for system (2) with a Lyapunov function on $\mathcal{X} \setminus \mathcal{Y}^\circ$.

**Theorem 8** (Björnsson et al. [2, Thm. 2.5])**.** *Consider $\mathcal{X}, \mathcal{Y} \in \mathfrak{N}(M)$, $\mathcal{Y} \subset \mathcal{X}^\circ$, and let $V$ be a Lyapunov function for $M$ on $\mathcal{X} \setminus \mathcal{Y}^\circ$. Let $c > 0$ be a constant such that $\mathcal{L}_{c,M} \neq \emptyset$. Then,*

T1 *$\mathcal{L}_{c,M}$, $\mathcal{L}_M^{\inf}$, and $\mathcal{L}_M^{\sup}$ are forward invariant sets,*
T2 *there is a constant $T > 0$ such that $x_0 \in \mathcal{L}_M^{\sup}$ implies $\Phi(T, x_0) \in \mathcal{L}_M^{\inf}$,*
T3 *$\mathcal{L}_M^{\inf} = \overline{\Psi_{a,M}}$ and $\mathcal{L}_M^{\sup} = \Psi_{b,M}$ where*

$$a = \max_{x\in\partial\mathcal{Y}} V(x) \text{ and } b = \sup\{c > 0 \,|\, \mathcal{L}_{c,M} \neq \emptyset\}.$$

## 3. Polytopic LMI approach for rational parameter-dependent constraints

In this section, we summarize our computational framework, first, inspired by Trofino and Dezuo [59], then, developed in Polcz et al. [43,44,46], finally, presented in details in Polcz [41, Chap. 5]. Though new results are not proposed in this section, all presented techniques are used in the forthcoming sections.

Consider a scalar inequality in the following form:

$$W(x) = \varphi^\top(x)\, Q\, \varphi(x) \ge 0 \text{ for all } x \in \mathcal{M}, \tag{12}$$

where $\varphi : \mathcal{M} \to \mathbb{R}^m$ is a fixed Lipschitz continuous *rational* function of the state, $\mathcal{M} \subset \mathbb{R}^n$ is a fixed compact *polytope* in $\mathcal{R}$, and $\mathcal{R} \subseteq \mathbb{R}^n$ is an $(n-1)$-dimensional affine submanifold (i.e., hyperplane) or $\mathbb{R}^n$ itself. In the further notations, we assume that the first coordinate of $\varphi(x)$ is 1. The constant entry in $\varphi(x)$ is used to allow checking the non-negativity of functions, that are not necessarily zero in the origin.

We are looking for a matrix $Q = Q^\top \in \mathbb{R}^{m\times m}$, that solves inequality (12) for all $x \in \mathcal{M}$.

Observe that a positive semidefinite matrix $Q$ is a possible solution for (12). In this case, function $W$ is a sum of squares, and it is non-negative for all $x \in \mathbb{R}^n$. However, the authors of Trofino and Dezuo [59] remarked that, it may be rather conservative to prescribe the SOS property for $W$ when its non-negativity should be provided *only* on a bounded set $\mathcal{M}$.

In the following corollary, we present a consequence of the parameter-dependent form Trofino and Dezuo [59] of Finsler's lemma [36], which formulates a still sufficient, but less conservative condition for (12).

**Corollary 1.** *Consider a constant matrix $S \in \mathbb{R}^{m\times m'}$, a rational function $\widehat{\varphi} : \mathbb{R}^n \to \mathbb{R}^{m'}$, and an affine function $N : \mathbb{R}^n \to \mathbb{R}^{s\times m'}$, such that*

1. *$S$ and $\widehat{\varphi}$ determine a **factorization** of $\varphi$ in $\mathcal{M}$, namely,*

$$\varphi(x) = S\widehat{\varphi}(x) \text{ for all } x \in \mathcal{M}, \tag{13}$$

2. *$N$ is an **annihilator** of $\widehat{\varphi}$ in $\mathcal{M}$, namely,*

$$N(x)\,\widehat{\varphi}(x) = 0 \text{ for all } x \in \mathcal{M}. \tag{14}$$

*Assume that there exists $\Lambda \in \mathbb{R}^{m' \times s}$, and $Q = Q^\top \in \mathbb{R}^{m\times m}$ such that*

$$S^\top Q\,S + \mathrm{He}\{\Lambda N(x)\} \succeq 0 \text{ for all } x \in \mathbf{Ve}(\mathcal{M}). \tag{15}$$

*Then, $Q$ solves (12).*

Observe that pre- and post-multiplying (15) by $\widehat{\varphi}^\top(x)$ and $\widehat{\varphi}(x)$ respectively, we get back inequality (12).

**Remark 3.** *The role of annihilator $N$* in (15) is to represent the algebraic interdependence between the coordinate functions of $\widehat{\varphi}$ (and hence of $\varphi$). Consider $\varphi(x) = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$ as an example with $S = I_3$. Though the nonlinear terms of $\varphi$ are eliminated from (15), some information about its structure is encoded in $N(x) = \begin{pmatrix} x & -1 & 0 \\ 0 & x & -1 \end{pmatrix}$. Note that other references call $\Lambda$ a slack [7] or scaling variable [59].

**Remark 4.** The factorization of $\varphi$ in (13) may have different roles depending on the size and structure of the coefficient matrix $S \in \mathbb{R}^{m\times m'}$.

1. First, it does not have any role if $S = I_m$.
2. When $S$ is a fat matrix ($m < m'$), function $\widehat{\varphi}$ represents a richer algebraic structure than $\varphi$. Therefore, compared to $\varphi$, function $\widehat{\varphi}$ may introduce important new degrees of freedom to the LMI (15) through its annihilator $N$. In this way, we may reduce the conservatism of (15) at the cost of increasing its dimension. In Sections 3.2 and 3.3, we present two possible alternatives to construct $S$ and $\widehat{\varphi}$ in a convenient form.
3. Assume that $S$ is a full-rank narrow matrix ($m > m'$). This fact encodes that function $\varphi$ is "redundant" in the sense that it can be reconstructed from function $\widehat{\varphi}$ having a smaller number of coordinate functions. In this case, matrix $S$ performs a dimension reduction on (15) without compromising the accuracy of the solution for $Q$. This aspect of Corollary 1 is explained in brief in Section 3.5.
4. Finally, a narrow matrix $S$ ($m > m'$) can help us to formulate a sufficient condition for (12) when the parameter ($x$) belongs to a hyperplane in $\mathbb{R}^n$. A possible construction of such a narrow matrix is presented in Section 3.4.

In the following subsections, we describe the four cases in more details but in a different (hopefully, didactic) order.

### 3.1. The trivial factorization

Let $S = I_n$, $\varphi = \widehat{\varphi}$, and $\mathcal{M} = \mathcal{X}$ denote a compact polytope in $\mathbb{R}^n$, such that it is an $n$-dimensional manifold in $\mathbb{R}^n$. Consider a function $N$ that satisfies $N(x)\varphi(x) = 0$ for all $x \in \mathcal{X}$. Practically, the terms in the coordinates of vector $N(x)\varphi(x)$ algebraically miss each other. Then, (15) simplifies to

$$Q + \mathrm{He}\{\Lambda N(x)\} \succeq 0 \text{ for all } x \in \mathbf{Ve}(\mathcal{X}). \tag{16}$$

Condition (16) is generally less conservative than simply prescribing $Q$ to be positive semidefinite.

### 3.2. Factorization with LFT to reduce conservatism

The major source of conservatism in (16) is that the algebraic coupling constraints between the coordinates of $\varphi(x)$ are not, or cannot be well represented by an affine annihilator. This is also the case when a few important monomials are missing from $\varphi(x)$. E.g., $\varphi(x) = (1, x, x^3)^\top$ does not admit an appropriate annihilator because $N(x) = (x, -1, 0)$ does not represent the coupling between the terms $x$ and $x^3$. In this way, LMI (16) describes a more general inequality, namely:

$$(1, x, y)Q(1, x, y)^\top \geq 0 \text{ for all } x \in \mathcal{X} \text{ and all } y \in \mathbb{R},$$

which is satisfied if and only of $Q$ is positive semidefinite.

We say that $\varphi$ admits a *preferred* annihilator if each coordinate of $\varphi(x)$ is involved in at least a single coupling constraint in $N(x)\varphi(x) = 0$ with a nonzero coefficient in $N(x)$, and $N(x)$ is not block diagonalizable by performing a simple symmetric row and column permutation. If $N(x)$ is block diagonalizable, then, at least two groups of basis functions in $\varphi(x)$ will be considered independent when solving (16). In [46], we illustrate that a rational function, which originates from a well-posed LFR, typically admits a preferred affine annihilator.

Consider function $\varphi$, that is defined as follows:

$$\varphi(x) = \mathcal{F}_l\left\{\left(\frac{g_{11}\,|\,G_{12}}{g_{21}\,|\,G_{22}}\right)\Delta_g(x)\right\} = H_g\varphi_{g,\mathrm{lfr}}(x), \tag{17}$$

where

$$H_g = (g_{11} \;\; G_{12}) \in \mathbb{R}^{m\times m_g} \text{ and} \tag{18}$$

$$\varphi_{g,\mathrm{lfr}}(x) = \begin{pmatrix} 1 \\ (I - \Delta_g(x)G_{22})^{-1}\Delta_g(x)g_{21} \end{pmatrix} \in \mathbb{R}^{m_g}. \tag{19}$$

Then, we say that $\varphi_{g,\mathrm{lfr}}$ originates from the LFR (17), and $\varphi_{g,\mathrm{lfr}}$ admits the following affine annihilator:

$$N_{g,\mathrm{lfr}}(x) = \begin{pmatrix} \Delta_g(x)g_{21} & \Delta_g(x)G_{22} - I \end{pmatrix}. \tag{20}$$

It can be shown that in each column of $N_{g,\mathrm{lfr}}(x)$ there exists a nonzero element, which involves the corresponding coordinate of $\varphi_{g,\mathrm{lfr}}(x)$ into a non-trivial coupling constraint. Considering the factorization (17) with annihilator (20), we can formulate the following LMI:

$$H_g^\top Q H_g + \mathrm{He}\{\Lambda_g N_{g,\mathrm{lfr}}(x)\} \succeq 0 \text{ for all } x \in \mathbf{Ve}(\mathcal{X}). \tag{21}$$

In [41, Section 5.5], we demonstrate through a few examples that LMI (21) is less conservative than (16) if $\varphi$ does not admit a preferred annihilator. However, we need to face with two possible drawbacks of this technique.

1. First, matrix $H_g$ is typically fat ($m < m_g$), thus, LMI (21) is a higher dimensional constraint than (16).
2. Though $N_{g,\mathrm{lfr}}$ (20) is typically a preferred annihilator for $\varphi_{g,\mathrm{lfr}}$ (19), it does not necessarily result in the least possible conservative affine LMI (21) to find a solution for (12).

To improve the proposed LFR-based factorization approach in (17), we use it in combination with a dimension reduction transformation and a different annihilator computation approach. These techniques are introduced later in Sections 3.5 and 3.6.

### 3.3. Canonical factorization to reduce conservatism

This approach can be considered as an alternative to the LFT-based factorization of Section 3.2. Suppose that the algebraic interdependence between the coordinates $\varphi$ cannot be described by only affine functions. Therefore, we are looking for a function $\varphi_g$, which admits a preferred annihilator and $\varphi(x) = H_g \varphi_g(x)$ for all $x \in \mathcal{X}$.

Let $\psi_d(x)$ comprise all at most $d$th degree monomials[1] of $x$, in a fixed order. Then,

$$\varphi(x) = \Theta\, \psi_d(x)\, q^{-1}(x), \tag{22}$$

is referred to as the *canonical factorization* of $\varphi$, where $q(x)$ denotes the lowest degree common denominator of the coordinates of $\varphi(x)$, and $d$ indicates the highest degree monomial, which can appear in $\varphi(x)q(x)$. It can be shown that for a fixed integer $d$, the canonical factorization of a rational function $\varphi$ is unique.

To compute the coefficient matrix

$$\Theta = \begin{pmatrix} \vartheta_{11} & \dots & \vartheta_{1m'} \\ \dots & \dots & \dots \\ \vartheta_{m1} & \dots & \vartheta_{mm'} \end{pmatrix}, \tag{23}$$

we introduce the vector $\gamma^\top = \begin{pmatrix} \gamma_1 & \dots & \gamma_m \end{pmatrix}$. First, we determine the unique irreducible factorized form of

$$\gamma^\top \varphi(x) = \frac{p(x;\gamma)}{q(x)}. \tag{24}$$

We should find matrix $\Theta$ such that

$$\gamma^\top \{\}(\varphi(x)q(x) - \Theta\psi_d(x)) = 0 \tag{25}$$

for all $x \in \mathcal{X}$ and all $\gamma \in \mathbb{R}^m$. Therefore, we collect the terms of (25) with respect to the monomials in $(x, \gamma)$, as follows:

$$\gamma^\top(\varphi(x)q(x) - \Theta\psi_d(x)) = \textstyle\sum_{k=1}^K c_k(\vartheta)\, p_k(x,\gamma), \tag{26}$$

where the coefficients $c_k$ are affine functions of unknown variables $\vartheta_{ij}$. Finally, we solve the system of linear equations $c_k(\vartheta) = 0$, $k = 1,\dots,K$ in $\vartheta_{ij}$ to obtain $\Theta$.

In [45, Section 3] we present a possible implementation of the computations above using MATLAB's Symbolic Math Toolbox [33].

### 3.4. Solution over a hyperplane

A specific factorization (13) of function $\varphi$ allows us to solve the parameter-dependent inequality (12) over the boundary of the polytopic $n$-dimensional manifold $\mathcal{X} \subset \mathbb{R}^n$. In this case, the inequality is reformulated for each facet $\mathcal{F}_k$, $k = 1,\dots,M_{\mathcal{X}}$ of polytope $\mathcal{X}$, where $M_{\mathcal{X}}$ denotes the number of facets of $\mathcal{X}$. Note that $\mathcal{F}_k$ is a subset of an $(n-1)$-dimensional hyperplane

$$\mathcal{R}_k = \{x \in \mathbb{R}^n \mid b_k + a_k^\top x = 0\} \subset \mathbb{R}^n, \tag{27}$$

where $a_k \in \mathbb{R}^n$ and $b_k \in \mathbb{R}$. It can be shown that, there exists a function $\varphi_k : \mathbb{R}^n \to \mathbb{R}^{m'_k}$, a full-rank narrow matrix $S_k \in \mathbb{R}^{m \times m'_k}$, and affine function $N_k : \mathbb{R}^n \to \mathbb{R}^{s_k \times m'_k}$ such that

$$\varphi(x) = S_k\, \varphi_k(x) \text{ and } N_k(x)\, \varphi_k(x) = 0, \tag{28}$$

for all $x \in \mathcal{F}_k$ (but not necessarily all $x \in \mathcal{X}$).

where $m > m'_k$. Consequently, the coordinates of vectors $\varphi(x) - S_k\,\varphi_k(x)$ and $N_k(x)\,\varphi_k(x)$ are not identically zeros in $\mathcal{X}$ but only on facet $\mathcal{F}_k$.

**Example 1.** Assume that the first $(n+1)$ coordinates of $\varphi(x)$ are $\psi_1(x) = (1\ x_1\ \dots\ x_n)^\top$. The following nonlinear coordinates of $\varphi(x)$ are referred to as $\varphi_1(x) \in \mathbb{R}^{m_1}$, $m_1 = m - n - 1$, namely,

$$\varphi(x) = \begin{pmatrix} \psi_1(x) \\ \varphi_1(x) \end{pmatrix}. \tag{29}$$

[1] Observe that each pair of monomials $p, q : \mathbb{R}^n \to \mathbb{R}$ corresponding to sibling lattice points in the Newton polytope Löfberg [27, Sec. III-A] of a polynomial are linearly coupled, namely, there exists a variable $x_i$ such that $p(x)x_i - q(x) = 0$ or $p(x) - q(x)x_i = 0$ for all $x$. In this way, it can be shown that any set of monomials corresponding to the lattice points in a Newton polytope admits a preferred annihilator.

To construct a possible factorization and an annihilator for $\varphi$ in $\mathcal{F}_k$ we follow Polcz [41, Cor. 4.7] and introduce the following notations:

1. Let $C_k^\perp = \begin{pmatrix} b_k & a_k^\top \end{pmatrix}$.
2. Consider $C_k \in \mathbb{R}^{(n+1)\times n}$ such that $C_k^\perp C_k = 0_{1\times n}$.
3. Denote the left pseudoinverse of $C_k$ by $C_k^\dagger \in \mathbb{R}^{n\times(n+1)}$, namely, $C_k^\dagger C_k = I_n$.
4. Introduce $\widehat{\psi}_k : \mathbb{R}^n \to \mathbb{R}^n$, $\widehat{\psi}_k(x) = C_k^\dagger \psi_1(x)$, and observe that
$$C_k \widehat{\psi}_k(x) = C_k C_k^\dagger \psi_1(x) = \psi_1(x) \text{ for all } x \in \mathcal{F}_k.$$
5. Then, $S_k = \begin{pmatrix} C_k & 0 \\ 0 & I_{m_1} \end{pmatrix}$ and $\varphi_k(x) = \begin{pmatrix} \widehat{\psi}_k(x) \\ \varphi_1(x) \end{pmatrix}$ satisfy $\varphi(x) = S_k \varphi_k(x)$ for all $x \in \mathcal{F}_k$.
6. Finally, $N_k(x) = \begin{pmatrix} r_k \\ N(x) \end{pmatrix} S_k$ is an affine annihilator for $\varphi_k$ in $\mathcal{F}_k$, namely, $\varphi_k(x) N_k(x) = 0$ for all $x \in \mathcal{F}_k$, where $r_k = \begin{pmatrix} C_k^\perp & 0_{1\times m_1} \end{pmatrix}$ and $N$ is an affine annihilator of $\varphi$ in $\mathcal{X}$.

The technique to compute $S_k$, $\varphi_k$, and $N_k$ is a slight generalization of that presented in Polcz [41, Cor. 4.7], in the sense that it allows $\mathcal{F}_k$ to intersect the origin. When $\mathcal{F}_k$ does not intersect the origin (i.e., $b_k \neq 0$), matrices $C_k$ and $C_k^\dagger$ may take the following values:

$$C_k = \begin{pmatrix} -b_k^{-1} a_k^\top \\ I_n \end{pmatrix} \text{ and } C_k^\dagger = \begin{pmatrix} 0_{n\times 1} & I_n \end{pmatrix}. \tag{30}$$

In this way, (15) simplifies to the following $(m-1)$-dimensional LMI:

$$S_k^\top Q S_k + \mathrm{He}\{\Lambda_k N_k(x)\} \succeq 0 \text{ for all } x \in \mathbf{Ve}(\mathcal{F}_k). \tag{31}$$

**Remark 5.** We note that (31) is a sufficient polytopic LMI condition for (12) over $\mathcal{M} = \mathcal{F}_k$, a polytopic segment of a hyperplane in $\mathbb{R}^n$. Condition (31) is rendered by matrix $S_k$ and functions $\varphi_k$, $N_k$ satisfying (28). Example 1 presents a simple approach to compute them.

In the following subsection, we outline a numerical technique, which makes possible to compute a factorization $\varphi = S_k\,\varphi_k$, which can be more advantageous from a computational point of view.

### 3.5. Dimension reduction

In Sections 3.2 and 3.3, we presented two different factorization techniques, where the coefficient matrix is fat. In this case, the dimension of LMI (15) is inflated compared to (16) to introduce additional free Lagrange variables and reduce the conservatism of the solution for $Q$.

In this section, we claim that a full-rank narrow matrix $S$ can reduce the dimension of the LMI without increasing its conservatism if function $\varphi$ can be reconstructed from $\widehat{\varphi} = S^\dagger\varphi$ as follows:

$$\varphi(x) = S\,\widehat{\varphi}(x) = S\,S^\dagger \varphi(x) \text{ for all } x \in \mathcal{M}. \tag{32}$$

Let $\mathcal{M}$ denote a polytope in $\mathcal{R}$, where $\mathcal{R}$ is an $(n-d)$-dimensional hyperplane in $\mathbb{R}^n$ $(d = 1)$ or $\mathbb{R}^n$ itself $(d = 0)$. Assume that matrix $S \in \mathbb{R}^{m\times m'}$ in (13) is a full-rank narrow matrix. We say that $S\widehat{\varphi}$ is a *minimal factorization* of $\varphi$ in $\mathcal{M}$ if the dimension of $\widehat{\varphi}$ (i.e., $m'$) is the possible minimum value such that $\varphi(x) = S\widehat{\varphi}(x)$ for all $x \in \mathcal{M}$.

According to Polcz [41, Thm. 5.17], we are able to formulate a sufficient polytopic LMI (15) for (12) if $m' < m - d$. Moreover, (15) is equivalent to, but smaller dimensional than (16) if $d = 0$ and (31) if $d = 1$.

### 3.6. Further remarks and computational aspects

**Remark 6** (Maximal affine annihilator)**.** Consider inequality (12) and its possible sufficient condition (15) with a fixed rational function $\widehat{\varphi}$ satisfying $\varphi = S\widehat{\varphi}$. In [41, Section 5.3], we introduced the notion of a *maximal affine annihilator* of function $\widehat{\varphi}$, which corresponds to the maximum achievable solution set (for $Q$) of the polytopic LMI condition in (15). To compute the maximal affine annihilator of function $\widehat{\varphi}$ over $\mathcal{M}$ satisfying (14), we refer to the technique presented in Polcz [41, Par. 5.3.2.1]. The computations rely on a parameter independent kernel computation of a parameter-dependent matrix. As a possible kernel computation method, one may consider:

(K1) the computer algebra approach Polcz [41, Proc. 5.12],
(K2) the numerical sample-based approach Polcz [41, Proc. 5.13].

Approach (K1) is applicable only when $\mathcal{M}$ is an $n$-dimensional polytope ($d = 0$).

**Remark 7.** To compute a *minimal factorization* $\varphi = S\widehat{\varphi}$ satisfying (13) we apply Polcz [41, Proc. 5.15] with an appropriate kernel computation approach (K1) or (K2).

**Remark 8.** The LFR-based factorization $\varphi = H_g\varphi_{g,\mathrm{lfr}}$ in Section 3.2 generates LMI (21), which can be less conservative than (16). Depending on the considered LFR realization of $\varphi$ in (17), function $\varphi_{g,\mathrm{lfr}}$ in (19) may admit a non-trivial minimal factorization $\varphi_{g,\mathrm{lfr}} = S_g\varphi_g$. Let $N$ denote a maximal affine annihilator of $\varphi_g$ and introduce $S = H_gS_g$. Then, (15) with $N$ and $S$ implies (12), but (15) is potentially less conservative than (16), and it is typically smaller dimensional than (21).

We note that the *minimal factorization* and the *maximal affine annihilator* computation of Polcz et al. [43, Section 5.3, Section 5.4] are more technical, and their detailed descriptions are not included in this paper.

**Remark 9** (Global analysis)**.** When a certain state variable $x_i$ is omitted from the annihilator $N(x)$, then, the LMI (15) implies (12) for all $x_i \in \mathbb{R}$. In this way, we are able to formulate global constraints with respect to selected variables. Furthermore, it is enough to test the LMI (15) in the corner points of a reduced smaller-dimensional polytope; a polyotope that is the convex hull of the projected corner points of $\mathcal{M}$ onto the space of the remaining variables.

### 3.7. Our approach in relation with the SOS methodology

The main mathematical apparatus behind the SOS techniques is the *positive locus theorem* or, as it is commonly called, the *Positivstellensatz*[2] (PS). Using PS, one can formulate sophisticated set containment constraints for semialgebraic (i.e., polynomial level-) sets. These constraints are formulated over the whole parameter space $\mathbb{R}^n$.

A possible *reinterpretation* of the extended SOS test of Chesi [7, Thm. 1], makes possible to test the non-negativity of $W$ over a hyperplane $\mathcal{R} \subset \mathbb{R}^n$ of the parameter space. Let $R(\lambda)$ be a linear parameterization of the set

$$\{R \mid \varphi^\top(x)R\varphi(x) = 0 \ \forall x \in \mathcal{R}, \ \underline{\text{but } \not\forall x \in \mathbb{R}^n}\}. \tag{33}$$

Then, the LMI $Q + R(\lambda) \succeq 0$ ([7, Eq. (8)]) ensures the non-negativity of $W$ but only in $\mathcal{R}$. E.g., consider an $(n-1)$-dimensional hyperplane $\mathcal{R}_k$ as defined in (27), a vector of monomials $\varphi(x)$, and its minimal factorization (28). Then, our approach gives a possible parameterized form of $R(\lambda)$, which makes possible to restrict the analysis to the hyperplane $\mathcal{R}_k$ by solving

$$Q + \mathrm{He}\left\{\tilde{\Lambda}_k S_k^\perp\right\} \succeq 0, \tag{34}$$

where $\tilde{\Lambda}_k$ is a free matrix Lagrange multiplier. According to de Oliveira and Skelton [36, Lemma 2, *ii*) $\Leftrightarrow$ *iv*)], condition (34) is equivalent to (31) if $N_k = 0$, and typically more conservative that (31) if $N_k \neq 0$. Furthermore, the LMI (34) has a higher dimension compared to (31).

In the SOS framework, the PS and the S-procedure Boyd et al. [6, Section 2.6] make possible to force local solutions, e.g., restricting the analysis to a bounded level set Prajna et al. [47, Prop. 10], an interval Wu and Prajna [62, Thm. 1], or a polytope Coutinho et al. [9, Eqs. (23)-(24)]. In this case, a new polynomial is formulated with additional Lagrange multiplier terms. But the non-negativity of the resulting polynomial (let's say, $W$ in (12)) is tested over the whole parameter space by checking its SOS property ($Q \succeq 0$, with the assumption that the coordinates of $\varphi(x)$ are linearly independent).

In comparison, Finsler's lemma with affine annihilators and the method of vertices make possible to test the non-negativity of $W$ over a *bounded* polytope of the parameter space $\mathbb{R}^n$ or a hyperplane $\mathcal{R} \subset \mathbb{R}^n$. In this sense, the method of vertices can be an alternative to the PS or S-procedure. Moreover, according to Trofino and Dezuo [59, Rem. 4.3], the method of vertices may lead to a less conservative solution compared to the SOS technique with PS. However, we emphasize that our approach does not exclude the possibility to consider additional PS or S-procedure constraints when it is necessary. An important result, where the S-procedure and the method of vertices with affine annihilators are combined, can be found in Coutinho et al. [9, Eqs. (23)-(24)].

From a computational point of view, the LFT allows us to manipulate with rational functions efficiently in a "natural" way (without "recasting" the *fraction* of polynomials into polynomials). The efforts to find an appropriate factorization for $\varphi$ in Sections 3.2–3.5 are made to reduce the conservatism while keeping the problem's dimension as low as possible. Note that the factorization of $\varphi$ directly affects the quadratic decomposition (12) of function $W$ to be tested. These techniques can be considered as the LFT [10,21]-based alternatives of the polynomial approaches of Löfberg [27], Papachristodoulou et al. [38] to find an optimal SOS decomposition (12) of polynomials.

## 4. Lyapunov function for a local attractor

In this section, we prescribe Lyapunov-type conditions for a candidate function $V$, which ensure the suitability of the Lyapunov function over a compact set $\mathcal{X}$.

Consider the system model in the following form:

$$\dot{x} = f(x) = A\pi(x), \ \text{ with } \pi(x) = \begin{pmatrix} 1 \\ \pi_1(x) \end{pmatrix}, \tag{35}$$

where $A \in \mathbb{R}^{n\times m}$ is a constant matrix and $\pi_1 : \mathbb{R}^n \to \mathbb{R}^{m-1}$ is a rational function of the state. We are looking for a rational Lyapunov function in the following parameterized quadratic expression:

$$V(x) = \pi^\top(x)P\pi(x), \tag{36}$$

where $P \in \mathbb{R}^{m\times m}$ is a free symmetric (not necessarily positive definite) matrix.

In the following theorem, we give sufficient conditions for the existence of a Lyapunov function for an invariant set $M$. Certain parts of the following theorem were inspired by Goluskin [18, Section 2.1].

**Theorem 9.** *Consider an autonomous system (2) over the compact set $\mathcal{X}$. Let $M_a$ be the largest invariant set in $\mathcal{X}$, and let $M \subseteq M_a$ be the*

[2] A detailed description of PS can be found in Papachristodoulou [37].

*largest invariant attractor in $\mathcal{X}$. Let $\alpha_1, \alpha_2 \geq 0$. Assume the existence of constants $0 < c_1 < c_2 < c_3$, $0 < \epsilon_2, \epsilon_3 \ll 1$, and function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the following conditions:*

(C1) *$V$ is continuous in $\mathcal{X}$ and continuously differentiable in $\mathcal{X}^\circ$,*
(C2) *$V(x) \geq 0$ for all $x \in \mathcal{X}$,*
(C3) *$\alpha_1 V(\xi) \leq c_1$ for all $\xi \in M$,*
(C4) *$L_f V(x) \leq -\alpha_2(V(x) - c_2 + \epsilon_2)$ for all $x \in \mathcal{X}$,*
(C5) *$V(x) \geq c_3 + \epsilon_3$ for all $x \in \partial\mathcal{X}$.*

*Then,*

(R1) *each connected subset of $\Omega_{c_3}$ is forward invariant with respect to (2).*
(R2) *If $M = \{\}x_*^{(i)}, i = 1, 2, \dots$ is a set of isolated equilibria, and (C3), (C4) are satisfied with $\alpha_1 > 0$ and $\alpha_2 = 0$, then, each equilibria $x_*^{(i)}$ is contained in a connected subset $\Omega_{c_3}^{(i)}$ of level set $\Omega_{c_3}$, and every solution starting in $\Omega_{c_3}^{(i)}$ approaches $M_a \cap \Omega_{c_3}^{(i)}$ as $t \to \infty$.*
(R3) *If $M$ is a periodic orbit with period $T < \infty$ (i.e., for all $\xi \in M$, $\xi = \Phi(T, \xi)$), and (C3) is satisfied for at least a single element $\xi \in M$ with $\alpha_1 > 0$, then, there exists a connected subset $\Omega_{c_3,M}$ of level set $\Omega_{c_3}$ that comprise $M$. Moreover, $V$ is a Lyapunov function for $M$ on $\Omega_{c_3,M} \setminus \Omega_{c_2,M}^\circ$ for (2) in the sense of Definition 7 if $\alpha_2 > 0$.*

**Proof.** (R1) When $\alpha_2 = 0$, the Lyapunov function is non-increasing along the trajectories in the whole polytope $\mathcal{X}$, therefore, a connected subset of any level set of $V$ contained in $\mathcal{X}^\circ$ is invariant. Furthermore, (C5) implies that the $c_3$-level set $\Omega_{c_3}$ is a subset of $\mathcal{X}^\circ$. When $\alpha_2 > 0$, condition (C4) implies the strict negativity of the Lie derivative in $\mathcal{X} \setminus \Omega_{c_2}^\circ$, and hence in $\Omega_{c_3} \setminus \Omega_{c_2}^\circ$. Though a connected subset of $\Omega_{c_2}$ is not necessarily invariant, it is surrounded by a connected subset of $\Omega_{c_3}$. Moreover, the Lie derivative in $\Omega_{c_3} \setminus \Omega_{c_2}^\circ$ is strictly negative. Then, any trajectory which leaves $\Omega_{c_2}$ and enters $\Omega_{c_3} \setminus \Omega_{c_2}^\circ$, will be immediately led back to $\Omega_{c_2}$. Consequently, each connected subset of $\Omega_{c_3}$ is invariant.

Proof for (R2). Let $E^{(i)}$ be the set of all points $x \in \Omega_{c_3}^{(i)}$, where $L_f V(x) = 0$, and observe that each equilibrium in $\Omega_{c_3}^{(i)}$ is necessarily an element of $E^{(i)}$. Therefore, $M \cap \Omega_{c_3}^{(i)}$ is the maximal invariant set in $E^{(i)}$, that is surrounded by the forward invariant set $\Omega_{c_3}^{(i)}$. Then, (R2) is a direct consequence of Theorem 4.

Proof for (R3). Assume that (C3) is satisfied for at least a single element $\xi \in M$ with $\alpha_1 > 0$. Then, according to (C1) and (C5), there exists a connected subset $\Omega_{c_3,M}$, which comprise $\xi$. Due to (R1), set $\Omega_{c_3,M}$ is invariant, therefore, $\Phi(t, \xi) \in \Omega_{c_3,M}$ for all $t \geq 0$. This implies that $M \subset \Omega_{c_3,M}$ if $M$ is a periodic orbit with period $T < \infty$. Moreover, a positive scalar $\alpha_2$ ensures that $L_f V(x) \leq -\alpha_2 \epsilon_2$ if $V(x) \geq c_2$. Due to the continuity property (C1), we have that $\Omega_{c_2,M} \subset \Omega_{c_3,M}^\circ$ if $\Omega_{c_2,M} = \Omega_{c_2} \cap \Omega_{c_3,M}$. Finally, the conditions

$$L_f V(x) \leq -\alpha_2 \epsilon_2, \;\; V(x) > 0 \;\; \forall x \in \Omega_{c_3,M} \setminus \Omega_{c_2,M}^\circ \tag{37}$$

with a nonempty $\mathcal{L}_M^{\inf} = \overline{\Omega_{c_2,M}}$ imply that $V$ is a Lyapunov function for $M$ on $\Omega_{c_3,M} \setminus \Omega_{c_2,M}^\circ$ for (2) in the sense of Definition 7. □

To ensure the necessary conditions for Definition 7, we force a few geometric properties for the Lyapunov function candidate in (C3), (C4), and (C5). First, the Lyapunov function is bounded from above by a positive scalar $c_1$ in $M$. Then, the Lyapunov function is forced to be higher than $c_3$ on the boundaries ($\partial\mathcal{X}$) of polytope $\mathcal{X}$. Finally, the $c_2$-level set of the Lyapunov function bounds the region where a negative Lie derivative is not required. When $\alpha_1, \alpha_2 > 0$, the conditions of Theorem 9 imply the following set containment relations:

$$M \subset \Omega_{c_2} \subset \Omega_{c_3} \subset \mathcal{X}^\circ. \tag{38}$$

**Remark 10.** When $M$ is a limit cycle and condition (C4) is satisfied with $\alpha_2 = 0$, then, the Lyapunov function should take the same (local minimum) value in set $M$, namely, $V(M) = \{v_0\}$, where $v_0 \leq V(x)$ for all $x \in \Omega_{c,M}$. This is only possible when the periodic orbit is described by a closed-form implicit equation, or the Lyapunov function is not given in a closed-form. Fortunately, condition (C4) with $\alpha_2 > 0$ allows the Lyapunov function candidate to vary along the orbit. Finally, note that statement (R1) of Theorem 9 also applies when $M$ is a strange attractor.

**Remark 11.** The negativity of the Lie derivative of the Lyapunov function is required over the whole polytope $\mathcal{X}$ except a (hopefully tight) neighborhood of $M$. Therefore, polytope $\mathcal{X}$ should be selected carefully when $M$ is only locally stable. Although it is not necessary for $\mathcal{X}$ to lie inside the DOA $B(M)$ of $M$, a feasible solution for (C2), (C3), (C4), and (C5) may not be found if $\mathcal{X} \setminus B(M)$ is significant.

## 5. Optimization-based Lyapunov function computation for local attractors

In this section, we formulate sufficient LMI conditions to compute a Lyapunov function for a dynamical system with multiple local attractors.

### 5.1. Convenient structure for the candidate function

First, observe that the parameterized structure of the candidate Lyapunov function is generated by function $\pi$, hence, it is called a "generator" in Polcz [41]. When selecting $\pi$, we focus on three objectives (so to say, trade-offs).

1. To compute a good estimation of $B(M)$, function $\pi$ should generate a relatively wide class of polynomial/rational functions,
2. however, the complexity of the optimization problem will increase combinatorially with the number of coordinates of $\pi$.
3. Finally, function $\pi$ admits a preferred annihilator.

**Remark 12.** In the literature [59], $\pi$ is often selected such that it contains all monomials of $x$ of given degrees (e.g., $\pi = \psi_d$).

**Remark 13.** Alternatively, we can construct a set of basis functions ($\pi$) by applying the LFR-based technique of Section 3.2 in combination with the dimension reduction method of Section 3.5. First, we factorize the vector field $f$ as proposed in (17):

$$f(x) = \mathcal{F}_l\left\{\begin{pmatrix} f_{11} | F_{12} \\ f_{21} | F_{22} \end{pmatrix}, \Delta(x)\right\} = (f_{11} \;\; F_{12})\,\pi_{\mathrm{lfr}}(x), \tag{39}$$

where $\pi_{\mathrm{lfr}}(x) = \begin{pmatrix} 1 \\ (I - \Delta(x)F_{22})^{-1}\Delta(x) f_{21} \end{pmatrix} \in \mathbb{R}^{\bar{m}+1}$. Then, we compute a minimal factorization $S\pi$ for $\pi_{\mathrm{lfr}}$ as proposed in Remark 7. Finally, the dynamics can be written in the form (35) with $A = (f_{11} \;\; F_{12})S$.

### 5.2. Lie derivative of the Lyapunov function

To formulate a convex constraint for the Lyapunov condition (C4) of Theorem 9, we rewrite the Lyapunov function's Lie derivative in the following form:

$$L_f V(x) = \mathrm{He}\left\{\pi^\top(x)\, P\, L_f \pi(x)\right\} = \pi_d^\top(x) \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \pi_d(x),$$

$$\text{where } \pi_d(x) = \begin{pmatrix} \pi(x) \\ L_f \pi(x) \end{pmatrix}. \tag{40}$$

As it is remarked in Polcz et al. [46, Section 4], the structure of $\pi_d$ may be unfortunate in the sense that we cannot find affine coupling between the first ($\pi$) and second block ($L_f\pi$) of $\pi$. Therefore, we select a well-structured function $\pi_g$ with a preferred affine anihilator $N_g$, and a fat matrix $H_g$ such that $\pi_d = H_g\pi_g$ in $\mathcal{X}$. To construct $H_g$ and $\pi_g$, we can follow Section 3.2 or Remark 8 applied to $\varphi = \pi_d$.

Finally, the Lie derivative can be written as follows:

$$L_f V(x) = \pi_g^\top(x)\,\mathrm{He}\left\{E_g^\top P A_g\right\}\pi_g(x). \tag{41}$$

where $E_g = (I_m\ 0_{m\times m})H_g$, $A_g = (0_{m\times m}\ I_m)H_g$.

### 5.3. Subdivision of polytope $\mathcal{X}$

The results in Theorem 9 do not guarantee a separate invariant domain for all locally asymptotically stable and isolated attractors $M^{(i)}$ in $M = \bigcup_i M^{(i)}$. To do so, we have the possibility to introduce further boundary conditions similar to (C5) in Theorem 9. Assume that polytope $\mathcal{X} = \bigcup_i \mathcal{X}^{(i)}$ can be divided into smaller polytope slices $\mathcal{X}^{(i)}$ satisfying $\mathcal{X}^{(i)\circ} \cap \mathcal{X}^{(j)\circ} = \emptyset$ for all $i \neq j$, such that each slice $\mathcal{X}^{(i)}$ contains a single invariant set $M^{(i)}$.

**Example 2.** Consider the gradient system

$$\dot{x} = -\nabla H(x),\ \text{ with } H(x) = \textstyle\prod_{i=1}^3 \left\|x - x_*^{(i)}\right\|^2, \tag{42}$$

having three locally asymptotically stable equilibrium points $x_*^{(i)} \in \{(0,1), (-1,2), (1,2)\}$. In (42), $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^{n_x}$. Let $\mathcal{X} = \mathbf{Co}(\{v_1, v_2, v_3\})$, where $v_1 = (-4,0)$, $v_2 = (4,0)$, $v_3 = (0,4)$. Then, a possible subdivision of $\mathcal{X}$ is

$$\mathcal{X}^{(1)} = \mathbf{Co}(\{v_1, v_4, v_6, v_3\}),\ \ \mathcal{X}^{(2)} = \mathbf{Co}(\{v_4, v_5, v_6\}),$$
$$\mathcal{X}^{(3)} = \mathbf{Co}(\{v_5, v_2, v_3, v_6\}), \tag{43}$$

where $v_4 = (-2,0)$, $v_5 = (2,0)$, $v_6 = (0,2)$. Observe that the three polytopes in (43) are "almost disjoint" (except their common edges). The secant segments between the polytope slices in (43) are

$$\mathcal{F}_4 = \mathbf{Co}(\{v_3, v_6\}),\ \ \mathcal{F}_5 = \mathbf{Co}(\{v_4, v_6\}),$$
$$\mathcal{F}_6 = \mathbf{Co}(\{v_5, v_6\}). \tag{44}$$

The indexing of the secant segments in (44) is started from 4 as the indices 1,2,3 are "reserved" for the faces of polytope $\mathcal{X}$, namely, $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 = \partial\mathcal{X}$. This notation for the secant segments allows us to reformulate condition (C5) of Theorem 9 for each slice in a convenient way, namely:

$$V(x) \geq c_3 + \epsilon_3 \text{ for all } x \in \mathcal{F}_k \text{ and all } k = 1,\dots,6. \tag{45}$$

**Example 3.** Consider the gradient dynamics

$$\dot{x} = -\nabla H(x), \tag{46}$$

$$\text{with } H(x) = x_1^4x_2^2 + x_1^2x_2^4 - 3x_1^2x_2^2 + 1,$$

where $H$ is the Motzkin polynomial [35]. System (46) has four locally asymptotically stable equilibrium points in $x_*^{(i)} \in \{(-1,-1), (1,-1), (1,1), (-1,1)\}$. Let $\mathcal{X} = [-2,2]\times[-2,2]$. Then, a possible subdivision of $\mathcal{X}$ can be obtained by considering the following secant segments:

$$\begin{aligned}\mathcal{F}_5 &= \mathbf{Co}(\{(-2,0), (2,0)\}),\\ \mathcal{F}_6 &= \mathbf{Co}(\{(0,-2), (0,2)\}).\end{aligned} \tag{47}$$

Observe that the four slices can be separated in an optimal way with only *two* secant segments. Again, the four sides of rectangle $\mathcal{X}$ are denoted by $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$.

One might see that in the $n$-dimensional Euclidean space the polytope slices $\mathcal{X}^{(i)}$ are separated by $(n-1)$-dimensional hyperplanes, such that the common sides are polytopes in the secant hyperplane. Let the number of slices be denoted by $m_{\mathrm{sd}}$ and the number of secant manifolds be denoted by $m_{\mathrm{sm}}$, where "sd" and "sm" stand for "subdivision" and "secant (polytopic) manifolds", respectively. Observe that, for a given set of subdivision $\{\mathcal{X}^{(i)}\}_{i=1,\dots,m_{\mathrm{sd}}}$, the set of secant manifolds are not unique, but they are determined heuristically (such that their number is minimized).

Condition (C5) in Theorem 9, prescribes a PD-LMI for each face $(\mathcal{F}_{k_i}^{(i)})$ of each slices $\mathcal{X}^{(i)}$ of polytope $\mathcal{X}$. However, as we suggested it in (45), it is enough to prescribe the boundary conditions only once along each secant manifold. Furthermore, it is convenient to detect those secant manifolds, which have an identical supporting hyperplane. The reason for we introduced the secant manifolds $\mathcal{F}_{m_\mathcal{X}+1}, \dots, \mathcal{F}_{m_\mathcal{X}+m_{\mathrm{sm}}}$ is to simplify the further notations and to reduce the number of convex constraints for the Lyapunov function computation.

### 5.4. Boundary conditions

Following the ideas of Polcz [41, Section 4.2.4] (first, proposed by [59]), we aim to scale the Lyapunov function such that the connected subset $\Omega_{1,M}^{(i)}$ of the unitary level set, which comprise $M^{(i)}$, is expanded as much as possible in each slice $\mathcal{X}^{(i)}$ of polytope $\mathcal{X}$. Therefore, instead of condition (C5) of Theorem 9, we prescribe an interval constraint on each bounding ($k = 1,\dots,m_\mathcal{X}$) and secant ($k = m_\mathcal{X}+1,\dots,m_\mathcal{X}+m_{\mathrm{sm}}$) manifold $\mathcal{F}_k$ as follows:

$$c_3 + \epsilon_3 \leq\ V(x) \leq \tau_k \qquad\qquad \text{for all } x \in \mathcal{F}_k \tag{48}$$

and all $k = 1,\dots,m_\mathcal{X}+m_{\mathrm{sm}}$, where $\tau_k$ are slack variables, which are minimized through the optimization.

The two inequalities in (48) are rewritten as follows:

$$\pi^\top(x)\,(P - (c_3+\epsilon_3)\,\mathbf{1}_m)\,\pi(x) \geq 0 \text{ for all } x \in \mathcal{F}_k,$$
$$\pi^\top(x)\,(P - \tau_k\,\mathbf{1}_m)\,\pi(x) \leq 0 \text{ for all } x \in \mathcal{F}_k, \tag{49}$$

where matrix $\mathbf{1}_m$ is introduced in (1).

Finally, a set of sufficient boundary LMIs for (49) can be formulated as presented in Section 3.4.

In the following subsection, we present a semidefinite program, which (if feasible) constructs a Lyapunov function for multiple local (not necessarily point-like) attractors.

### 5.5. The resulting optimization problem for computing Lyapunov functions

Consider an autonomous system (2) over $\mathcal{X} = \bigcup_{i=1}^{m_{\mathrm{sd}}} \mathcal{X}^{(i)}$, in each slice with the largest invariant attractor $M^{(i)}$. Consider a Lyapunov function candidate (36) and a possible quadratic factorization (41) for its Lie derivative. Observe that inequality (C4) of Theorem 9 can be written in the form

$$\pi_g^\top(x)\,P_g\,\pi_g(x) \leq 0 \text{ for all } x \in \mathcal{X} \text{ with} \tag{50}$$

$$P_g = \mathrm{He}\left\{E_g^\top P A_g\right\} + \alpha_2\{()E_g^\top P E_g - H_g^\top \mathbf{1}_{2m} H_g(c_2 - \epsilon_2)),$$

where matrices $E_g$, $A_g$, and $H_g$ are defined in Section 5.2.

Compute two maximal affine annihilators

$$N : \mathcal{X} \to \mathbb{R}^{s\times m} \text{ and } N_g : \mathcal{X} \to \mathbb{R}^{s_g\times m_g}, \tag{51}$$

as proposed in Remark 6 with (K1) or (K2), such that

$$N\pi \equiv 0 \text{ and } N_g\pi_g \equiv 0 \text{ on } \mathcal{X}. \tag{52}$$

Then, for each bounding and secant facet $\mathcal{F}_k$, compute

1. a minimal factorization $\pi = S_{f,k}\pi_{f,k}$ in $\mathcal{F}_k$ as proposed in Remark 7 with (K2), where $S_{f,k} \in \mathbb{R}^{m\times m_{f,k}}$ and $\pi_{f,k} : \mathcal{F}_k \to \mathbb{R}^{m_{f,k}}$.
2. a maximal affine annihilator $N_{f,k} : \mathcal{F}_k \to \mathbb{R}^{s_{f,k}\times m_{f,k}}$ as proposed in Remark 6 with (K2), such that $N_{f,k}\pi_{f,k} \equiv 0$ on $\mathcal{F}_k$,

where $m_{f,k} \leq m-1$, $k = 1,\dots,m_{\mathcal{X}} + m_{\text{sm}}$.

In the following corollaries, we present the LMI constraints, which determine a Lyapunov function for system (2). These results are direct consequences of Corollary 1.

**Corollary 2.** *Assume that there exist matrices* $\Lambda \in \mathbb{R}^{m\times s}$ *and* $\Lambda_g \in \mathbb{R}^{m_g\times s_g}$ *such that:*

$$\begin{aligned} &P + \text{He}\{\Lambda N(x)\} \succeq 0 && \text{for all } x \in \mathbf{Ve}(\mathcal{X}),\\ &P_g + \text{He}\{\Lambda_g N_g(x)\} \preceq 0 && \text{for all } x \in \mathbf{Ve}(\mathcal{X}). \end{aligned} \tag{53}$$

*Furthermore, assume that for all facets* $\mathcal{F}_k$, $k = 1,\dots,m_{\mathcal{X}} + m_{\text{sm}}$ *there exists a full matrix* $\Lambda_{f,k}^{(1)} \in \mathbb{R}^{m_{f,k}\times s_{f,k}}$ *such that*

$$S_{f,k}^{\top}(P - (c_3+\epsilon_2)\,\mathbf{1}_m)S_{f,k} + \text{He}\left\{\Lambda_{f,k}^{(1)}N_{f,k}(x)\right\} \succeq 0 \tag{54}$$

*is satisfied for all* $x \in \mathbf{Ve}(\mathcal{F}_k)$. *Finally, assume that P satisfies the following ordinary LMI:*

$$V(\xi^{(i)}) = \pi^{\top}(\xi^{(i)})P\pi(\xi^{(i)}) \leq c_1,\ \ i = 1,\dots,m_{\text{sd}}, \tag{55}$$

*for some* $0 < c_1 \ll 1$ *and* $\xi^{(i)} \in M^{(i)}$.

*Then, the conditions of Theorem 9 are satisfied in each polytope slice* $\mathcal{X}^{(i)}$, *where* $i = 1,\dots,m_{\text{sd}}$, *and V is a Lyapunov function for system (2) on* $\mathcal{X} \setminus \Omega_{c_2}^{\circ}$.

The four convex constraints in Corollary 2 together ensure the conditions of Theorem 9 and provide a common Lyapunov function on the whole polytope $\mathcal{X}$, and a positively invariant domain for each locally asymptotically stable invariant set $M^{(i)}$. In addition, we may introduce further constraints, which may shape the Lyapunov function more conveniently, e.g., enlarge the invariant domains.

**Corollary 3.** *In addition to Corollary 2, assume that, for each bounding and secant facet* $\mathcal{F}_k$, $k = 1,\dots,m_{\mathcal{X}} + m_{\text{sm}}$, *there exists a matrix* $\Lambda_{f,k}^{(2)} \in \mathbb{R}^{m_{f,k}\times s_{f,k}}$ *such that*

$$S_{f,k}^{\top}\,(P - \tau_k\,\mathbf{1}_m)\,S_{f,k} + \text{He}\left\{\Lambda_{f,k}^{(2)}N_{f,k}(x)\right\} \preceq 0 \tag{56}$$

*is satisfied for all* $x \in \mathbf{Ve}(\mathcal{F}_k)$. *Furthermore, assume that* $\tau_k$ *are the minimal possible values that fulfill (56). Then, the Lyapunov function satisfies the second inequality in (48), and* $\Omega_{c_3,M}^{(i)} = \mathcal{L}_{M^{(i)}}^{\inf}$ *is the largest invariant level set of V in* $\mathcal{X}^{(i)}$, *which contains* $M^{(i)}$.

## 6. Illustrative examples

To find a possible LFR realization of a rational function, we used the object-oriented recursive LFT implementation of the Enhanced LFT Toolbox for MATLAB [28]. The operations of the referred LFT implementation are presented in details in Polcz [41, Section 3.6]. To model and solve semidefinite programs, we used YALMIP [26] with Mosek [34] solver.

In the case studies, the computed matrices/functions are often obtained as complicated structures of large matrices (e.g., high order LFR realization). These values are not provided in this manuscript, but we refer to the online repository [42], where all intermediate results can be computed with a MATLAB implementation for each case study.

### *6.1. Gradient dynamics of the Motzkin polynomial*

In this section, we revisit the gradient dynamics (46) of the Motzkin polynomial [35], which was presented in Example 3. The shape of function $H$ is illustrated in panel (a) of Fig. 1. Function $H$ has four local minima, which correspond to the four isolated (locally asymptotically stable) equilibrium points of dynamics (46).

A possible Lyapunov function for system (46) is the Motzkin polynomial $H$ itself. The Motzkin polynomial is a counterexample for a non-negative polynomial, which is not a sum of squares of polynomials[3] due to the negative coefficient of term $x_1^2x_2^2$. Now, consider a candidate function $W$, such that $W(x) = H(x) + 0.1$. It can be shown that $W$ is still not an SOS, however, our approach allows to check its non-negativity in $\mathcal{X} = [-2,2]\times[-2,2]$. A possible decomposition for $W$ is presented in Polcz [42, Section 5]

Though function $W$ can prove stability, we compute a synthetic Lyapunov function following the procedure described in Section 5. It is an interesting question, how the computed Lyapunov function will follow the shape of the Motzkin polynomial.

During the computations, we considered polytope $\mathcal{X}$ and its subdivision as presented in Example 3. The two secant manifolds $\mathcal{F}_5$ and $\mathcal{F}_6$ in (47) are illustrated by the green dashed segments in panel (b) of Fig. 1.

To construct function $\pi$ for the Lyapunov function, we considered Remark 13. First, we computed an LFR realization of the system Eq. (46), which generates $\pi_{\text{lfr}} : \mathbb{R}^2 \to \mathbb{R}^{27}$. Then, we computed a minimal factorization of $\pi_{\text{lfr}} = S\pi$, where $\pi : \mathbb{R}^2 \to \mathbb{R}^{15}$. The coordinates of the computed vector $\pi(x)$ are the following monomials: 1, $x_1^3x_2^2$, $x_1^2x_2^2$, $x_1x_2^2$, $x_1x_2^4$, $x_1^2x_2^3$, $x_1x_2^3$, $x_1^2x_2$, $x_1x_2$, $x_1^4x_2$, $x_1^3x_2$, $x_2^2$, $x_2$, $x_2^4$, $x_2^3$. Then, we followed Remark 8 to find a convenient factorization for function $\pi_d$ in (40).

To compute an invariant domain for each $x_*^{(i)}$, we need to assure the geometrical properties of the Lyapunov function around each equilibrium point. For this, we introduced two secant lines in (47), on which the boundary conditions (54) and (56) should be satisfied as well as on the boundary of polytope $\mathcal{X}$.

We proceeded the steps of Section 5.5 to compute annihilators $N$, $N_g$, and then $S_{f,k}$, $\pi_{f,k}$, and $N_{f,k}$ for each bounding and secant lines $\mathcal{F}_k$.

We solved the LMIs in Corollaries 2 and 3 with $\alpha_1 = 1$, $\alpha_2 = 0$, $c_1 = 0.3$, and $c_3 = 1$. The role of constants $\epsilon_2$ and $\epsilon_3$ in Theorem 9 and Corollary 2 are technical, their positive values can be selected arbitrarily small. The value of $c_2$ is not relevant when $\alpha_2 = 0$. Furthermore, we minimized the values of $\tau_k$, $k = 1,\dots,6$ to enlarge the invariant level sets of $V$ as much as possible in each slice of $\mathcal{X}$. The computed a Lyapunov function and its invariant level set are presented in panel (b) of Fig. 1. The area of a connected subset of the unitary level set, which contains $x_*^{(1)}$ is 2.1350.

It is worth mentioning that the area of the simply connected subset of the $c = 0.99$ level set of the Motzkin polynomial (filled region in the negative orthant in panel (a) of Fig. 1) is 1.9908. This area tends to $\frac{\pi\sqrt{3}}{2} \simeq 2.7207$ as $c \to 1$.

In a higher-dimensional state-space, the polytope decomposition is a complex task, therefore, we illustrate how the shape of the computed function is altered when the polytope is not or only partly decomposed. In panel (c) of Fig. 1, the Lyapunov function is presented when polytope $\mathcal{X}$ is decomposed into two slices by the secant segment $\mathcal{F}_6$ in (47). In panel (d) of Fig. 1, the computed Lyapunov function is illustrated when $\mathcal{X}$ is not decomposed. Observe that both connected subsets of the $c_3$-level set of $V$ in panel (d) of Fig. 1 is a common invariant domain for two separate equilibria.

### *6.2. Genetic toggle switch in Escherichia coli*

Let us consider the dynamics of the genetic toggle switch in Escherichia coli [12]:

$$\dot{x}_1 = \frac{\mu_1}{1 + x_2^{\beta}} - x_1, \quad \dot{x}_2 = \frac{\mu_2}{1 + x_1^{\gamma}} - x_2, \tag{57}$$

[3] The Motzkin polynomial, as any other non-negative polynomial, is a sum of squares of rational functions [1]. The non-negativity of a polynomial $p(x_1,\dots,x_n)$ that is not an SOS can be proved by finding the smallest integer $r$ such that $(1 + x_1^2 + \ldots + x_n^2)^r p(x_1,\dots,x_n)$ is an SOS [49].

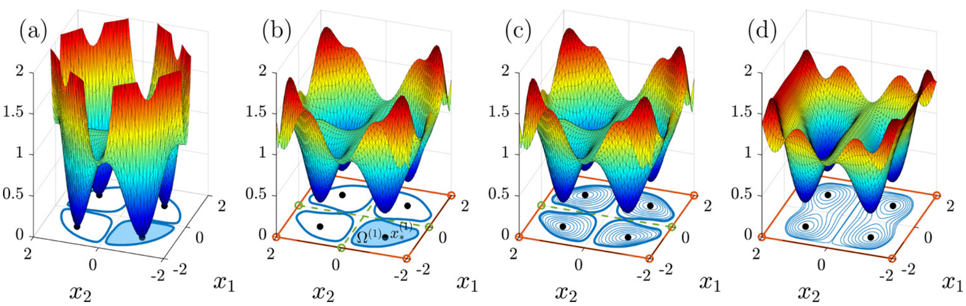


**Fig. 1.** Motzkin polynomial (46) and its $c = 0.99$ level set (a). Computed Lyapunov function for the gradient dynamics (46) and its $c = 1$ level set when: (b) polytope $\mathcal{X}$ is split into 4 slices corresponding to the four equilibria; (c) polytope $\mathcal{X}$ is split into 2 slices each containing two equilibria; (d) polytope $\mathcal{X}$ is not split at all. In (b)–(d), polytope $\mathcal{X}$ is illustrated by the red solid rectangle, whereas, the green segments constitute the secant segments $\mathcal{F}_5$ and $\mathcal{F}_6$. The filled blue region in panel (b) highlights the computed positively invariant domain $\Omega^{(1)}$ for $x_*^{(1)}$.

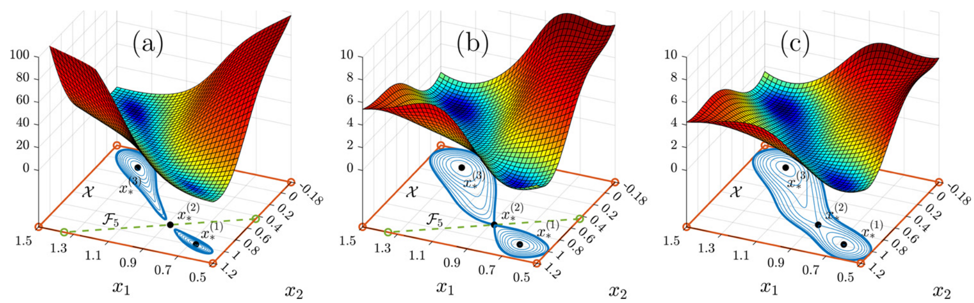


**Fig. 2.** Three different Lyapunov functions computed for system (57) with (b) and (c) and without (a) the boundary condition of (56). In all panels, the black dots are the equilibria, the blue contour lines are the level sets of the Lyapunov functions, the red rectangle is polytope $\mathcal{X}$. In panels (a) and (b) a Lyapunov function is computed for two polytope slices split by the secant segment $\mathcal{F}_5$ highlighted by the green dotted line. In panel (c), polytope $\mathcal{X}$ is not decomposed.

where $\mu_1 = 1.3$, $\mu_2 = 1$, $\beta = 3$, $\gamma = 10$. These dynamics are bistable with two locally asymptotically stable equilibrium points $x_*^{(1)}$ and $x_*^{(3)}$ separated by a separatrix through equilibrium $x_*^{(2)}$.

Let $\mathcal{X} = [0.5, 1.5] \times [-0.18, 1.2]$. Furthermore, we considered two slices $\mathcal{X}^{(1)} \cup \mathcal{X}^{(2)} = \mathcal{X}$, which are separated by the secant line $\mathcal{F}_5 = \{\}\lambda x_*^{(2)} \,\big|\, \lambda \in \mathbb{R} \cap \mathcal{X}$ (green dashed segment in Fig. 2).

In this example, the computational steps and the constants of Corollary 2 are the same as in Section 6.1. The structure of the Lyapunov function is determined by

$$\pi(x) = \begin{pmatrix} 1 & x_1 & x_2 & \frac{x_1}{q_1} & \frac{x_1^2}{q_1} & \frac{x_1^3}{q_1} & \frac{x_1^4}{q_1} & \frac{x_1^5}{q_1} & \frac{x_1^6}{q_1} & \frac{x_1^7}{q_1} & \frac{x_1^8}{q_1} & \frac{x_1^9}{q_1} & \frac{x_1^{10}}{q_1} & \frac{x_2}{q_2} & \frac{x_2^2}{q_2} & \frac{x_2^3}{q_2} \end{pmatrix}^\top,$$

where $q_1 = 1 + x_1^{10}$, $q_2 = 1 + x_2^3$.

First, we computed a Lyapunov function by solving the LMIs of Corollary 2 only. Secondly, a Lyapunov function is computed by considering Corollary 3, where the values of $\tau_k$, $k = 1, \dots, 5$ are minimized. Finally, we repeated the second computation without considering a decomposition for $\mathcal{X}$.

In Fig. 2, we illustrate the shape of both Lyapunov functions obtained. The outermost blue contour lines in Fig. 2 illustrate the forward invariant levels set of the Lyapunov functions.

#### 6.2.1. Comparative evaluation with the SOS approach

Observe that system (57) contains rational functions with a 3rd and 10th order denominator. Furthermore, we consider a rational Lyapunov function with a surprisingly high (28th and 26th) degree of numerator ($V_{\mathrm{num}}(x)$) and denominator ($V_{\mathrm{den}}(x) > 0$). This rich algebraic structure of the candidate function is obtained by only $m = 16$ distinct basis *rational* functions in $\pi$. To solve $L_f V(x) \le 0$ with polynomial optimization, we should multiply $L_f V(x)$ by its common denominator $(x_1^{10} + 1)^3 \cdot (x_2^3 + 1)^3$, which results in a 41st degree polynomial. The SOS decomposition of $V_{\mathrm{num}}(x)$ and $L_f V(x)$ comprise at least 59 and 101 distinct monomials, respectively, which were selected as proposed in Löfberg [27]. Due to the dense representation of both $V_{\mathrm{num}}(x)$ and $L_f V(x)$, the block diagonalization of their SOS decomposition is not possible. Therefore, the final conditions ensuring the non-negativity of $V(x)$ and $-L_f V(x)$ are 59 and 101-dimensional LMIs. In comparison, the dimensions of $\pi$ and $\pi_g$ in (41), and hence the dimension of LMIs in (53) are only $m = 16$ and $m_g = 41$, respectively.

### 6.3. The periodic ring

In this example, we compute a Lyapunov function for a variant of the periodic ring oscillator taken from Tedrake [56, Ex. 16.2]. The equations of the system in their general form can be written as:

$$\begin{cases} \dot{x}_1 = x_2 + x_1\, h(r), \\ \dot{x}_2 = -x_1 + x_2\, h(r), \end{cases} \tag{58}$$

where $r = \sqrt{x_1^2 + x_2^2}$. In this case study, we consider $h(r) = R_0^2 - r^2$, with $R_0 = 2$. The equations can be transformed into the polar coordinates as follows [48]:

$$\begin{cases} \dot{r} = r\, h(r), \\ \dot{\theta} = -1, \ \text{where } \theta = \operatorname{atan}_2(x_2, x_1). \end{cases} \tag{59}$$

The system equations in the form (59) allows us to guess a possible Lyapunov function candidate [56]:

$$\bar{V}(r) = \tfrac{1}{2}(R_0 - r)^2. \tag{60}$$

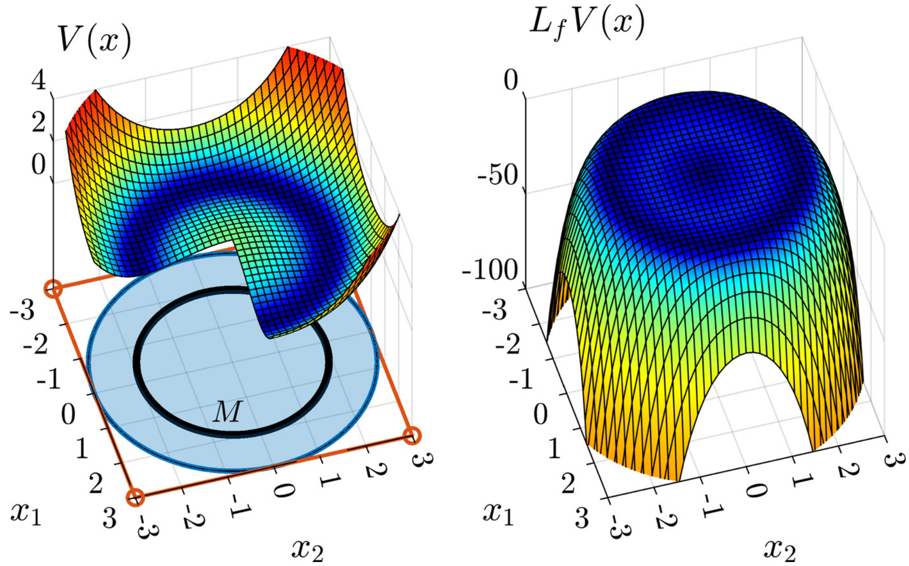


**Fig. 3.** Computed 4th degree polinomal Lyapunov function for the periodic ring system (left) and its Lie derivative (right). The red rectangle illustrates polytope $\mathcal{X}$, the solid black line is the limit cycle $M$, the light blue disc is the $c$-level set $\Omega_{c,M}$ of the Lyapunov function, where $c = 1.3919$. (As in Tedrake [56, Fig. 16.1], we removed a small segment from $V$ for the purposes of visualization).

The Lie derivative of function $\bar{V}$ w.r.t. $\bar{f}(r) = \begin{pmatrix} r\,h(r) \\ -1 \end{pmatrix}$ is

$$\begin{aligned} L_{\bar{f}}\bar{V}(r) &= -r(R_0 - r)\,h(r) \\ &= -r(R_0 - r)^2(R_0 + r), \end{aligned} \tag{61}$$

which is negative for all $r \in (0, \infty) \setminus \{R_0\}$.

In the following subsections, we present a possible synthetic construction of a Lyapunov function for system (58).

### 6.3.1. Synthetic Lyapunov function construction

In this example, we follow Remark 12 and consider a fourth degree Lyapunov function candidate (36) with the fixed set of monomials

$$\pi(x) = \psi_2(x) = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{pmatrix}. \tag{62}$$

Then, the Lie derivative of $V = \pi^\top P \pi$ w.r.t. (58) can be written in the form (41), where

$$A_g = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 8 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 \end{pmatrix},$$

$$E_g = \begin{pmatrix} I_6 & 0_{9\times 9} \end{pmatrix}, \;\; \pi_g = \psi_4 : \mathbb{R}^2 \to \mathbb{R}^{15}. \tag{63}$$

The canonical factorization for

$$\pi_d(x) = \begin{pmatrix} \pi(x) \\ L_f \pi(x) \end{pmatrix} = A_g \pi_g(x) \tag{64}$$

was performed by computer algebra manipulations of MATLAB's Symbolic Math Toolbox [33]. The symbolic computations follow the procedure proposed in Polcz et al. [45, Section 3]. To compute the maximal affine annihilators for $\pi$ and $\pi_g$, we applied Remark 6 with (K1).

Then, we solved the semidefinite feasibility problem formulated in Corollary 2 with $\mathcal{X} = [-3, 3] \times [-3, 3]$. In (53), (54), and (55), we used of $\alpha_1 = 1$, $\alpha_2 = 0$, $c_1 = 0.1$, and $c_3 = 1$. During the computations the upper-bound constraints of Corollary 3 along the facets $\mathcal{F}_k$ are not considered. We note that the feasibility problem with no objective has multiple solutions, and the computed solution depends on the implementation of the optimization solver.

In Fig. 3, we illustrate the computed Lyapunov function and its Lie derivative. Apparently, the circular "valley" of function $V$ follows the limit cycle.

### 6.4. Van der Pol oscillator

In this section, we demonstrate the operations on the Van der Pol system [40]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \mu\,(1 - x_1^2)\,x_2 - x_1, \end{cases} \tag{65}$$

where $\mu = 1$ in this case study.

We considered an 8th degree Lyapunov function, and the selected function $\pi = \psi_4$ comprise all the possible monomials of $x$ of degree 0 (constant) up to degree 4.

Then, we computed the canonical factorized form of $\pi_d(x) = A_g \pi_g(x)$ in (40), where $\pi_g = \psi_6$ comprise the monomials of degree 0 up to degree 6.

To find a Lyapunov function, first we fixed polytope

$$\mathcal{X} = [-4, 4] \times [-5, 5]. \tag{66}$$

We solved the semidefinite problem described in Corollary 2 with constants, $c_2 = 0.1$, and $c_3 = 1$.

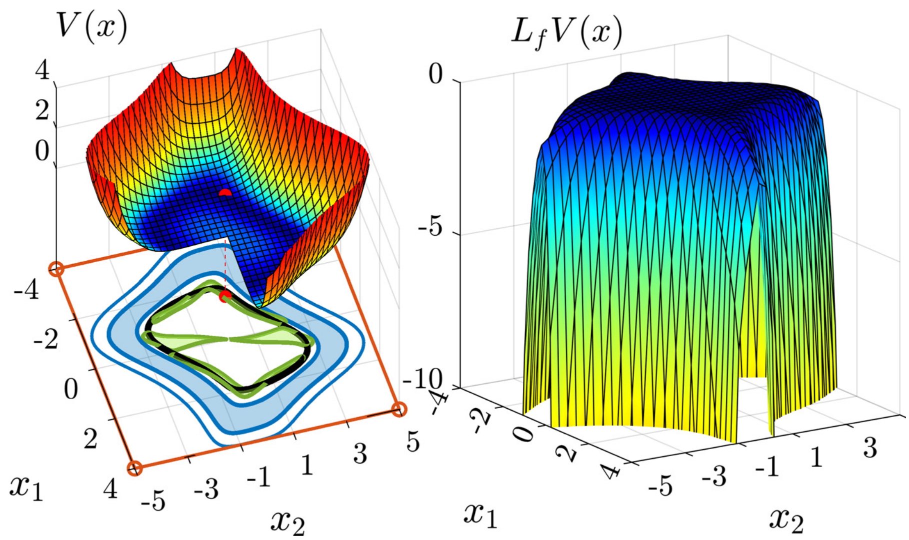


**Fig. 4.** Computed Lyapunov function $V$ for the Van der Pol system (65) and its Lie derivative $L_f V$. The red rectangle denotes polytope $\mathcal{X}$, the solid black line is the limit cycle, the outermost blue contour line illustrates $\mathcal{L}_M^{\sup}$, the maximal level set in $\mathcal{X}$, the shaded light blue region is $\Omega_{c_3,M} \setminus \Omega^{\circ}_{c_2,M}$, the red dot points out the minimum value of the Lyapunov function. The shaded light green regions highlight the set, in which the Lie derivative of the Lyapunov function is positive.

With $\alpha_1 = 0$, we neglect the central condition (C3), but with a nonzero $\alpha_2 = 0.1$, we relax the Lyapunov inequality. As we noted in Remark 10, this relaxation is necessary as the limit cycle of Van der Pol system does not have a closed-form expression [48].

The values of the Lyapunov function along the periodic orbit are between [0.0309,0.0386]. The maximal level set inside $\mathcal{X}$ belongs to $c_{\max} = 2.8612$ (Fig. 4). The minimum value of the computed Lyapunov function and the maximum value of its Lie derivative in $\mathcal{X}$ are

$$\begin{aligned} &V(-1.5765, 0.4412) = 0.0266, \\ &L_f V(1.3176, -0.7941) = 0.0069. \end{aligned} \tag{67}$$

### 6.5. The Lorenz system

Here, we illustrate the proposed approach on the well-known Lorenz system with a change of coordinates as proposed by Jones and Peet [23]. A tight attractive forward invariant domain (i.e., attractor) is computed both by Goluskin [18], Jones and Peet [23] with a 8th degree polynomial. In this case study, we do not address to find the minimum volume attracting cover of the Lorenz attractor, but demonstrate the operations of our approach, and compute a Lyapunov function in the form of a fraction of 6th and 4th degree polynomials.

We consider the dynamics of state variables $x_1 = z_1/50$, $x_2 = z_2/50$, and $x_3 = (z_3 - 25)/50$, where $z = (z_1, z_2, z_3)$ satisfies the following dynamic equations:

$$\begin{cases} \dot{z}_1 = \sigma(z_2 - z_1), \\ \dot{z}_2 = z_1(\rho - z_3) - z_2, \\ \dot{z}_3 = z_1 z_2 - \beta z_3. \end{cases} \tag{68}$$

where $\rho = 28$, $\sigma = 10$, and $\beta = 8/3$. The final dynamical model is as follows:

$$\begin{cases} \dot{x}_1 = \sigma\,(x_2 - x_1), \\ \dot{x}_2 = x_1\,(\rho - 50\,x_3 - 25) - x_2, \\ \dot{x}_3 = 50\,x_1 x_2 - \beta\,(x_3 + 0.5). \end{cases} \tag{69}$$

The system has a chaotic attractor, “to which almost every trajectory tends” [18]. To compute a Lyapunov function, we consider the following vector of rationals:

$$\pi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x)\, q^{-1}(x) \end{pmatrix}, \quad \text{where } q(x) = 1 + x^\top x. \tag{70}$$

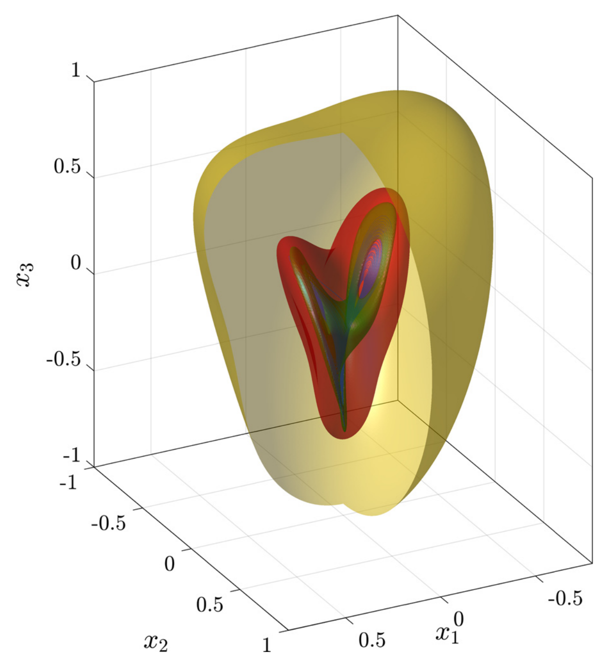


**Fig. 5.** Level sets of the Lyapunov function computed for the Lorenz system. The outermost yellow surface illustrates $\mathcal{L}_M^{\sup}$, the maximal level set in $\mathcal{X}$, whereas, the red surface is the boundary of $\mathcal{L}_M^{\inf}$. The inner green surface bounds the set where the Lie derivative is positive. The blue trajectory inside the red surface is the numerically approximated solution of the Lorenz system illustrating the shape of the strange attractor. The blue trajectory constitutes a numerically approximated solution of the Lorenz system.

Note that $q$ is the polynomial, which is advised by Reznick [49] to help proving non-negativity of polynomials that are not sums of squares of polynomials. It can be shown, that $\pi$ admits a preferred annihilator.

The quadratic decomposition (41) of the Lie derivative is computed as proposed in Remark 8. The factorizations (28) for the boundary LMIs (54), and the affine annihilators were computed numerically using samples (K2) and LFR operations as described in Remark 6. The dimensions of the final LMIs in (53) are 14 and 51, respectively, the dimension of the boundary LMI in (54) is 8.

We solved the semidefinite problem over

$$\mathcal{X} = [-0.8, 0.8] \times [-1, 1] \times [-1, 1], \tag{71}$$

with $\alpha_1 = 0$, $\alpha_2 = 8/3$ (as suggested by Goluskin [18]), and $c_3 = 1$. Furthermore, we were looking for the smallest possible value $c_2$, which turned to be 0.032. Though the value of $c_2$ was minimized through the optimization, its optimal value does not guarantee the least possible volume for $\mathcal{L}_M^{\text{inf}}$. The minimal (red) and maximal (yellow) invariant level set of the Lyapunov function, and the region where the Lie derivative is positive (green) are illustrated in Fig. 5. The volume of the minimal attracting level set (red) is approximately 0.3 cubic units.

## 7. Conclusions

In this paper, we formulated Lyapunov-type conditions to compute a common closed-form local Lyapunov function for multiple local (point-like, periodic, or strange) attractors of a nonlinear autonomous system. The presented method is a non-trivial extension of the DOA computation approaches of Polcz et al. [44,46], Trofino and Dezuo [59] with the novel numerical computational framework of Polcz et al. [43]. Instead of the classical Lyapunov stability concept Isidori [22, Thm. 10.1.3] that is used in Polcz et al. [44,46], Trofino and Dezuo [59], we considered the extended notion of a Lyapunov function introduced in Björnsson et al. [2].

Unlike the polynomial approaches, the use of LFT framework makes possible to cope with models in a form of fractions of polynomials and compute rational Lyapunov functions in a natural, efficient way. The method of corner points, the affine annihilators, and Finsler's lemma allow to formulate local LMI conditions for stability analysis. The approach is illustrated on four planar benchmark models. Differently from the state-of-the art SOS approaches [18,23], we computed a (6,4)-degree *rational* Lyapunov function for the Lorenz system with a specific denominator suggested by Reznick [49].

One drawback of the presented approach is that it requires a preliminary knowledge on the behavior of the system to select a polytope, in which the analysis is performed. A possible computational difficulty of the presented approach is the exponential growth of LMIs as the number of corner points increases exponentially with the number of state variables. In the case of a higher order system, it is suggested to use simple interval constraints on the state variables if necessary, and perform global analysis with respect to certain state variables when possible (Remark 9).

Further research will be focused on the computational stability analysis of semistable non-negative dynamical models (e.g., biochemical reaction networks) having a complex structure of equilibria. It is also motivating to study the possibilities how the advantageous features of the SOS methodology and the polytopic framework with Finsler's lemma can be combined.

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### CRediT authorship contribution statement

**Péter Polcz:** Conceptualization, Data curation, Formal analysis, Writing – original draft, Writing – review & editing, Validation. **Gábor Szederkényi:** Conceptualization, Data curation, Formal analysis, Writing – original draft, Writing – review & editing, Validation.

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### References

[1] E. Artin, Über die zerlegung definiter funktionen in quadrate, Abh. Math. Semin .Univ. Hambg. 5 (1) (1927) 100–115, doi:10.1007/bf02952513.
[2] J. Björnsson, P. Giesl, S.F. Hafstein, C.M. Kellett, and, Computation of Lyapunov functions for systems with multiple local attractors, Discrete Contin. Dyn. Syst. - A 35 (9) (2015) 4019–4039, doi:10.3934/dcds.2015.35.4019.
[3] J. Björnsson, P. Giesl, S. Hafstein, C.M. Kellett, H. Li, Computation of continuous and piecewise affine Lyapunov functions by numerical approximations of the Massera construction, in: 53rd IEEE Conference on Decision and Control, 2014, pp. 5506–5511, doi:10.1109/CDC.2014.7040250.
[4] J. Björnsson, S.F. Hafstein, Efficient Lyapunov function computation for systems with multiple exponentially stable equilibria, Procedia Comput. Sci. 108 (2017) 655–664, doi:10.1016/j.procs.2017.05.285. International Conference on Computational Science, ICCS 2017, 12–14 June 2017, Zurich, Switzerland
[5] R. Bobiti, M. Lazar, Automated-sampling-based stability verification and DOA estimation for nonlinear systems, IEEE Trans. Autom. Control 63 (11) (2018) 3659–3674, doi:10.1109/TAC.2018.2797196.
[6] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear matrix inequalities in system and control theory, Studies in Applied Mathematics, vol. 15, SIAM, Philadelphia, PA, 1994.
[7] G. Chesi, LMI techniques for optimization over polynomials in control: a survey, IEEE Trans. Autom. Control 55 (11) (2010) 2500–2510, doi:10.1109/TAC.2010.2046926.
[8] G. Chesi, Rational Lyapunov functions for estimating and controlling the robust domain of attraction, Automatica 49 (4) (2013) 1051–1057.
[9] D.F. Coutinho, M. Fu, A. Trofino, P. Danès, L2-gain analysis and control of uncertain nonlinear systems with bounded disturbance inputs, Int. J. Robust Nonlinear Control 18 (1) (2008) 88–110, doi:10.1002/rnc.1207.
[10] R. D'Andrea, S. Khatri, Kalman decomposition of linear fractional transformation representations and minimality, in: Proceedings of the 1997 American Control Conference (Cat. No. 97CH36041), volume 6, Albuquerque, New Mexico, USA, 1997, pp. 3557–3561, doi:10.1109/ACC.1997.609484. vol.6
[11] D.S. De Madeira, J. Adamy, Output feedback control of rational nonlinear systems: a new approach based on passivity indices, in: 2016 IEEE 55th Conference on Decision and Control (CDC), Las Vegas, NV, USA, 2016, pp. 3880–3885, doi:10.1109/CDC.2016.7798855.
[12] A.I. Doban, M. Lazar, Computation of Lyapunov functions for nonlinear differential equations via a Massera-type construction, IEEE Trans. Autom. Control 63 (5) (2018) 1259–1272, doi:10.1109/TAC.2017.2736961.
[13] Y. Ebihara, D. Peaucelle, D. Arzelier, S-variable Approach to LMI-Based Robust Control, Springer, 2015.
[14] L. El Ghaoui, G. Scorletti, Control of rational systems using linear-fractional representations and linear matrix inequalities, Automatica 32 (9) (1996) 1273–1284, doi:10.1016/0005-1098(96)00071-4.
[15] G. Fantuzzi, D. Goluskin, D. Huang, S.I. Chernyshenko, Bounds for deterministic and stochastic dynamical systems using sum-of-squares optimization, SIAM J. Appl. Dyn. Syst. 15 (4) (2016) 1962–1988, doi:10.1137/15M1053347.
[16] R. Goebel, R.G. Sanfelice, A.R. Teel, Hybrid Dynamical Systems: Modeling, Stability, and Robustness, Princeton University Press, 2012.
[17] D. Goluskin, Bounding averages rigorously using semidefinite programming: mean moments of the Lorenz system, J. Nonlinear Sci. 28 (2) (2017) 621–651, doi:10.1007/s00332-017-9421-2.
[18] D. Goluskin, Bounding extrema over global attractors using polynomial optimisation, Nonlinearity 33 (9) (2020) 4878–4899, doi:10.1088/1361-6544/ab8f7b.
[19] S.F. Hafstein, An algorithm for constructing Lyapunov functions, Electron. J. Differ. Equ. 08 (2007) 1–100. Monograph 08. URL https://ejde.math.txstate.edu/Monographs/08/hafstein.pdf.
[20] J. Hauser, C.C. Chung, Converse Lyapunov functions for exponentially stable periodic orbits, Syst. Control Lett. 23 (1) (1994) 27–34, doi:10.1016/0167-6911(94)90078-7.
[21] S. Hecker, A. Varga, Symbolic techniques for low order LFT-modelling, IFAC-PapersOnline 16 (1) (2005) 523–528, doi:10.3182/20050703-6-CZ-1902.01032. 16th IFAC World Congress
[22] A. Isidori, Nonlinear Control Systems II, Springer-Verlag London, London, UK, 1999, doi:10.1007/978-1-4471-0549-7.
[23] M. Jones, M.M. Peet, Using SOS for optimal semialgebraic representation of sets: finding minimal representations of limit cycles, chaotic attractors and unions, in: 2019 American Control Conference (ACC), 2019, pp. 2084–2091, doi:10.23919/ACC.2019.8814848.

[24] C.M. Kellett, Classical converse theorems in Lyapunov's second method, Discrete Contin. Dyn. Syst. - Ser. B 20 (8) (2015) 2333–2360, doi:10.3934/dcdsb.2015.20.2333.
[25] H.K. Khalil, Nonlinear Systems, Pearson Education, Prentice Hall, 2002.
[26] J. Löfberg, YALMIP : a toolbox for modeling and optimization in MATLAB, in: Proceedings of the CACSD Conference, Taipei, Taiwan, 2004, doi:10.1109/CACSD.2004.1393890.
[27] J. Löfberg, Pre- and post-processing sum-of-squares programs in practice, IEEE Trans. Autom. Control 54 (5) (2009) 1007–1011, doi:10.1109/TAC.2009.2017144.
[28] J.-F. Magni, Linear fractional representation toolbox (version 2.0) for use with Matlab, 2006a,
[29] J.-F. Magni, User manual of the Linear Fractional Representation Toolbox: version 2.0, 2006b.
[30] I.R. Manchester, Transverse dynamics and regions of stability for nonlinear hybrid limit cycles, IFAC Proc. Vol. 44 (1) (2011) 6285–6290, doi:10.3182/20110828-6-IT-1002.03063. 18th IFAC World Congress
[31] I.R. Manchester, M.M. Tobenkin, M. Levashov, R. Tedrake, Regions of attraction for hybrid limit cycles of walking robots, IFAC Proc. Vol. 44 (1) (2011) 5801–5806, doi:10.3182/20110828-6-IT-1002.03069. 18th IFAC World Congress
[32] J.L. Massera, On Liapounoff's conditions of stability, Ann. Math. 50 (3) (1949) 705, doi:10.2307/1969558.
[33] MathWorks, Symbolic Math Toolbox: User's guide (r2020b), 2020. Mathworks.
[34] MOSEK ApS, The MOSEK optimization toolbox for MATLAB manual. version 7.1 (revision 28), 2015. http://docs.mosek.com.
[35] T.S. Motzkin, The arithmetic-geometric inequality, in: Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965), 1967, pp. 205–224.
[36] M.C. de Oliveira, R.E. Skelton, Perspectives in Robust Control, Springer, 2001, pp. 241–257.
[37] A. Papachristodoulou, Scalable Analysis of Nonlinear Systems Using Convex Optimization, California Institute of Technology, 2005 Ph.D. thesis.
[38] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, P.A. Parrilo, SOSTOOLS: sum of squares optimization toolbox for MATLAB, 2013. http://arxiv.org/abs/1310.4716, Available from http://www.eng.ox.ac.uk/control/sostools.
[39] A. Papachristodoulou, S. Prajna, On the construction of Lyapunov functions using the sum of squares decomposition, in: Proceedings of the 41st IEEE Conference on Decision and Control, 2002., vol. 3, 2002, pp. 3482–3487 vol.3, doi:10.1109/CDC.2002.1184414.
[40] B. van der Pol, On relaxation-oscillations, Lond., Edinb., Dublin Philos. Mag. J. Sci. 2 (11) (1926) 978–992, doi:10.1080/14786442608564127.
[41] P. Polcz, Computational Analysis of Nonlinear Uncertain Systems, Pázmány Péter Catholic University, Faculty of Information Technology and Bionics, 2020 Ph.D. thesis. Available on-line at http://polcz.itk.ppke.hu/files/docs/PhD_dissertation
[42] P. Polcz, Lyapunov function computation for local attractors, 2021, (GitHub repository, https://github.com/ppolcz/LF_for_local_attractors).
[43] P. Polcz, T. Péni, B. Kulcsár, G. Szederkényi, Induced L2-gain computation for rational LPV systems using Finsler's lemma and minimal generators, Syst. Control Lett. 142 (2020) 104738, doi:10.1016/j.sysconle.2020.104738.
[44] P. Polcz, T. Péni, G. Szederkényi, Improved algorithm for computing the domain of attraction of rational nonlinear systems, Eur. J. Control 39 (2017) 53–67, doi:10.1016/j.ejcon.2017.10.003.
[45] P. Polcz, T. Péni, G. Szederkényi, Reduced linear fractional representation of nonlinear systems for stability analysis, IFAC-PapersOnLine 51 (2) (2018) 37–42, doi:10.1016/j.ifacol.2018.03.007. 9th Vienna International Conference on Mathematical Modelling
[46] P. Polcz, T. Péni, G. Szederkényi, Computational method for estimating the domain of attraction of discrete-time uncertain rational systems, Eur. J. Control 49 (2019) 68–83, doi:10.1016/j.ejcon.2018.12.004.
[47] S. Prajna, A. Papachristodoulou, F. Wu, Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach, in: 2004 5th Asian Control Conference (IEEE Cat. No. 04EX904), vol. 1, 2004, pp. 157–165Vol.1.
[48] Private communication with Prof. Barnabás Garay, full professor at Pázmány Péter Catholic University, Faculty of Information Technology and Bionics. 2021,
[49] B. Reznick, Uniform denominators in Hilbert's seventeenth problem, Math. Z. 220 (1) (1995) 75–97, doi:10.1007/bf02572604.
[50] S. Rozgonyi, K.M. Hangos, G. Szederkényi, Determining the domain of attraction of hybrid non-linear systems using maximal Lyapunov functions, Kybernetika 46 (2010) 19–37. 10338.dmlcz/140048
[51] M. Sato, D. Peaucelle, Comparison between SOS approach and slack variable approach for non-negativity check of polynomial functions: multiple variable case, in: 2007 European Control Conference (ECC), 2007a, pp. 3016–3025, doi:10.23919/ECC.2007.7069009.
[52] M. Sato, D. Peaucelle, Comparison between SOS approach and slack variable approach for non-negativity check of polynomial functions: single variable case, in: 2007 American Control Conference, 2007b, pp. 6139–6146, doi:10.1109/ACC.2007.4282698.
[53] A. Shiriaev, L. Freidovich, I. Manchester, Can we make a robot ballerina perform a pirouette? Orbital stabilization of periodic motions of underactuated mechanical systems, Annu. Rev. Control 32 (2) (2008) 200–211, doi:10.1016/j.arcontrol.2008.07.001.
[54] E. Sidorov, M. Zacksenhouse, Lyapunov based estimation of the basin of attraction of Poincare maps with applications to limit cycle walking, Nonlinear Anal. 33 (2019) 179–194, doi:10.1016/j.nahs.2019.03.002.
[55] G. Stengle, A nullstellensatz and a positivstellensatz in semialgebraic geometry, Math. Ann. 207 (2) (1974) 87–97, doi:10.1007/BF01362149.
[56] R. Tedrake, Underactuated robotics: algorithms for walking, running, swimming, flying, and manipulation (Course notes for MIT 6.832), 2021, Downloaded on 2021-03-02 from http://underactuated.mit.edu/limit_cycles.html.
[57] I. Tobasco, D. Goluskin, C.R. Doering, Optimal bounds and extremal trajectories for time averages in nonlinear dynamical systems, Phys. Lett. A 382 (6) (2018) 382–386, doi:10.1016/j.physleta.2017.12.023.
[58] U. Topcu, A.K. Packard, P. Seiler, Local stability analysis using simulations and sum-of-squares programming, Automatica 44 (10) (2008) 2669–2675, doi:10.1016/j.automatica.2008.03.010.
[59] A. Trofino, T.J.M. Dezuo, LMI stability conditions for uncertain rational nonlinear systems, Int. J. Robust Nonlinear Control 24 (18) (2013) 3124–3169, doi:10.1002/rnc.3047. Cited By 14
[60] G. Valmorbida, J. Anderson, Region of attraction estimation using invariant sets and rational Lyapunov functions, Automatica 75 (2017) 37–45, doi:10.1016/j.automatica.2016.09.003.
[61] A. Vannelli, M. Vidyasagar, Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems, Automatica 21 (1) (1985) 69–80, doi:10.1016/0005-1098(85)90099-8.
[62] F. Wu, S. Prajna, SOS-based solution approach to polynomial LPV system analysis and synthesis problems, Int. J. Control 78 (8) (2005) 600–611, doi:10.1080/00207170500114865.
[63] V.I. Zubov, L.F. Boron, Methods of A. M. Lyapunov and Their Application, Noordhoff Groningen, 1964.