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Filling in pattern designs for incomplete pairwise comparison matrices: (Quasi-)regular graphs with minimal diameter☆

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ABSTRACT

Pairwise comparisons have become popular in the theory and practice of preference modelling and quantification. In case of incomplete data, the arrangements of known comparisons are crucial for the quality of results. We focus on decision problems where the set of pairwise comparisons can be chosen and it is designed completely before the decision making process, without any further prior information. The objective of this paper is to provide recommendations for filling patterns of incomplete pairwise comparison matrices based on their graph representation. The proposed graphs are regular and quasi-regular ones with minimal diameter (longest shortest path). Regularity means that each item is compared to others for the same number of times, resulting in a kind of symmetry. A graph on an odd number of vertices is called quasi-regular, if the degree of every vertex is the same odd number, except for one vertex whose degree is larger by one. We draw attention to the diameter, which is missing from the relevant literature, in order to remain the closest to direct comparisons. If the diameter of the graph of comparisons is as low as possible (among the graphs of the same number of edges), we can decrease the cumulated errors that are caused by the intermediate comparisons of a long path between two items. Contributions of this paper include a list containing (quasi-)regular graphs with diameter 2 and 3 up until 24 vertices. Extensive numerical tests show that the recommended graphs indeed lead to better weight vectors compared to various other graphs with the same number of edges. It is also revealed by examples that neither regularity nor small diameter is sufficient on its own, both properties are needed. Both theorists and practitioners can utilize the results, given in several formats in the appendix: plotted graph, adjacency matrix, list of edges, 'Graph6' code.

1. Introduction

Pairwise comparisons form the basis of preference measurement, ranking, psychometrics and decision modelling [1–3]. Multicriteria Decision Making is indeed an important tool both at an individual and at an organizational level. We can think about different kind of ranking of alternatives or weighting of criteria, like tenders, selection among schools or job offers, selection among the implementation of different projects in an enterprise, etc.

One of the most commonly used techniques in connection with Multi-criteria Decision Making is the method of the pairwise comparison matrices. One can apply this technique both for determining the weights of the different criteria and for the rating of the alternatives according to a criterion. Usually we denote the number of criteria or alternatives by n, which means the pairwise comparison matrix is an $n \times n$ matrix, often denoted by A. In this case the ij-th element of the A matrix, a_{ij} shows how many times the i-th item is larger/better than the j-th element.

Formally, matrix A is called a pairwise comparison matrix (PCM) if it is positive ($a_{ij} > 0$ for $\forall$ i and j) and reciprocal ($1/a_{ij} = a_{ji}$ for $\forall$ i and j) [4], which also indicates that $a_{ii} = 1$ for $\forall$ i.

Dealing with incomplete data gets more and more attention in the literature. When some elements of a PCM are missing we call it an incomplete PCM. There could be many different reasons why these elements are absent, some data could have been lost or the comparisons are simply not possible (for instance in sports [5]).

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The most interesting case for us is when the decision makers do not have time, willingness or the possibility to make all the $n(n-1)/2$ comparisons.

In this article we would like to study which comparisons should be made, or more precisely what patterns of comparisons are recommended in order to get good approximation of the decision makers' preferences calculated from the whole set of comparisons. The graph representation of the pairwise comparisons is a natural and convenient tool to examine our question, thus we will use this throughout the paper.

In many cases the set of comparisons can be adaptive, i.e., the next questions depend on the answers to the previous ones as in, e.g., [6–8]. However, we assume in the paper that the whole set of comparisons is designed completely before the decision making process, and we do not have any further prior information about the items to be compared. Thus the 'confidence level' of every single comparison is the same in our problems, the probability of their 'errors' is identical. For instance the (pre-)compilation of questionnaires in connection with decision making problems can be named as an indeed common practical example that satisfies these conditions.

There are already known special structures proposed for incomplete pairwise comparison matrices in the literature, which include:

(i) spanning tree, in particular if one row/column is filled in completely (its associated graph is the star graph)

(ii) two rows/columns are filled in completely (its associated graph is the union of two star graphs) [9]

(iii) a method of 2-cyclic designs, the union of two edge-disjoint n-cycles, has been also recommended to select $2n$ paired comparisons from n number of objects [10]

(iv) more or less regular graphs, for example the regularity of the comparisons' graph appears in the designs of [11] and [12].

Regularity results in a kind of symmetry that is also desirable in case of sport competitions [13], where the number of matches played equals for every player or team, at least in the first phase (before the knockout stages). This also appears in other sport tournaments, where they use the so-called Swiss system, in which besides a lot of other requirements, every player or team plays the same number of matches (if possible) [14–16]. Thus the resulting representing graph of the comparisons is regular [17].

A special type and extension of regular graphs is considered by [18]. They proposed the (quasi-)strongly regular designs based on (quasi-)strongly regular graphs in order to select pairs to be compared within incomplete information. A graph is called strongly regular with parameters (n, k, λ, μ), if each of the n vertices has degree k, and (i) for any pair of adjacent vertices u and v, the number of vertices adjacent to both u and v is λ; (ii) for any pair of not adjacent vertices u and v, the number of vertices adjacent to both u and v is μ. Since these properties are rather restrictive, a linear algebraic generalization, the so called quasi-strongly regular graphs are also taken into consideration. By simulation, they showed that both designs give better results (based on a logarithmic distance function defined on the weight vectors) than other random designs of the same cardinality.

[19] create an incompleteness index based on the number of missing pairwise comparisons and their arrangements. Using different kind of Monte Carlo simulations they conclude that inconsistency and incompleteness both have crucial effect on sensitivity, and the regularity of the PCM also has a huge effect both on the quantitative and the qualitative results.

Note that the first three examples above lack regularity. Regularity means that each item is compared to others for the same number of times (if the cardinality of the items to compare is odd, one of the degrees can be smaller or greater – in our analysis, greater – by one), resulting in a kind of symmetry, as we mentioned earlier. Despite the fact that regularity has been recognized as an important property in connection with the representing graph of the comparisons, the above-mentioned examples do not examine it as generally as we do, their definitions on regularity is more restrictive and their instances are less systematic.

Diameter, the other key concept of the paper besides regularity, shows how far items can be from each other in the sense that how many comparisons are needed in order to have an indirect comparison between them. The well known telephone game or effect [20], also known as The Whisper Game [21] shows small errors are cumulated along a sufficiently long series. If a message passes through a line of people, in a whisper, the original and the final versions differ a lot, despite the neighboring versions are usually quite similar. A classical example for the non-transitivity of indifference [22] is the addition of very small portions of sugar to the same cup of coffee. No one can distinguish between two consecutive steps, however, if this sequence is long enough, the indifference disappears [23].

In the set of connected graphs, diameter can be considered as a measure of closeness, or a stronger type of connectedness. It is not properly studied in the literature, however, for instance in [24] the estimation of the matrix of comparison probabilities is investigated for several graph structures and some research questions, e.g., on a possible relation of the graph's diameter and the worst-case approximation error, are raised. One of our notable findings is to determine the diameter of the representing graph as a crucial property for filling in pattern designs of incomplete PCMs.

Note that regular graphs can have large diameter, e.g., a cycle on n vertices is 2-regular and has diameter $d = \lfloor n/2 \rfloor$. The star graph, mentioned among the examples, has minimal diameter 2, but it is far from being regular. Our aim is to find the graphs, among (quasi-)regular ones, with minimal diameter. We are especially interested in the smallest nontrivial values of the diameter, namely $d = 2$ and $d = 3$. Intuition suggests, and it is confirmed by the graphs found, that for a fixed n, higher regularity, i.e., more edges, makes the diameter smaller.

The rest of the paper is structured as follows. Basic mathematical concepts are introduced in Section 2. Later on we assume that we know the number n of alternatives or criteria, it is also a key assumption through our paper that the graph representing the MCDM problem is k-(quasi-)regular and we also know (or with the help of the other inputs we can determine) the diameter d of the graph. In Section 3 (which is complemented by Appendix A (online)) we provide a systematic collection of suggested incomplete pairwise comparisons' patterns with the help of the above-mentioned inputs and all/some graphs for the examined cases. We would like to emphasize that this list is a major contribution of our paper. Section 4 presents a motivational example showing that the diameter of a regular graph can be large and the result can be very sensitive to the errors of the matrix elements. A wide range of numerical simulations, using the distances of the weights computed with different filling in patterns respect to the weights calculated from the complete PCMs, is also provided in order to validate our recommendations. Finally, Section 5 concludes and provides further research questions closely connected to the discussed topic. Results of Sections 3 and 4 are given in more details in the appendices. B (online) includes the recommended graphs themselves. For practitioners, this list might serve as a 'recipe' in designing questionnaires based on pairwise comparisons. Appendix D (online) includes the results of the comparisons of weight vectors calculated from the different graphs. Appendices A, B, C and D can be found in the online supplementary material.

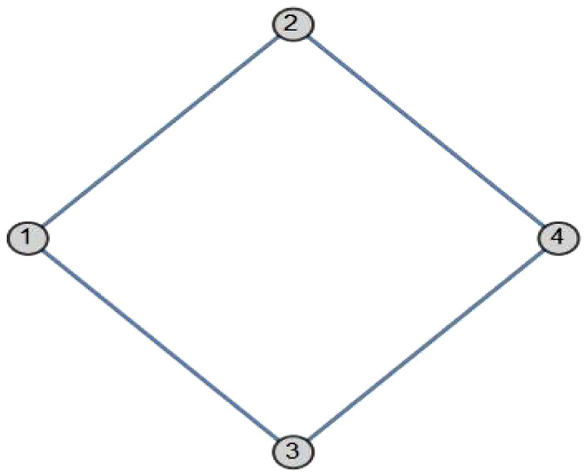


Fig. 1. Graph representation example

2. Basic concepts of the graph representation

The graph representation of paired comparisons has already been used in the 1940s [25]. Of course after the widespread application of PCMs and incomplete PCMs it has become a common method in the literature, see for instance [26], [27] or [28].

Usually in these articles the authors use directed graphs for the representation, because they distinguish the preferred item from the less preferred one in every pair. In our approach the only important thing is whether there exists a comparison between the two elements. This means that we use undirected graphs, where the vertices denote the criteria or the alternatives. There is an edge between two vertices if and only if the decision makers made their comparison for the two respective items (the appropriate element of the PCM is known). In order to understand the concepts so far, there is a small example below:

Example 1. Let us assume that there are 4 criteria ($n = 4$) and our decision maker already answered some questions, denoted their locations in the matrix by $\bullet$ and their reciprocal values by $\circ$, which lead to the following incomplete PCM:

$$A = \begin{bmatrix} 1 & \bullet & \bullet & \\ \circ & 1 & & \bullet \\ \circ & & 1 & \bullet \\ & \circ & \circ & 1 \end{bmatrix}$$

This incomplete PCM is represented by the graph in Figure 1.

As we can see there is no edge between the first and the fourth vertices, where the PCM has missing values and there is no edge between the second and third vertices, where the situation is the same. There is an edge between every other pair, where we have no missing values in the PCM.

It is important to emphasize that as the known elements of the PCM determine the representing graph, it is also true in the other way around. Thus, the graph in Figure 1 shows which comparisons are known in the PCM. This is the key property that we use in this paper, as we present the representing graphs that show the filling in patterns, the comparisons that should be made. We assume that the representing graphs are connected and k-(quasi-)regular through our paper, thus we need some definitions to make these concepts clear.

Definition 1 (Connected graph)**.** In an undirected graph, two vertices u and v are called connected if the graph contains a path from u to v. A graph is said to be connected if every pair of vertices in the graph is connected.

Definition 2. (***k*-regular graph**) A graph is called k-regular if every vertex has k neighbours, which means that the degree of every vertex is k.

Definition 3. (***k*-quasi-regular graph**) A graph is called k-quasi-regular if exactly one vertex has degree $k+1$, and all the other vertices have degree k.

The k-regularity basically means that the vertices are not distinguished, there is no particular vertex as, for example, in the case of the star graph, thus we would like to avoid the cases when the elimination of relatively few vertices would lead to the disintegration of the whole comparison system [29]. Besides regularity, the connectedness of the representing graph is indeed important, because to approximate the decision makers' preferences well, we need to have at least indirect comparisons between the different criteria, otherwise we cannot say anything about the relation between certain elements [30].

However, it is also notable that we would like to avoid the cases when two items are compared only indirectly through a very long path, because this could aggregate the small, tolerable errors of the different comparisons and we could end up with an intolerably large error in the relation between the two elements. Such an example was found in [29], where the graph generated from the table tennis players matches included a long shortest path between two vertices (players), and the calculated result appeared to be misleading. The diameter of the representing graph is a very suitable mathematical tool to measure this problem:

Definition 4 (The diameter of a graph)**.** The diameter (denoted by d) of a graph G is the length of the longest shortest path between any two vertices:

$$d = \max_{u,v \in V(G)} \ell(u,v),$$

where $V(G)$ denotes the set of vertices of G and $\ell(.,.)$ is the graph distance between two vertices, namely the length of the shortest path between them.

We also define here the twisted product, a graph construction method that is used by us extensively to find the proposed graphs:

Definition 5 (Twisted product of two graphs)**.** ([31])

Let $G = (V, E)$ and $G' = (V', E')$ be two undirected graphs, where V and V' are the vertex sets, while E and E' are the edge sets of the respective graphs. Let $\overrightarrow{E}$ denote the set of arcs in an arbitrary orientation of G. For each arc $(i, j) \in \overrightarrow{E}$, let $\pi_{(i,j)}$ be a one to one mapping from V' to itself. The twisted product of graphs G and G', denoted by $G * G'$ is defined as follows: its vertex set is the Cartesian product $V \times V'$, and there is an edge between vertices (i, i') and (j, j') if either $[i = j$ and $(i', j') \in E']$ or $[(i, j) \in \overrightarrow{E}$ and $j' = \pi_{(i,j)}(i')]$.

Note that the twisted product with π =identity results in the Cartesian product.

Briefly from now on we will examine graphs representing MCDM problems defined by the following inputs: (n, k, d), where n is the number of vertices (criteria), k shows the level of regularity of the graph and d is the diameter of the graph.

3. (Quasi-)regular graphs with minimal diameter

In this section, we present one of the most important findings of the paper, the examined (quasi-)regular graphs themselves. First of all, it is a key step to determine which cases are interesting for us considering our inputs. It is important to emphasize that we deal with unlabelled graphs, because we are trying to find out what kind of patterns are needed in the comparisons for different instances. Thus if we exchange the 'names' of two criteria (for instance '1' and '2' in Example 1) the pattern would be the same.

Then we can consider the regularity parameter k. The $k=1$ case is possible only when n is even, but they are not connected except for $n=2$, so this is not interesting for us. When $k=2$ there is only one connected graph for every n, namely the cycle, for which $d=\lfloor n/2 \rfloor$ as already mentioned in the introduction.

The larger regularity parameters could be interesting, but of course we need a reasonable upper bound for the number of criteria, n, which is also an indirect upper bound for k. In our research we examined the $n=5,6,\dots,24$ cases, because on the one hand for larger n parameters, some computations become very difficult, and on the other hand the largest 5-regular graph with diameter 2 contains 24 vertices, so this is a nice theoretical bound, as well. It is also true that in the majority of the fields of application it is sufficient to examine the number of alternatives (vertices) up until 24.

The smaller the d parameter is, the more stable or trustworthy our system of comparisons is. This means that in an optimal case we would like to minimize this parameter, while the number of the criteria (n) is always a fixed exogenous parameter in our MCDM problems. As we mentioned above, k is crucial to avoid the cases when some criteria (vertices) would be too important in the system, however it also shows us how many comparisons have to be made, because every vertex has a degree of k, which means the number of edges is $nk/2$. Thus if our decision makers would like to spend the shortest time with the creation of the PCM, we should choose a small k parameter. But, of course, as usually happens in these situations, there is a trade off between the parameters, because for many criteria (large n) the smaller regularity (k) will cause a bigger diameter (d), namely, a more fragile system of comparisons.

In this paper we would like to provide a list of graphs which shows the patterns of the comparisons that have to be made in case of different parameters. We used computational and constructing methods to determine the graph(s) with the smallest diameter (d parameter) for a given (n,k) pair. With the help of these results it was easy to determine which k is the smallest that is needed to reach a given d for a given n. We found that, with the chosen upper bound of n (24) the interesting values for the regularity are $k=3,4,5$, while the interesting values for the diameter of the graph are $d=2,3$. Of course $d=1$ would mean a complete graph that is not reachable for many (n,k) pairs, and it represents a complete PCM, thus it is not interesting for us. For a general MCDM problem probably instead of k, it would give more information if we considered an indicator that shows how far we are from the 'extreme' case, when the decision makers have to make all the comparisons. This would mean $n(n-1)/2$ comparisons instead of our $nk/2$ in case of regular graphs or $(nk+1)/2$ in case of quasi-regular graphs, therefore the completion ratio is defined as follows:

$$c=\begin{cases}\frac{nk/2}{n(n-1)/2} & \text{if n or k is even}\\ \frac{(nk+1)/2}{n(n-1)/2} & \text{if n and } k \text{ are odd}\end{cases}$$

that we will calculate for every instance.

Here we will present the graphs with the smallest diameter for a given (n,k) pair, it is important to emphasize that it is recommended to read this section together with Appendix A, as a large part of our list (Tables 2, A1a, A1b, A2, A3, A4a, and A4b) takes place there, because of the length of the tables. The finding for the different graphs in our list consisted of several methods, sources and layers:

1. As a starting reference point, we checked the built in graphs in Wolfram Mathematica [32], which are even complete catalogues in case of small number of vertices, thus we selected the ones with minimal diameter among them.
2. For smaller and middle-sized graphs, when Mathematica's built in examples cover only a sample of the cases, we used nauty and Traces [33] and IGraph/M [34] to generate all the possible (quasi-)regular graphs and select the needed ones.
3. Our results contain many well known graphs as well, like the Petersen graph [35], that we collected from different kind of articles indicated in the respective tables as 'Source'. We also collected further information, like uniqueness, about those graphs that we got with the help of Mathematica and are well known cases. We cite these information as 'See also' in our tables.
4. For larger graphs we were not able to generate all the possible regular cases, thus we used several construction techniques such as the twisted product, integer linear programming or merging and extending methods with the help of some already known graphs. Many of these cases were challenging and time-consuming to find, the same idea rarely worked twice.
5. It is also important that as k-quasi-regularity was defined by us, all of the quasi-regular graphs are our findings (or at least we are the first to use them in this kind of context), but we do not denote this separately in the tables.

Table 1
The summary of our list of graphs: the different sets of graphs based on the regularity level k and the number of vertices n can be found in the indicated tables, from which Table 2 can be found in the main text, while the other tables take place in Appendix A in the supplementary material. Lightgray denotes $d=2$ and gray denotes $d=3$.

	k		
n	3	4	5
$n=5,\dots,10$	Table 2		
$n=11,\dots,15$	Table A1a	Table A2	
$n=16,\dots,20$	Table A1b		Table A4a
$n=21,\dots,24$		Table A3	Table A4b

Table 1 presents a table of tables that provides an overview of our list of graphs.

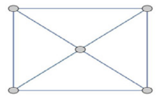
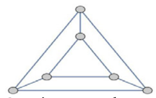
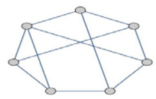
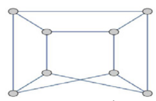
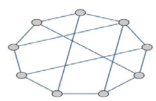
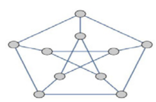
Table 2 shows the cases when $k=3$ and $d=2$ is the minimal value of the parameter. It is important to note that $k=3$ is only possible when n is even, but when it is odd, we examine 3-quasi-regular graphs, where all vertices have degree 3 except one where it has 4, because these are the closest to 3-regularity.

We can see that with $k=3$ the minimal diameter can be 2 until we have 10 vertices. Of course for $n\le 3$ the 3-regularity is not possible, and for $n=4$ the diameter is 1, because this is a complete graph, that is why we skip those in the table. It is also notable that the completion ratio (c) even reaches 1/3 when we have 10 vertices (it is obviously decreasing in n). We should emphasize the fact that there are only a few graphs for every (n,k) pair with the minimal diameter. Some of them are bipartite graphs, which have special spectral properties [27, Lemma 4, Theorem 2, Proposition 3], and they also indicate that there are two groups which are always compared through the other ones.

If we go on to larger graphs ($n>10$), then we will find that the smallest reachable diameter changes to $d=3$, but it is also true that at first we have so many graphs that satisfy these properties. However, as we examine the $n=18$ or the $n=20$ cases, we can see that there is only one graph that fulfils our assumptions [36]. The results in case of larger graphs, with 3-regularity and 3 as the minimal diameter can be found in Tables A1a and A1b in Appendix A.

As we can see the completion ratio is still decreasing in n and on larger graphs it can be taken below 0.2. It is also true that we

Table 2
$k = 3$-(quasi-)regular graphs on n vertices with minimal diameter $d = 2$.

$k = 3$	Graph	Further information
$n = 5$		• $c = 8/10 = 0.8$ • Unique graph
$n = 6$	 3-prism graph ($C_3 \times K_2$)	• $c = 9/15 = 0.6$ • 2 graphs • Source: [36] • The other solution is the bipartite graph $K_{3,3}$
$n = 7$		• $c = 11/21 \approx 0.524$ • 4 graphs
$n = 8$	 Wagner graph	• $c = 12/28 \approx 0.429$ • 2 graphs • See also: [37] • The other solution is the X_8 graph [31]
$n = 9$		• $c = 14/36 \approx 0.389$ • 2 graphs
$n = 10$	 Petersen graph	• $c = 15/45 \approx 0.333$ • Unique graph • See also: [38]

do not need to answer for more than 30 questions for an MCDM problem even with 20 criteria, which can be indeed useful.

If we go on to larger graphs, the minimal diameter would change to $d = 4$, however, in this paper we only consider the graphs with $d \leq 3$, so we discussed the interesting cases for $k = 3$. The former results mean that, if we would like to examine the graphs where $k = 4$, it is obvious that the minimal diameter would be 2 until $n = 10$, but it is not so important to make so many comparisons because this property can be reached with $k = 3$, as well. Thus for $k = 4$ the interesting cases start above 10 vertices, and the question is if we can reach a smaller diameter (a more stable system of comparisons) with the rise of the answered questions. We found that with $k = 4$ we can get 2 as the minimal diameter until $n = 15$, but for larger values of n, it will be 3 again, which can be also reached by $k = 3$, thus we would not recommend these combinations of parameters. The results for $(11 \leq n \leq 15, k = 4)$ are shown in Table A2. It is also important to note that $k = 4$ is possible in case of both odd and even values of n, thus now we do not have to pay special attention to this.

As we can see, the completion ratio is increasing in k, so we cannot get so small c values as in Table A1a, however the system of comparisons will be more stable even on many vertices, because the smallest diameter is 2 here. It is also interesting that, for larger graphs and regularity levels, the number of connected graphs increases very rapidly. For instance, when we have 15 vertices, there are 805 491 connected 4-regular graphs (that means 805 491 possible filling patterns of the PCM), and only one has 2 as its diameter. Our results and methodology has a strong relationship with the so-called degree/diameter problem that is well known in the literature of mathematics ([39], [40], [41]), but they are looking for the largest possible n for a given diameter and a given level of maximum degree. Several construction techniques have been proposed for graphs in connection with the degree/diameter problem [31,42,43], and one can also find extended tables with the known results [44]. For an indeed extensive summary of the problem, see [45]. The scientific results in this field support our findings, too, because for $(k = 3, d = 2)$ the largest n is 10, while for $(k = 3, d = 3)$ it is 20. In the case of $(k = 4, d = 2)$ the largest n is 15, but for $(k = 4, d = 3)$ it is proven that the largest graph is much above our bound, while the optimal number of the vertices in this case is still an open question.

As we mentioned earlier, there is no point in finding 4-regular graphs when $16 \leq n \leq 20$, thus Table A3 contains the 4-regular graphs for $21 \leq n \leq 24$ for which the diameter is 3. When the tables contain '$\geq \ldots$ graphs', that means we have not checked all the possible cases with minimal diameter, but in connection with decision making problems, it is enough to see that there is one pattern that satisfies the needed properties.

Finally, we can increase the regularity level to 5 in order to find out if we are able to get 2 as the smallest diameter for larger graphs. The answer is yes, actually it is also proven that $d = 2$ is reachable for 5-regular graphs until 24 vertices, but of course we are interested in the specific graphs that could help us determine the adequate comparison patterns. Our results can be found in Tables A4a and A4b. The $k = 5$ parameter is only possible when n is even again, so when it is odd, we let one vertex to have 6 as its degree.

The 5-quasi-regular graph on 21 vertices has been found by us as a twisted product $K_3 * X_7$, where X_7 is a graph with diameter 2 on 7 vertices, in which all vertices have degree 3, except one, where it has 2. The 5-regular graph on 22 vertices has been found by [46] with the help of the following integer linear programming problem:

Let $N = \{1, \ldots, 22\}$ be the nodes, and let $P = \{i \in N, j \in N : i < j\}$ be the set of node pairs. For $(i, j) \in P$, let binary decision variable $X_{i,j}$ indicate whether (i, j) is an edge. For $(i, j) \in P$ and $k \in N \setminus \{i, j\}$, let binary decision variable $Y_{i,j,k}$ indicate whether k is a common neighbor of i and j. For $(i, j) \in P$ let binary decision variable $SLACK_{i,j}$ be a slack variable.

$$\min \sum_{(i,j)\in P} SLACK_{i,j} \quad (1)$$

$$\sum_{(i,j)\in P:k\in\{i,j\}} X_{i,j} = 5 \qquad \text{for } k \in N \quad (2)$$

$$X_{i,j} + \sum_{k\in N\setminus\{i,j\}} Y_{i,j,k} + SLACK_{i,j} \geq 1 \qquad \text{for } (i,j) \in P \quad (3)$$

$$Y_{i,j,k} \leq [i<k]X_{i,k} + [k<i]X_{k,i} \qquad \text{for } (i,j)\in P \text{ and } k \in N\setminus\{i,j\} \quad (4)$$

$$Y_{i,j,k} \leq [j<k]X_{j,k} + [k<j]X_{k,j} \qquad \text{for } (i,j)\in P \text{ and } k \in N\setminus\{i,j\} \quad (5)$$

Constraint **(2)** enforces 5-regularity. Constraint **(3)** enforces diameter 2. Constraints **(4)** and **(5)** enforce that $Y_{i,j,k} = 1$ implies k is a neighbor of i and j, respectively. A desired graph exists if and only if the integer linear program has a solution with $SLACK_{i,j} = 0$ for $\forall\ (i,j) \in P$.

The authors of this paper are still looking for a 5-quasi-regular graph on 23 vertices with diameter 2, but managed to find a graph, which has 23 vertices, and its diameter is 2, but it has one more edge than it should, namely three vertices have degree 6 and all the others have 5.

As we can see in Tables A4a and A4b there are higher completion ratios again, and for instance when we have 24 vertices, the decision makers should make 60 comparisons, which in certain situations can be too many. One can also note that in this table we report that there are some graphs with the needed properties, but never indicate the number of them. The reason behind this is simple: the very high number of the potential connected 5-regular graphs (for instance in the case of $n = 24$ there are roughly $2 \cdot 10^{22}$ possibilities).

This means that we have examined all the cases that we previously called interesting. According to our results, if we use the (n,k,d) parameters, then for smaller MCDM problems the $k = 3$ is enough to get 2 as the diameter of the representing graph, which leads to a small completion ratio and a stable system of the comparisons. In larger problems, when we have more alternatives or criteria, we can choose if we use $k = 3$, when the completion ratio is smaller, but our approximation can be unstable, or choose higher level of regularity (and completion ratio) with more reliable results. We also showed examples and graphs with the needed properties for the different cases, which can help anyone in a MCDM problem to decide which comparisons have to be made. One can find the summary of our results in Table 3, which shows how many graphs we know for given (n,k,d) parameters. It is also true that if there is a graph for (n,k,d) in the table, then, on the one hand, no graph exists with the parameters $(n,k,d-1)$, and, on the other hand, graphs for (n,k,D), where $D > d$, are not counted, and the corresponding cells are left empty. We omitted the cases when $k = 4$ and $n \leq 10$, because the minimal diameter is the same as it was in the case of $k = 3$. There is the same reasoning behind the emptiness of the table when $k = 5$ and $n \leq 15$. We have not included the cases when $k = 4$ and $16 \leq n \leq 20$, because $d = 3$ can be achieved by 3-regular graphs, but for $d = 2$ at least 5-regularity is needed. We also not included the $k = 3$ and $n \geq 20$ cases, because we were examining graphs with $d = 2$ and 3 only.

Table 3
The summary of the results: the number of k-(quasi-)regular graphs on n nodes with diameter d. Lightgray denotes $d = 2$ and gray denotes $d = 3$, '≥' means that there are at least as many graphs as indicated, but we could not check all the possible cases.

	k		
n	**3**	**4**	**5**
5	1		
6	2		
7	4		
8	2		
9	2		
10	1		
11	134	37	
12	34	26	
13	353	10	
14	34	1	
15	290	1	
16	14		≥ 3
17	51		≥ 1
18	1		≥ 1
19	4		≥ 1
20	1		≥ 1
21		≥ 3	≥ 1
22		≥ 1	≥ 1
23		≥ 1	?
24		≥ 1	≥ 1

All the graphs in Tables 2, A1a, A1b, A2, A3, A4a, and A4b are given in several forms in Appendix B: graph, adjacency matrix (that directly shows which comparisons should be made, which PCM elements are required), list of edges and 'Graph6' format. The list of edges also present the needed comparisons, for instance the graph on 5 vertices in Figure B1 in Appendix B (see it also in Table 2) shows that the decision maker should fill in the following elements of the PCM: a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{35} and a_{45}. Upon request the other graphs of each family are available from the authors in these and other forms, as well.

4. Numerical example and simulations

The regularity of the representing graphs has been extensively studied in connection with incomplete pairwise comparisons' designs, while the diameter has only been investigated partially in the literature, as it was mentioned in the introduction. We would like to present what kind of problems can occur even with regular graphs, if we do not take into account the diameter, through a motivational example.

A wide range of simulations has also been performed in order to validate our recommendations, the applied methodology and the gained results are discussed in many details below. We would like to emphasize that, in this section we rely on the framework of the pairwise comparison matrices, though, our recommendations can be adopted in many other fields, as well.

4.1. Simulation methodology

It is important to see if the filling in pattern designs recommended by us are truly useful, thus we applied extensive simulations to have a better understanding of the problem. As for the calculation techniques of the weights derived from the PCMs, we used the well-known Logarithmic Least Squares Method (LLSM) and the Eigenvector Method based on the CR-minimal completion (CREV) [30]. We applied two metrics to determine the differences from the weights calculated from the complete PCMs, that is the Euclidean distance (d_{euc}) and the maximum absolute distance (d_{max}, also known as Chebyshev distance), given by the following formu-

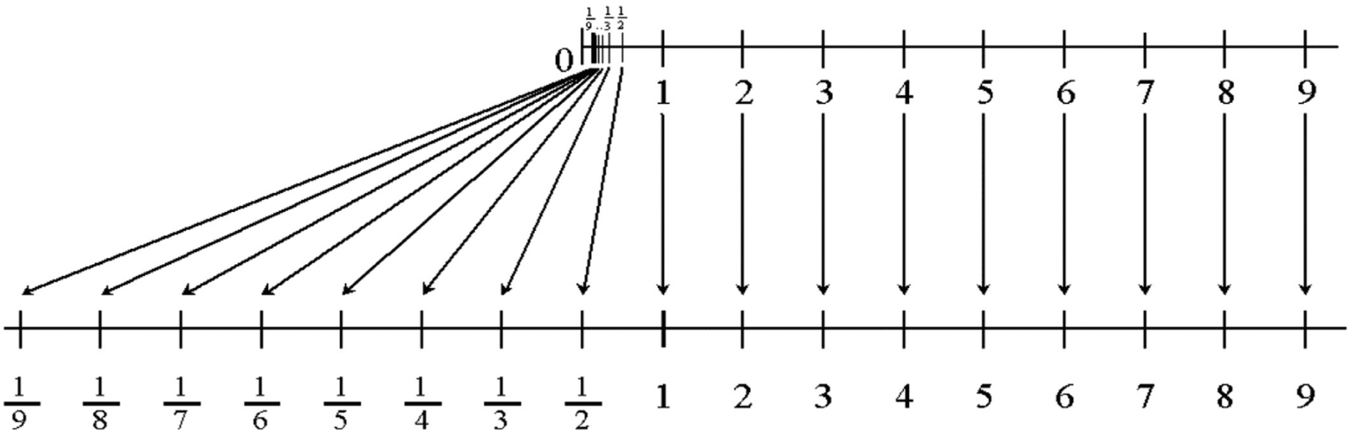


Fig. 2. The scaling on different ranges

las:

$$d_{euc}(u, v) = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$$

$$d_{max}(u, v) = \max_{i \in 1,\ldots,n} |u_i - v_i|,$$

where u denotes the weight vector calculated from a certain filling in design, while v is the weight vector calculated from the complete PCM. u and v are normalized by $\sum_{i=1}^{n} u_i = 1$ and $\sum_{i=1}^{n} v_i = 1$, respectively, while v_i and u_i denote the ith element of the appropriate vectors.

The process of the simulation for a given (n, k) pair consisted of the following steps:

1. We generated random $n \times n$ complete and consistent pairwise comparison matrices. The elements of these matrices were given as $a_{ij} = w_i / w_j$, where $w_i \in [1, 9]$ is a uniformly distributed random real number for $\forall i$.
2. Then we perturbed the elements of our consistent matrices three different ways, to get inconsistent PCMs with three distinguishable inconsistency levels. We call these levels weak, modest and strong given with the following formulas:

$$b_{ij} = \max\left(\frac{1}{2},\ a_{ij} + \Delta\right) \qquad \Delta \in [-1, 1] \qquad \text{(weak)}$$

$$b_{ij} = \max\left(\frac{1}{2},\ a_{ij} + \Delta\right) \qquad \Delta \in [-2, 2] \qquad \text{(modest)}$$

$$b_{ij} = \max\left(\frac{1}{3},\ a_{ij} + \Delta\right) \qquad \Delta \in [-3, 3] \qquad \text{(strong)}$$

Where b_{ij} is the element of the perturbed matrix, a_{ij} is the element of the consistent matrix, $a_{ij} \geq 1$, and Δ is uniformly distributed in the given ranges. The motivation behind this structure is the following, we can get perturbed data even from an ordinal point of view, when $b_{ij} < 1$. However, in order to get meaningful results, we should use a different scale for the range of (0,1) compared to the range of (1,9] in connection with PCMs, as Figure 2 suggests. That is why the maximum function and the lower bounds (1/2, 1/2 and 1/3, respectively) appear in the definition. These element-wise perturbation methods correlate with the well known Consistency Ratio (CR), as it is shown in Figure 3. We tested several combinations of parameters, and found that these, more or less balanced perturbations around 1, result in the most relevant levels of inconsistency.

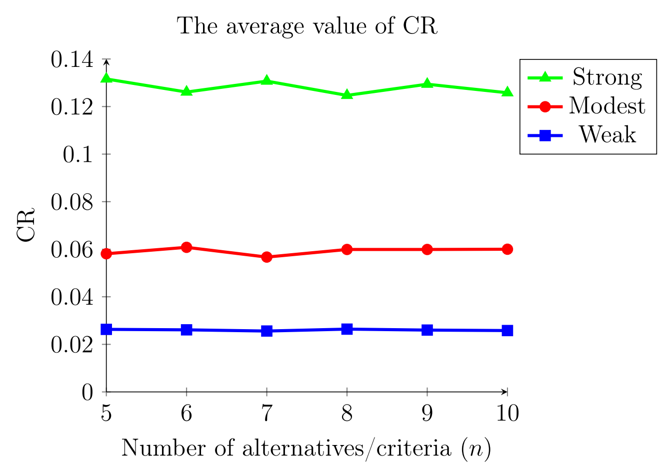
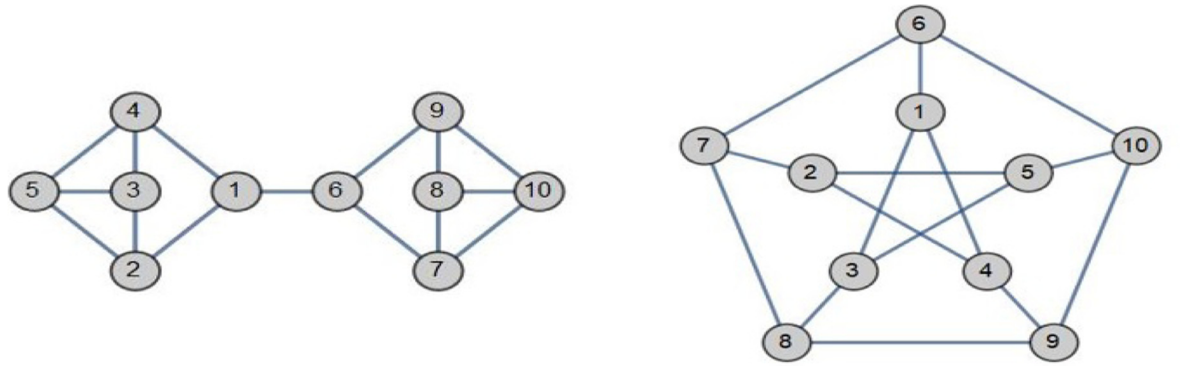


Fig. 3. The connections between CR and our element-wise perturbations. Each point shows the average CR of 1000 randomly generated perturbed pairwise comparison matrices.

3. We deleted the respective elements of the matrices in order to get the filling in pattern that we were examining, and applied the LLSM and the CREV techniques to get the weights. We always computed the certain designs' distances from the weights that we calculated from the complete inconsistent matrices. We used 1000 PCMs for every level of inconsistency and applied the following filling in patterns to compare them with each other:
(i) Our recommendations: k-(quasi-)regular graphs of minimal diameter, detailed in Section 3 and Appendix A,
(ii) Random connected graphs with the same number of edges as our recommendation (1000 graphs per inconsistency level per simulation),
(iii) Connected k-(quasi-)regular graphs, but not of minimal diameter (1000 graphs per simulation),
(iv) Randomly generated, connected, of minimal diameter, but not regular graphs with the same number of edges (1000 graphs per simulation),
(v) Minimal diameter, modified/extended star graphs with the same number of edges (1000 graphs per inconsistency level per simulation).
4. Finally, we saved the mean and standard deviation of the distances for the different weight calculation methods and filling in designs.



(a) An alternative 3-regular graph (b) Our recommendation: the Petersen graph

Fig. 4. The graph representation of two 3-regular designs

We restricted the connected k-(quasi-)regular graphs to the Hamiltonian ones during the generation. With this we excluded the k-(quasi-)regular graphs with the largest diameters as well. This was also interesting, because all of our recommendations in Section 3 and Appendix A are Hamiltonian except the Petersen graph and the Tietze graph, but these two are well-known exceptions [47,48].

In case of (iv), we basically generated random connected graphs and selected the ones with minimal diameter (the same diameter as our recommendation), until we had 1000 such graphs, at least in the cases where we have found so many instances in a reasonably long time.

As for (v), when the diameter of our recommendation was 2, then we generated a random star graph, and complemented it with the needed number of random edges. While in case of diameter 3, we did the same, but at the end, we deleted one edge from the star and replaced it with another one, so that the diameter of the graph became 3.

It is important to note that we considered only the graph presented in Section 3 and Appendix A for a given (n, k) pair in (i), and not all the k-(quasi-)regular graphs with minimal diameter. This is due to the fact that in many cases we were able to find one graph with the needed properties, but could not find all of them or even could not determine the exact number of such graphs.

Before the results of the simulations, we show a motivational example, in which we compare two different filling in pattern designs similarly as in the case of the simulations. This numerical instance shows that it is also important to take into account the minimal diameter property, and not just regularity.

4.2. Motivational example

Let us demonstrate the simulation process, as well as the importance of the diameter, when we have 10 alternatives, and we examine only two different filling in structures.

We generate 1000 $n \times n$ consistent PCMs with elements $a_{ij} = w_i/w_j$, where $w_i, w_j \in [1, 9]$ are uniformly distributed random real numbers. Then we perturb all of the elements of these PCMs three different ways as described in Equations weak, modest and strong.

We would like to compare the differences of the calculated weights from the ones that we get from these complete perturbed PCMs, when we consider the two filling in patterns represented by the graphs in Figure 4. The filling structures related to these graphs can be seen in Table 4, which means that we delete all the other elements, when we compute the weights according to the given pattern.

As for the two representing graphs, the Petersen graph has minimal diameter among 3-regular graphs on 10 vertices, its diameter is 2, while the Alternative 3-regular graph's diameter is 5. As one can see there are common elements of the two filling in patterns, as for instance the bridge-edge between vertices 1 and 6 (a_{16}, bridge set [49]), which connects the two symmetric components of the Alternative graph. It is also worth to mention that the special structure of this graph (also highlighted by the two separate parts of the related PCM in Table 4) ensures that the weights of 1 and 6 are always determined exactly by b_{16}.

Table 5 summarizes the mean (denoted by M) and the standard deviation (σ) of distances (d_{euc} and d_{max}) of the weights calculated from the two filling patterns respect to the complete case for the three inconsistency (perturbation) levels (Weak, Modest and Strong).

One can see that there are significant contrasts between the outcomes of the examined filling in patterns. In case of both the Euclidean and maximum absolute (Chebyshev) metrics, the distances of the weights computed from the Alternative graph respect to the ones we got from the complete PCM are approximately 1.5 times larger, than the same for the Petersen graph, however both the relative and absolute differences are smaller in case of the absolute maximum distance. The same results are true when we consider the standard deviation of the distances. This means that the Petersen graph tends to provide small errors and a consistent performance (small standard deviation) depending on the perturbations, compared to the filling pattern represented by the Alternative graph.

We think that this example can give a deeper understanding of the simulation method. Besides that, the main message of this sub-section is that, one should consider the diameter of the graph as an important parameter in these designs, because even among regular graphs, there can be large differences.

4.3. Simulation results

The results of the simulations seem to mainly depend on the value of k, and barely on n, as well as the patterns of the outcomes seem to be the same for every case.

The tables for all parameters (n, k, d) calculated are available in Appendix D, while we have chosen to visualize only the following representative examples: $(n = 16, k = 3, d = 3)$, $(n = 11, k = 4, d = 2)$ and $(n = 24, k = 5, d = 2)$. The first one is the largest 3-regular case, where we could apply (iv), and it is the only one that can be found in the main text due to the length of the figures. The second one is the smallest 4-regular, and the last one is the largest 5-regular case that we examined. The results of the simu-

Table 4
The known elements of the given PCM in case of the two different filling in patterns represented by the graphs in Figure 4. The design related to the Alternative graph can be seen to the left, while the filling structure of the Petersen graph is shown in the PCM to the right.

	1	2	3	4	5	6	7	8	9	10		1	2	3	4	5	6	7	8	9	10
1	1	b_{12}		b_{14}		b_{16}					1	1		b_{13}	b_{14}		b_{16}				
2	$\frac{1}{b_{12}}$	1	b_{23}		b_{25}						2		1		b_{24}	b_{25}		b_{27}			
3		$\frac{1}{b_{23}}$	1	b_{34}	b_{35}						3	$\frac{1}{b_{13}}$		1		b_{35}			b_{38}		
4	$\frac{1}{b_{14}}$		$\frac{1}{b_{34}}$	1	b_{45}						4	$\frac{1}{b_{14}}$	$\frac{1}{b_{24}}$		1					b_{49}	
5		$\frac{1}{b_{25}}$	$\frac{1}{b_{35}}$	$\frac{1}{b_{45}}$	1						5		$\frac{1}{b_{25}}$	$\frac{1}{b_{35}}$		1					b_{510}
6	$\frac{1}{b_{16}}$					1	b_{67}		b_{69}		6	$\frac{1}{b_{16}}$					1	b_{67}			b_{610}
7						$\frac{1}{b_{67}}$	1	b_{78}		b_{710}	7		$\frac{1}{b_{27}}$				$\frac{1}{b_{67}}$	1	b_{78}		
8							$\frac{1}{b_{78}}$	1	b_{89}	b_{810}	8			$\frac{1}{b_{38}}$				$\frac{1}{b_{78}}$	1	b_{89}	
9						$\frac{1}{b_{69}}$		$\frac{1}{b_{89}}$	1	b_{910}	9				$\frac{1}{b_{49}}$				$\frac{1}{b_{89}}$	1	b_{910}
10							$\frac{1}{b_{710}}$	$\frac{1}{b_{810}}$	$\frac{1}{b_{910}}$	1	10					$\frac{1}{b_{510}}$	$\frac{1}{b_{610}}$			$\frac{1}{b_{910}}$	1

Table 5
The average distances and their standard deviation for the different designs. The following notations are used: M-mean, σ-standard deviation, 'Weak', 'Modest' and 'Strong' refer to the level of perturbation.

Weak	**LLSM d_{euc} M**	**CREV d_{euc} M**	**LLSM d_{max} M**	**CREV d_{max} M**	**LLSM d_{euc} σ**	**CREV d_{euc} σ**	**LLSM d_{max} σ**	**CREV d_{max} σ**
Petersen	0.0424	0.0422	0.0275	0.0274	0.0285	0.0283	0.0193	0.0191
Alternative	0.0605	0.0604	0.0370	0.0369	0.0468	0.0467	0.0286	0.0285
Modest								
Petersen	0.0673	0.0669	0.0450	0.0445	0.0378	0.0376	0.0278	0.0274
Alternative	0.0956	0.0956	0.0604	0.0602	0.0610	0.0611	0.0400	0.0399
Strong								
Petersen	0.0967	0.0952	0.0665	0.0652	0.0527	0.0519	0.0402	0.0390
Alternative	0.1318	0.1314	0.0881	0.0877	0.0825	0.0826	0.0590	0.0592

lations for them are shown in Figures 5, C1 and C2, respectively, and it is also recommended to read this section together with Appendix C, as the latter two cases are presented there. The figures show the mean of the different metrics (M) and the standard deviation (σ) as well. We refer to the different levels of the perturbation as 'Weak', 'Modest' and 'Strong', as before.

It is clear from the outcomes of the simulations that the stronger perturbation causes larger distances, and the higher regularity level leads to smaller differences. As one can see, our recommendations have the smallest means and standard deviations among the different designs in case of both metrics and both weight calculation methods for every (n, k) pair, which suggests that the results are not solely dependent on the used techniques and parameters. The smallest mean shows that the k-(quasi-)regular graphs with minimal diameter provide the closest weights to the complete PCM on an average level. On the other hand, the smallest standard deviation also implies that our recommendations are commonly not connected to huge errors, and that these filling in pattern designs perform at a very consistent level regarding the deviations from the results of the complete PCMs. It is also true that the randomly generated minimal diameter graphs (denoted by (iv)) tend to have smaller means and standard deviations compared to the simple random graphs. Again, this suggests that, the diameter of the representing graph is relevant. The k-(quasi-)regular graphs (denoted by (iii)) always have the second smallest means and standard deviations in their distances, thus the already known fact, that regularity is a key property, confirmed here as well. It is also important to note that we have excluded the k-(quasi-)regular cases with the largest diameters, because of the Hamiltonian construction as we mentioned earlier, thus we expect random (quasi-)regular graphs to have a bit even 'worse' results compared to our recommendations. The case of the modified star graphs (denoted by (v)) is interesting. In case of $k = 3$, they always have smaller means and standard deviations compared to the simple random graphs, but for $k = 4$ they always have larger means, and in some cases even their standard deviations are higher. For $k = 5$ the modified star graphs tend to have the largest means and standard deviations among the examined designs. This also suggests that considering only the diameter is not sufficient in these problems. Finally, we would like to emphasize that these patterns and findings, are the very same for all studied (n, k) pairs, especially regarding the dominance of the k-(quasi-)regular graphs, thus our recommendations seem to perform indeed well in the framework of pairwise comparison matrices.

5. Conclusions and further research

5.1. Summary

The main contribution of the paper is a systematic collection of recommended filling patterns of incomplete pairwise comparisons' using the graph representation of the PCMs. The proposed (quasi-)regular graphs with minimal diameter have not only pure graph theoretical relevance, but their importance in multi-criteria decision making is also demonstrated via the comparisons to other incomplete filling in patterns of the same cardinality.

Graphs are included in several formats in Appendix B, which can show practitioners the comparisons that should be made, i.e. the PCM elements to be filled in. We presented our results using the number n of criteria or alternatives, regularity level k and diameter d of the representing graph as parameters. We identified the diameter, that was missing from the relevant literature of decision theory and preference modelling, as an important parameter in these problems. It has been shown that relatively small diameters $d = 2, 3$ can be achieved with relatively small completion ratios, and examples has been provided for every case up until 24 vertices.

We also validated our recommendations with the help of numerical simulations. 1000 perturbed PCMs were used in case of 3 different inconsistency (perturbation) level to compare several fill-

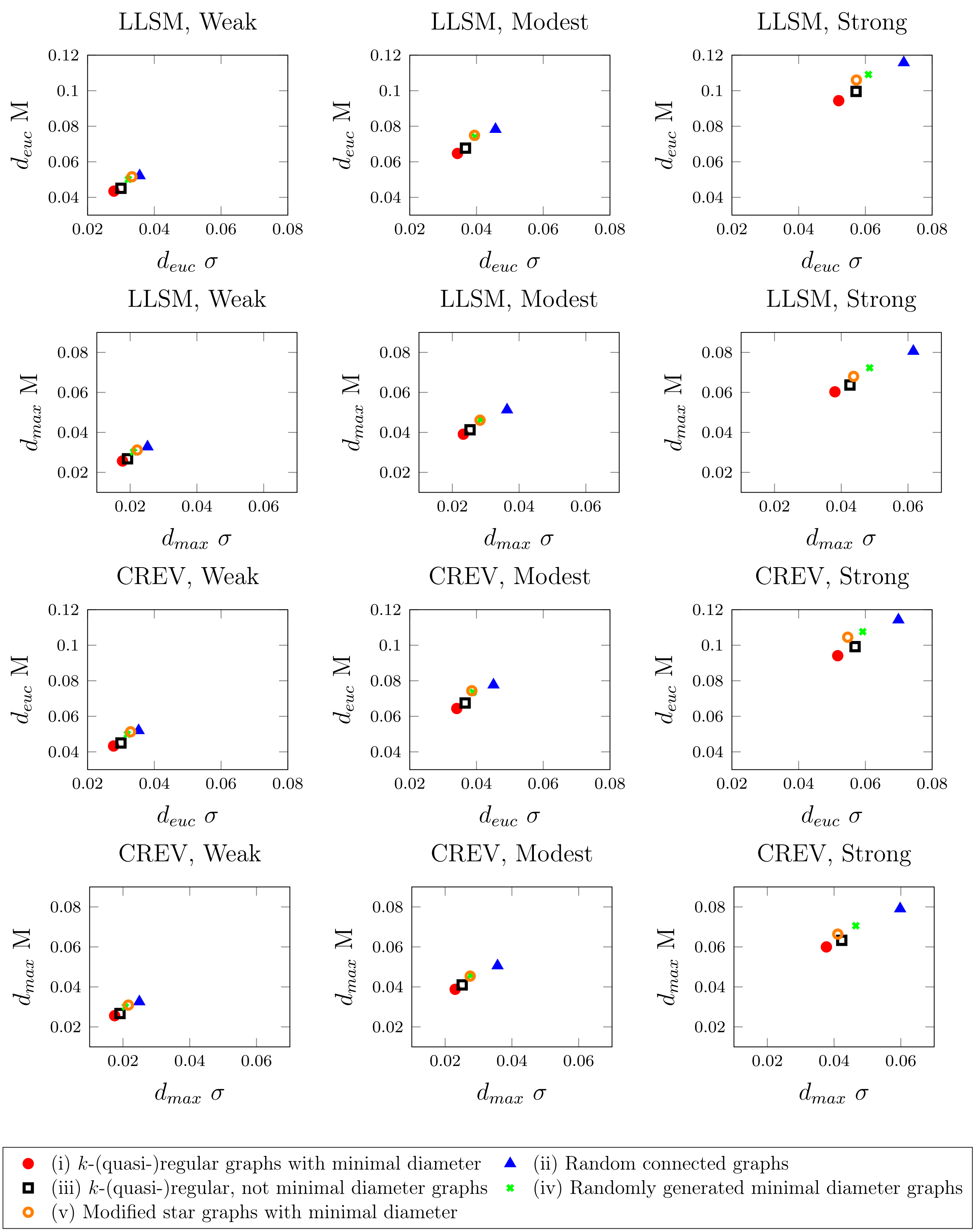


Fig. 5. The results of the simulation for (n = 16, k = 3, d = 3). The following notations are used: M-mean, σ-standard deviation, d_{euc}-Euclidean distance, d_{max}-maximum absolute distance, Weak, Modest and Strong refer to the level of perturbation. See Table D13 in Appendix D for numerical details.

ing in patterns with the proposed ones for every examined parameter combinations (all in all 34). The recommended filling structures provided the closest weight vectors to the complete case on average, with the smallest standard deviation, according to 2 distances (Euclidean and Chebyshev), in case of both the incomplete LLSM and Eigenvector weight calculation techniques (for detailed results of all parameter combinations, see Appendix D). Examples also show that neither regularity nor small diameter is sufficient on its own, both of these properties are needed.

5.2. Limitations and further research

Simulations show that the proposed (quasi)-regular graphs with minimal diameter are better, in the sense of the metrics we considered, than e.g., the random ones, or the ones having only one of the two properties, regularity and minimal diameter, instead of both. However, it certainly does not mean that other, yet undiscovered or unidentified structures could not be even better.

The investigation of the robustness of the results, namely what is 'between' the different regularity levels (when the degrees of different vertices are not the same), could be the topic of a further research, as well as the cases with larger minimal diameters. Similarly, what is between diameters $d+1$ and d, in particular 2 and 1 (i.e. the complete graph)? According to Tables 2, A2, A4a and A4b, diameter 2 is achieved at relatively low completion ratios, especially for larger n parameters, so the *game* of having better weight vectors by adding more comparisons is continuing rather than ending at $d=2$, as the values in Tables D2–D7, D18–D22 and D27–D35 show.

It is also an interesting problem to concentrate directly on the completion ratio as a parameter instead of the regularity of the representing graph. If the (n, c) pair is given (and $(n-1)c$, the average degree is not necessarily integer), then which comparisons should be made?

Our approach definitely has a strong connection with other metrics based on the lengths of shortest paths (e.g. their average) as well as centrality measures [50]. When there are several graphs with the needed properties, we can reduce their number based on some chosen centrality measures. We would like to deal with these questions in our future works.

Group decision making [51] is a potential application area of our results, as we may assume that the individual preferences can be colorful enough, so we cannot suppose any prior information. In other words: we treat the items to be compared in a symmetric way, therefore our recommended graphs can be applied.

Although our results were presented within the framework of pairwise comparison matrices, they are applicable in a wider range. A lot of other models based on pairwise comparisons can utilize our findings. For example ranking of sport players or teams based on their matches leads to the problem of tournament design: which pairs should play against each other (without the use of prior knowledge or estimation of their strength)?

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.omega.2021.102557

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