

Frequency-Domain Data-Driven Controller Synthesis for Unstable LPV Systems ^{*}

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Abstract: Synthesizing controllers directly from frequency-domain measurement data is a powerful tool in the linear time-invariant framework. Ever-increasing performance requirements necessitate extending these approaches to account for plant variations. The aim of this paper is to develop frequency-domain analysis and synthesis conditions for local internal stability and \mathcal{H}_∞ -performance of single-input single-output linear parameter-varying systems. The developed synthesis procedure only requires frequency-domain measurement data of the system and does not need a parametric model of the plant. The capabilities of the synthesis procedure are demonstrated on an unstable nonlinear system.

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1. INTRODUCTION

Frequency response function (FRF) measurements have traditionally been used to manually design controllers directly from measurement data. A frequency response function estimate provides an accurate nonparametric description of the system that is relatively fast and inexpensive to obtain (Pintelon and Schoukens, 2012). This has enabled the use of classical techniques such as loop-shaping, alongside graphical tools including the Bode diagram or Nyquist plot, to design such controllers (Maciejowski, 1989). These controllers often have a proportional-integral-derivative (PID) structure in addition to higher-order filters to compensate parasitic dynamics (Steinbuch and Norg, 1998). Loop-shaping can also be applied to multivariable systems through decoupling or sequential loop closing (Oomen and Steinbuch, 2017). However, these methods have in common that the design procedure can be difficult as they are based on design rules, insight and experience.

As an alternative, control design based on nonparametric models has been further developed towards automated procedures that utilize FRF measurements to synthesize linear time-invariant (LTI) controllers. At first, these methods were developed along the lines of the classical control theory to synthesize PID controllers (Grassi et al., 2001). More recently, these methods have been tailored towards more general control structures that focus on \mathcal{H}_∞ -performance, with many successful applications within the LTI domain (Karimi and Galdos, 2010; Khadraoui et al., 2014). This was further extended to a framework in which model uncertainties can be incorporated into the control design, such that a robustly stabilizing controller is synthesized to accommodate for the variations in the

plant (Karimi et al., 2007, 2018). However, this typically comes at the cost of performance.

The paradigm of linear parameter-varying (LPV) systems has been developed to provide a systematic framework for the analysis and design of gain-scheduled controllers for nonlinear systems (Shamma and Athans, 1990). An LPV system is characterized by a linear input-output (IO) map, similar to the LTI framework, where now the dynamics depend on an exogenous time-varying signal whose values can be measured on-line. This so-called scheduling variable p can be used to capture the nonlinear or operating condition-dependent dynamics of a system. Typically, a priori information on the scheduling variable is known, such as the range of variation. The class of LPV systems is supported by a well-developed model-based control and identification theory, with approaches that can be viewed as extensions of LTI control methodologies, see, e.g., (Hoffmann and Werner, 2015; Mohammadpour and Scherer, 2012) and the references therein. Also, data-driven control design techniques in the time-domain exist (Formentin et al., 2016). With respect to data-driven controller synthesis based on frequency response functions, only a handful of methodologies exist (Kunze et al., 2007; Karimi and Emedi, 2013; Bloemers et al., 2019). These methods have in common that an LPV controller is synthesized such that, locally for every operating point, stability and performance can be guaranteed.

Although data-driven controller synthesis based on FRF data enables systematic design approaches in the LTI framework, within the LPV framework, these are conservative and limited to stable systems only for. Within the LTI literature, necessary and sufficient frequency-domain analysis conditions exist for robust stability (Rantzer and Megretski, 1994). These conditions have been used in (Karimi et al., 2018) to synthesize controllers for even unstable LTI systems, guaranteeing stability and \mathcal{H}_∞ -performance. The aim of this paper is to overcome the

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limitations currently present for data-driven LPV controller synthesis in the frequency-domain by (i) developing necessary and sufficient analysis and synthesis conditions for (possibly) unstable systems and controllers, and (ii), allowing a rational LPV controller parameterization.

The main contributions of this paper are (C1) a procedure to synthesize LPV controllers for possibly unstable single-input single-output plants that achieve local internal stability and \mathcal{H}_∞ -performance guarantees. This is achieved by the following sub-contributions.

C2 Development of a local LPV frequency-domain stability analysis condition.

C3 Development of an LPV frequency-domain performance analysis condition.

The results in Rantzer and Megretski (1994) are recovered as a special case for stable systems, constituting to C2. Contribution C3 is achieved by developing new insights into the performance conditions presented in (Karimi et al., 2018), that relate to the robust control theory and consequently to LPV systems by means of the main loop theorem, see e.g., (Zhou et al., 1996). Furthermore, the results in (Karimi et al., 2018) are recovered as a special case when the scheduling disappears. Finally, C1 is achieved by utilizing a global parameterization of the LPV controller, for which local stability and performance guarantees are provided by means of C2 and C3.

The paper is organized as follows. In Section 2 the problem setting is defined and the problem of interest is formulated. Then, in Section 3 analysis conditions for stability and performance are derived, constituting to C2 and C3. This is followed by the derivation of a synthesis procedure and the main contribution C1 in Section 4. In Section 5, the capabilities of the proposed methodology are demonstrated by means of a simulation example. Finally, conclusions are drawn in Section 6.

Throughout this paper, \mathbb{R} denotes the set of real numbers and \mathbb{C} is the set of complex numbers. The imaginary axis is denoted by \mathbb{C}_0 and the right half-plane is denoted by \mathbb{C}_+ . The real part of a complex number $z \in \mathbb{C}$ is denoted by $\Re\{z\}$. The imaginary unit is denoted by $i = \sqrt{-1}$. The set of real rational proper and stable transfer functions is denoted as \mathcal{RH}_∞ , while the continuous frequency set associated with the Fourier transform is given by $\Omega := \{\mathbb{R} \cup \{\infty\}\}$.

2. PROBLEM FORMULATION

2.1 Preliminaries

Consider the single-input single-output (SISO), continuous-time (CT) LPV system, with LPV state-space representation (Tóth, 2010):

$$G_p : \begin{cases} \dot{x}(t) &= A(p(t))x(t) + B(p(t))u(t), \\ y(t) &= C(p(t))x(t) + D(p(t))u(t), \end{cases} \quad (1)$$

where $x : \mathbb{R} \rightarrow \mathbb{X} \subseteq \mathbb{R}^{n_x}$ denotes the state variable, $u : \mathbb{R} \rightarrow \mathbb{U} \subseteq \mathbb{R}$ is the input signal, $y : \mathbb{R} \rightarrow \mathbb{Y} \subseteq \mathbb{R}$ is the output signal and $p : \mathbb{R} \rightarrow \mathbb{P} \subseteq \mathbb{R}^{n_p}$ the scheduling variable.

When the scheduling signal $p(t) \equiv p$ is frozen in time, the scheduling-dependent matrices in (1) become time-invariant, i.e., with slight abuse of notation

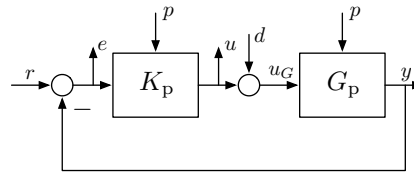


Fig. 1. Feedback interconnection, including 4-block shaping problems, depending on the scheduling signal.

$$G_p = \left(\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & D(p) \end{array} \right) \quad (2)$$

represents the LPV system with state-space form (1) for constant scheduling p . For a given $p \in \mathbb{P}$, (2) describes the local behavior of (1). Hence, (2) is referred to as the frozen behavior of (1).

Taking the Laplace transform of (2) with zero initial conditions results in

$$\hat{y}(s) = (C(p)(sI - A(p))^{-1}B(p) + D(p)) \hat{u}(s), \quad (3)$$

where $G_p(s) = C(p)(sI - A(p))^{-1}B(p) + D(p)$ and s is the Laplace variable. The frozen behavior (2) also has a corresponding Fourier transform

$$Y(i\omega) = G_p(i\omega)U(i\omega), \quad (4)$$

where i is the complex unit, $\omega \in \mathbb{R}$ is the frequency and $G_p(i\omega)$ represents the frozen Frequency Response Function (fFRF) of (1) for every constant $p(t) \equiv p \in \mathbb{P}$ (Schoukens and Tóth, 2019).

2.2 Problem statement

The problem addressed in this paper is to design an LPV controller directly from fFRF measurement data. We denote the data $\mathcal{D}_{N,p_\tau} = \{G_p(i\omega_k), p_\tau\}_{k=1}^N$, obtained at the set of operating points $\mathcal{P} = \{p_\tau\}_{\tau=1}^{N_{\text{loc}}} \subset \mathbb{P}$. Consider the feedback interconnection in Figure 1. The objective is to design a controller K_p such that the following requirements are satisfied:

- R1 The closed-loop system in Figure 1 is internally stable in the local sense for all $p(t) \equiv p \in \mathbb{P}$.
- R2 The performance channels $(r, d) \mapsto (e, u)$ in Figure 1 are bounded in the local \mathcal{H}_∞ -norm sense by $\gamma > 0$ for all $p \in \mathbb{P}$.

In the next section, a rational controller parameterization is introduced that allows for a specific formulation of internal stability. This forms the basis to develop analysis conditions for internal stability and \mathcal{H}_∞ -performance. The theory is first formulated for $p \in \mathbb{P}$ for the sake of generality. This also ensures R1 and R2 for $p \in \mathcal{P}$.

3. STABILITY AND PERFORMANCE ANALYSIS CONDITIONS

This section presents local LPV stability and performance analysis conditions. This constitutes to requirements R1 and R2 and contributions C2 and C3, respectively. Based on these results, a data-driven synthesis procedure is developed. Throughout this section, first the results are presented with a continuous frequency spectrum $\Omega = \{\mathbb{R} \cup \{\infty\}\}$, which will be restricted later by a finite frequency grid $\Omega_N = \{\omega_k\}_{k=1}^N$ corresponding to the data \mathcal{D}_{N,p_τ} .

3.1 Stability

Figure 1 corresponds to the internal stability problem (Doyle et al., 1992, Chapter 3). For a frozen $p \in \mathbb{P}$, let the IO map $T(G_p, K_p) : (r, -d) \mapsto (e, u)$ in Figure 1 be defined by

$$T(G_p, K_p) = \begin{bmatrix} S_p & S_p G_p \\ K_p S_p & T_p \end{bmatrix}, \quad (5)$$

with $S_p = (1 + G_p K_p)^{-1}$ and $T_p = 1 - S_p$. If $G_p, K_p \in \mathcal{RH}_\infty$, then $T(G_p, K_p)$ is internally stable if all elements in the IO map $T(G_p, K_p)$, defined by (5), are stable (Doyle et al., 1992, Chapter 3). If $T(G_p, K_p) \in \mathcal{RH}_\infty$ holds for all frozen $p \in \mathbb{P}$ then the closed-loop LPV system is called locally internally stable. To assess internal stability for unstable G_p or K_p , introduce

$$G_p = N_{G_p} D_{G_p}^{-1}, \quad \{N_{G_p}, D_{G_p}\} \in \mathcal{RH}_\infty. \quad (6)$$

The two transfer functions $\{N_{G_p}, D_{G_p}\}$ are a coprime factorization over \mathcal{RH}_∞ if there exist two other transfer functions $\{X_p, Y_p\} \in \mathcal{RH}_\infty$ such that they satisfy the Bézout identity

$$N_{G_p} X_p + D_{G_p} Y_p = 1. \quad (7)$$

Correspondingly, K_p admits the coprime factorization

$$K_p = N_{K_p} D_{K_p}^{-1}, \quad \{N_{K_p}, D_{K_p}\} \in \mathcal{RH}_\infty. \quad (8)$$

Using these definition, (5) can be represented by

$$T(G_p, K_p) = D_p^{-1} \begin{bmatrix} D_{G_p} D_{K_p} & N_{G_p} D_{K_p} \\ D_{G_p} N_{K_p} & N_{G_p} N_{K_p} \end{bmatrix}, \quad (9)$$

with characteristic equation

$$D_p = D_{G_p} D_{K_p} + N_{G_p} N_{K_p}. \quad (10)$$

The feedback system in Figure 1 is internally stable if and only if $D_p^{-1} \in \mathcal{RH}_\infty$. This follows from the Bézout identity, i.e., set $N_{K_p} = X_p$ and $D_{K_p} = Y_p$, then the characteristic equation (10) equals the Bézout identity (7) and the feedback system is internally stable. Similarly, the closed-loop LPV system is called locally internally stable if these conditions hold for all $p \in \mathbb{P}$.

For the channel transfer $w \mapsto z$, where $w \in \{r, -d\}$ and $z \in \{e, u\}$, let

$$T_{z,w}(G_p, K_p) = N_p D_p^{-1}, \quad (11)$$

with $\{N_p, D_p\} \in \mathcal{RH}_\infty$ and $T_{z,w}(G_p, K_p) \in \mathcal{RH}_\infty$, define the corresponding SISO element of (9). For example, $T_{r,e}(G_p, K_p) = N_p D_p^{-1}$ with $N_p = D_{G_p} D_{K_p}$ defines the sensitivity S_p in (5) and (9).

The following theorem presents analysis conditions to verify internal stability of a closed-loop LPV system locally, given the plant and controller only.

Theorem 1. Let G_p and K_p be as defined in (6) and (8), respectively, and let $D_p \in \mathcal{RH}_\infty$ be as defined in (10). Then the following conditions are equivalent. For all $p \in \mathbb{P}$

- 1a) $D_p^{-1} \in \mathcal{RH}_\infty$.
- 1b) $D_p(s) \neq 0, \forall s \in \mathbb{C}_+ \cup \mathbb{C}_0 \cup \{\infty\}$.
- 1c) There exists a multiplier $\alpha_p, \alpha_p^{-1} \in \mathcal{RH}_\infty$ such that

$$\Re\{D_p(i\omega)\alpha_p(i\omega)\} > 0, \forall \omega \in \Omega.$$

The proof can be found in Appendix A. Theorem 1 provides an analysis condition to verify local stability for the closed-loop system if instead of a parametric model N_{G_p} and D_{G_p} are only given in terms of local frequency-

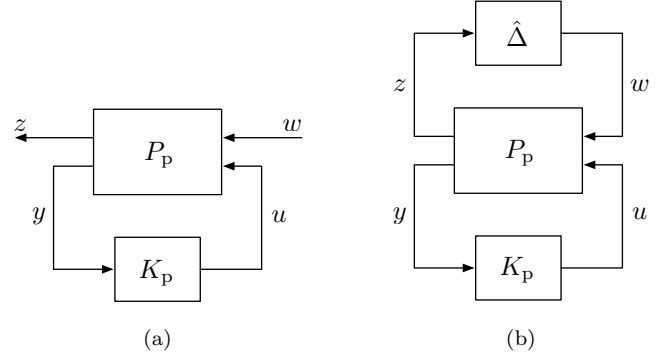


Fig. 2. Generalized LPV plant (a); and performance of the SISO closed-loop map $w \mapsto z$ (b).

domain data. The test relates to the Nyquist stability theorem, however without the need to visualize the data in terms of a plot and counting encirclements. Instead, if a transfer function $\alpha_p, \alpha_p^{-1} \in \mathcal{RH}_\infty$ can be found such that statement 1c) holds, then Nyquist stability holds and the system is internally stable. The next subsection presents the extension towards a performance analysis condition.

3.2 Performance

This subsection presents an analysis condition to assess locally the \mathcal{H}_∞ -performance of an LPV system. This constitutes contribution C3. To derive performance analysis conditions, we first present the main loop theorem.

Consider the transfer function $T_{z,w}(G_p, K_p) \in \mathcal{RH}_\infty$ of interest in Figure 2a, such that $w \mapsto z : T_{z,w}(G_p, K_p)$, and let $\hat{\Delta} \in \mathbf{B}\hat{\Delta}$, with

$$\mathbf{B}\hat{\Delta} := \left\{ \hat{\Delta} \in \mathcal{RH}_\infty \mid |\hat{\Delta}(i\omega)| < 1, \forall \omega \in \Omega \right\} \quad (12)$$

a fictitious uncertainty that represents the \mathcal{H}_∞ -performance criterion. Then, \mathcal{H}_∞ -performance of the system in Figure 2a is equivalent to Figure 2b (Skogestad and Postlethwaite, 2001, Theorem 8.7). This is captured by the following theorem.

Theorem 2. (Main loop theorem). Let $W_T \in \mathcal{RH}_\infty$ and $T_{z,w}(G_p, K_p)$ be defined as in (11). The following statements are equivalent. For all $p \in \mathbb{P}$

- 2a) $\sup_{\omega \in \Omega} |W_T(i\omega)T_{z,w}(G_p, K_p)(i\omega)| \leq \gamma$.
- 2b) $1 - \gamma^{-1}W_T(i\omega)T_{z,w}(G_p, K_p)(i\omega)\hat{\Delta}(i\omega) \neq 0,$
 $\forall \omega \in \Omega, \forall \hat{\Delta} \in \mathbf{B}\hat{\Delta}.$

Theorem 2 is a special case of Zhou et al. (1996, Theorem 11.7), where the weighting filter W_T is introduced to specify the frequency-dependent design requirements on the map $w \mapsto z$. The theorem connects nominal performance to robust stability through the interconnection of the performance channels with a fictitious uncertainty block, see Figure 2b.

The main loop theorem provides useful insight into performance. In the data-driven setting, the absence of a parametric model of $T_{z,w}(G_p, K_p)$ makes it difficult to turn statement 2b) into a convex constraint as it is generally done in model-based LPV synthesis approaches for gain-scheduling (Hoffmann and Werner, 2015). Hence, in that case statement 2b) is needed to be evaluated for an infinite set of realizations of the fictitious uncertainty

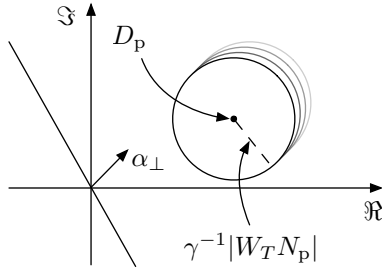


Fig. 3. Illustration of stability and \mathcal{H}_∞ -performance. The transfer function α_p represents, for each frequency, a line passing through the origin. If this line does not intersect with the disks $D_p - \gamma^{-1}|W_T N_p|$ for each frequency, then the disks exclude the origin and (2b) must hold.

$\hat{\Delta}$, for example, as in (van Solingen et al., 2018). The contribution in this paper is to utilize Theorem 1 together with Theorem 2 to derive a single theorem to analyze both stability and performance without the need to sample $\hat{\Delta}$.

Theorem 3. Let $W_T \in \mathcal{RH}_\infty$ and $T_{z,w}(G_p, K_p)$ be defined as in (11). Requirements R1 and R2 are satisfied if and only if there exists a multiplier $\alpha_p \in \mathcal{RH}_\infty$ with $\alpha_p^{-1} \in \mathcal{RH}_\infty$ such that

$$\Re\{(D_p(i\omega) - \gamma^{-1}|W_T(i\omega)N_p(i\omega)|)\alpha_p(i\omega)\} > 0, \quad (13)$$

$$\forall \omega \in \Omega, \forall p \in \mathbb{P}.$$

The proof is given in Appendix B. Theorem 3 states that the performance condition 2a) is satisfied if and only if for each frequency $\omega \in \Omega$ and scheduling value $p \in \mathbb{P}$ the disks with radius $\gamma^{-1}|W_T N_p|$, centered at D_p , do not include the origin. This holds if there exists a transfer function $\alpha_p, \alpha_p^{-1} \in \mathcal{RH}_\infty$, representing for each frequency a line passing through the origin, that does not intersect with the disks. This is illustrated in Figure 3. Theorem 3 also implies internal stability because $\Re\{D_p(i\omega)\alpha(i\omega)\} > 0$ implies internal stability by Theorem 1.

The analysis condition is especially useful as it provides a local stability and performance result given only a controller and the data $\mathcal{D}_{N,p,\tau}$. Similar to the stability analysis condition, a parametric model is not required.

3.3 Synthesis

We give an equivalent formulation of Theorem 3 that is useful for controller synthesis.

Theorem 4. Given $G_p = N_{G_p} D_{G_p}^{-1}$, with $\{N_{G_p}, D_{G_p}\} \in \mathcal{RH}_\infty$ coprime, as defined in (6), and a weighting filter $W_T \in \mathcal{RH}_\infty$, the following statements are equivalent.

- 4a) There exists a proper rational controller K_p that achieves internal stability and performance as defined in requirements R1 and R2, respectively.
- 4b) There exists a controller $K_p = N_{K_p} D_{K_p}^{-1}$, with $\{N_{K_p}, D_{K_p}\} \in \mathcal{RH}_\infty$, as defined in (8), such that

$$\Re\{D_p(i\omega)\} > \gamma^{-1}|W_T(i\omega)N_p(i\omega)|, \quad (14)$$

$$\forall \omega \in \Omega, \forall p \in \mathbb{P}.$$

The proof can be found in Appendix C. Theorem 4 is the main result in this paper and presents a local \mathcal{H}_∞ -optimal controller synthesis condition given only data $\mathcal{D}_{N,p,\tau}$. This is further developed in Section 4, where an optimization

problem is formulated and the controller parameterization is discussed.

4. CONTROLLER SYNTHESIS

In this section we develop a procedure to synthesize LPV controllers directly from the frequency-domain measurement data $\mathcal{D}_{N,p,\tau}$. First, an optimization problem is set up in Section 4.1 that characterizes the synthesis problem based on Theorem 4. This is followed by a discussion on the controller parameterization in Section 4.2.

4.1 Controller synthesis

Given the data $\{\mathcal{D}_{N,p,\tau}, p \in \mathcal{P}\}$ and a controller parameterization $K_p = N_{K_p} D_{K_p}^{-1}$, the following optimization problem is formulated to satisfy Requirements R1 and R2:

$$\begin{aligned} \min_{\theta, \gamma} \quad & \gamma \\ \text{s.t.} \quad & \gamma \Re\{D_p(i\omega, \theta)\} > |W_T(i\omega)N_p(i\omega, \theta)| \\ & \forall \omega \in \Omega, p \in \mathcal{P}, \end{aligned} \quad (15)$$

where θ are the controller parameters.

The optimization problem (15) is generally non-convex. However, a linear parameterization of $\{N_{K_p}, D_{K_p}\}$ results in a quasi-convex form of (15) in the controller parameters θ and the performance indicator γ . A bisection algorithm can be used to solve the quasi-convex program. This results in an iterative approach, where for every fixed value of γ a second-order cone program is solved.

To provide stability and performance guarantees, the constraints in (15) need to be satisfied on the infinite set $\omega \in \Omega$, leading to a semi-infinite program. One solution is to solve (15) for a finite grid of frequencies $\Omega_N = \{\omega_k\}_{k=1}^N \subset \Omega$. The frequencies in this grid have to be chosen dense enough such that a Nyquist curve can be interpreted from the data.

4.2 Controller parameterization

In Section 3 the rational controller factorization is introduced. This section presents the controller parameterization and the requirements that are need to be satisfied.

- i) The controller must admit the factorization (8). This enables tuning of both the poles and zeros of the controller, in contrast to previous data-driven frequency-domain LPV tuning methods (Kunze et al., 2007; Karimi and Emedi, 2013; Bloemers et al., 2019).
- ii) The scheduling-dependency must be chosen such that $\{N_{K_p}, D_{K_p}\} \in \mathcal{RH}_\infty$ for all $p \in \mathbb{P}$.
- iii) A linear parameterization of N_{K_p} and D_{K_p} is preferred to keep (15) quasi-convex.
- iv) The controller structure must be such that the multiplier α_p can be absorbed. This requirement can be alleviated, but consequently results in a bi-linear optimization problem between the controller parameters and multiplier α_p .
- v) A monic structure of D_{K_p} avoids a trivial solution to (15). Furthermore, this ensures that $D_{K_p}^{-1}$ is well-defined for all $p \in \mathbb{P}$.

An orthonormal basis function (OBF)-based representation (Tóth, 2010) is a natural choice to parameterize the controller factors

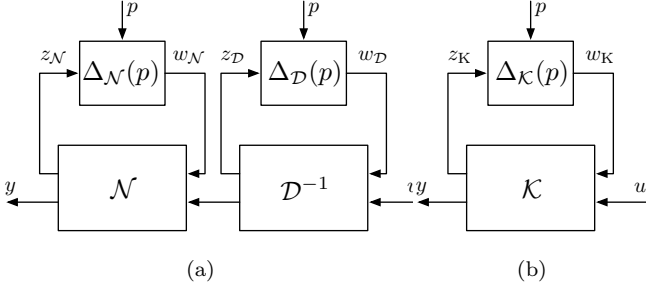


Fig. 4. Controller realization through (a) the series connection of the LFR representations $N_{K_p} = \mathcal{F}_u(\mathcal{N}, \Delta_{\mathcal{N}}(p))$ and $D_{K_p} = \mathcal{F}_u(\mathcal{D}^{-1}, \Delta_{\mathcal{D}}(p))$ and (b) $K_p = \mathcal{F}_u(\mathcal{K}, \Delta_{\mathcal{K}})$, with $\Delta_{\mathcal{K}} = \text{diag}(\Delta_{\mathcal{N}}, \Delta_{\mathcal{D}})$.

$$N_{K_p}(s) = \sum_{i=0}^{n_N} w_i(p) \phi_i(s), \quad (16a)$$

$$D_{K_p}(s) = \sum_{i=0}^{n_D} v_i(p) \varphi_i(s), \quad (16b)$$

such that the requirements (i)-(v) are satisfied. Here, $\{\phi_i\}_{i=0}^{n_N}$ and $\{\varphi_i\}_{i=0}^{n_D}$ with $\phi_0 = \varphi_0 = 1$ and $n_D \geq n_N$ are the sequence of basis functions, with coefficient functions

$$w_i(p) = \sum_{\ell=1}^m \check{w}_i^\ell \psi_\ell(p), \quad (17)$$

and similarly for $v_i(p)$. Here, the coefficient functions are formed through a chosen functional dependence, e.g., affine, polynomial or rational dependence characterized by the basis functions $\{\psi_\ell\}_{\ell=1}^m$. See (Tóth, 2010, Chapter 9.2) for an overview of OBF based LPV model structures and their properties.

Remark. The concept in this paper is to shape the global behavior of the controller by tuning the parameter-dependent coefficient functions based on their local behavior, i.e., for constant p .

4.3 Controller implementation

The OBF parameterizations admit a linear fractional representation (LFR)-structure. In this structure, the dependency on the scheduling variable p is extracted by formulating (16a) and (16b) in terms of LTI systems, denoted by \mathcal{N} and \mathcal{D} , such that $N_{K_p} = \mathcal{F}_u(\mathcal{N}, \Delta_{\mathcal{N}}(p))$ and $D_{K_p} = \mathcal{F}_u(\mathcal{D}^{-1}, \Delta_{\mathcal{D}}(p))$, respectively, where \mathcal{F}_u is the upper linear fractional transformation Zhou et al. (1996), see Figure 4a. As a consequence of controller restriction (v), the inverse input-output map \mathcal{D}^{-1} exists for all $p \in \mathbb{P}$. This inverse is obtained through partial inversion of the IO map, see, e.g., (Zhou et al., 1996, Chapter 10). The controller is formed through the series connection of the LFRs \mathcal{N} and \mathcal{D}^{-1} , resulting in the LFR \mathcal{K} such that $K_p = \mathcal{F}_u(\mathcal{K}, \text{diag}(\Delta_{\mathcal{N}}, \Delta_{\mathcal{D}}))$, see Figure 4b.

5. RESULTS

Consider a DC motor with mass imbalance corresponding to the following dynamic behavior

Table 1. Parameters of the unbalanced disk.

Parameter		Value	Unit
Motor torque constant	K	0.0536	Nm/A
Motor resistance	R	9.50	Ω
Motor impedance	L	$0.84 \cdot 10^{-3}$	H
Disk inertia	J	$2.2 \cdot 10^{-4}$	Nm ²
Viscous friction	b	$6.6 \cdot 10^{-5}$	Nms/rad
Additional mass	M	0.07	kg
Mass - center disk distance	l	0.042	m

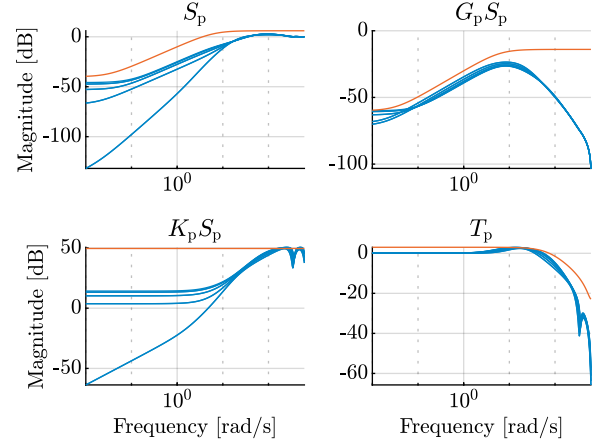


Fig. 5. Magnitude plots of the frozen closed-loop sensitivity functions in blue. Their respective weighting filters are shown in orange.

$$\begin{bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \\ \dot{I}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{Mgl}{J} p(t) & -\frac{b}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{1}{L} \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ I(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad (18a)$$

$$y(t) = \theta(t), \quad (18b)$$

where $\theta \in [-\pi, \pi] = \mathbb{Y}$ denotes the rotation angle of the disk and $u \in \mathbb{U}$ is the input voltage. Furthermore, we define by $p(t) = \text{sinc}(\theta(t)) \in \mathbb{P}$ the scheduling variable. The parameters of the unbalanced disk are given in Table 1. The unbalanced disk, intrinsically an unstable system, can be thought of as an inverted pendulum rotating around its origin. A set of FRF data of the coprime factors $\{N_{G_p}, D_{G_p}\}$ (derived analytically) is obtained at $N_{\text{loc}} = 9$ equidistantly distributed frozen operating points $p \in \mathcal{P} \subset \mathbb{P}$ and $N_\omega = 400$ logarithmically spaced frequency points $\omega \in \Omega_N \subset [10^{-2}, 200\pi]$ rad/s. The data is obtained in a discrete-time setting under a zero-order-hold assumption at a sampling-rate $T_s = 0.005$ sec.

The control objective is to design a discrete-time controller that achieves good reference tracking and disturbance rejection. The chosen control architecture is that of Figure 1, i.e., a four-block problem. The performance specifications are captured in terms of the weighting filters W_T , which are shown in Figure 5. The controller factors $\{N_{K_p}, D_{K_p}\}$ are parameterized by 5th order pulse basis functions, i.e., $\{\phi_i(s)\}_{i=0}^{n_N=5} = \{\varphi_i(s)\}_{i=0}^{n_D=5} = \{z^{-i}\}$. The controller coefficients are chosen to have affine dependence on p , resulting in $\{\psi(p)\}_{\ell=1}^{m=2} = \{1, p\}$.

Table 2. Controller parameters of $\{N_{K_p}, D_{K_p}\}$

$\ell \setminus i$	0	1	2	3	4	5
$\tilde{w}_i^{\ell=1}$	143.74	-113.36	-24.37	-40.16	-72.00	106.74
$\tilde{w}_i^{\ell=2}$	74.97	-6.25	-72.88	-44.02	-6.82	55.59
$\tilde{v}_i^{\ell=1}$	1	-0.51	-0.017	-0.24	-0.19	-0.049
$\tilde{v}_i^{\ell=2}$	0	0.39	-0.25	-0.13	-0.25	0.24

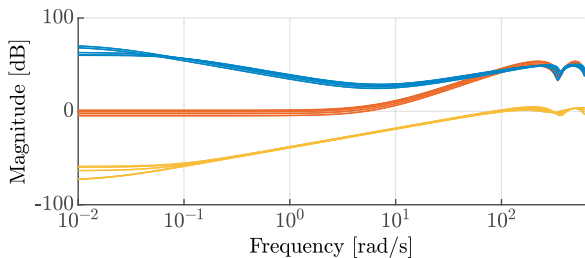


Fig. 6. Magnitude plot of the controller K in blue and $\{N_{K_p}, D_{K_p}\}$, each being parameterized with 5th order pulse bases, in orange and yellow, respectively. The plots are shown at frozen operating points $p \in \mathcal{P}$.

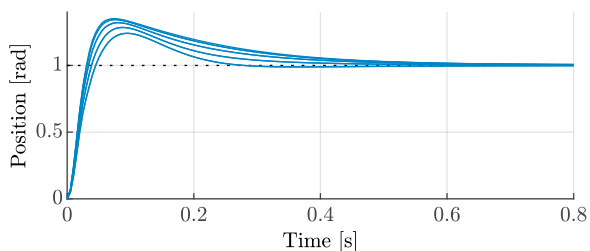


Fig. 7. Step responses of the closed-loop system using the data-driven LPV control design. The step responses are shown at frozen operating points $p \in \mathcal{P}$.

The controller design results in an LPV controller that achieves a performance of $\gamma = 1.15$. The controller parameters are given in Table 2 and the magnitude plots of the controller and its factorization are given in Figure 6. The local step responses in Figure 7 shows satisfactory performance, indicating that the LPV controller is able to adapt itself to the operating condition changes of the system based on the available information of the scheduling signal.

Figure 8 shows the reference tracking performance of the closed-loop nonlinear system with the designed LPV controller. Remark that, in contrast to before, the scheduling variable is varying over time. It can be observed that stability as well as good performance in terms of reference tracking is achieved for time-varying scheduling trajectories. However, due to the considered local stability and performance setting in this paper, stability can only be guaranteed for sufficiently slow variations of the scheduling parameter.

6. CONCLUSION

This paper presents an LPV controller synthesis approach which enables the design of operating condition-dependent controllers directly from frequency-domain measurement data. This approach enables the design of rational LPV controllers, in contrast to existing data-

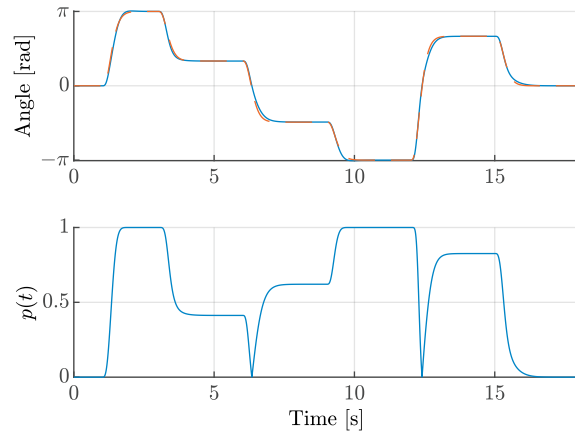


Fig. 8. Simulation of the closed-loop nonlinear system with the designed LPV controller. The top plot shows the reference trajectory and angle of the disk in dashed orange and blue, respectively. The bottom plot shows the variation of the scheduling variable.

driven methods in the literature. The capabilities of this approach are presented through a case study on an unstable nonlinear system. We emphasize that only estimates of frozen frequency response functions of the plant are required, no parametric plant model is needed.

REFERENCES

- Bloemers, T., Tóth, R., and Oomen, T. (2019). Towards Data-Driven LPV Controller Synthesis Based on Frequency Response Functions. In *Proc. of the 58th IEEE Conference on Decision and Control*. Nice, France.
- Doyle, J.C., Francis, B.A., and Tannenbaum, A.R. (1992). *Feedback Control Theory*. Macmillan Publishing Co.
- Formentin, S., Piga, D., Tóth, R., and Savaresi, S.M. (2016). Direct learning of LPV controllers from data. *Automatica*, 65, 98–110.
- Grassi, E., Tsakalis, K.S., Dash, S., Gaikwad, S.V., MacArthur, W., and Stein, G. (2001). Integrated system identification and PID controller tuning by frequency loop-shaping. *IEEE Transactions on Control Systems Technology*, 9(2), 285–294.
- Hoffmann, C. and Werner, H. (2015). A survey of linear parameter-varying control applications validated by experiments or high-fidelity simulations. *IEEE Transactions on Control Systems Technology*, 23(2), 416–433.
- Karimi, A. and Emedi, Z. (2013). \mathcal{H}_∞ gain-scheduled controller design for rejection of time-varying narrow-band disturbances applied to a benchmark problem. *European Journal of Control*, 19(4), 279–288.
- Karimi, A. and Galdos, G. (2010). Fixed-order \mathcal{H}_∞ controller design for nonparametric models by convex optimization. *Automatica*, 46(8), 1388–1394.
- Karimi, A., Kunze, M., and Longchamp, R. (2007). Robust controller design by linear programming with application to a double-axis positioning system. *Control Engineering Practice*, 15(2), 197–208.
- Karimi, A., Nicoletti, A., and Zhu, Y. (2018). Robust \mathcal{H}_∞ controller design using frequency-domain data via convex optimization. *International Journal of Robust and Nonlinear Control*, 28, 3766–3783.

- Khadraoui, S., Nounou, H., Nounou, M., Datta, A., and Bhattacharyya, S.P. (2014). A model-free design of reduced-order controllers and application to a DC servomotor. *Automatica*, 50(8), 2142–2149.
- Kunze, M., Karimi, A., and Longchamp, R. (2007). Gain-scheduled controller design by linear programming. In *Proc. of the European Control Conference*, 5432–5438. Kos, Greece.
- Maciejowski, J.M. (1989). Multivariable feedback design. *Addison-Wesley*, 6, 85–90.
- Mohammadpour, J. and Scherer, C.W. (2012). *Control of linear parameter varying systems with applications*. Springer-Verlag, New York.
- Oomen, T. and Steinbuch, M. (2017). Model-based control for high-tech mechatronic systems. *The Handbook on Electrical Engineering Technology and Systems*, 5.
- Pintelon, R. and Schoukens, J. (2012). *System Identification: A Frequency Domain Approach*. John Wiley & Sons, 2 edition.
- Rantzer, A. and Megretski, A. (1994). A Convex Parameterization of Robustly Stabilizing Controllers. *IEEE Transactions on Automatic Control*, 39(9), 1802–1808.
- Schoukens, M. and Tóth, R. (2019). Frequency response functions of linear parameter-varying systems.
- Shamma, J.S. and Athans, M. (1990). Analysis of gain scheduled control for nonlinear plants. *IEEE Transactions on Automatic Control*, 35(8), 898–907.
- Skogestad, S. and Postlethwaite, I. (2001). *Multivariable Feedback Control Analysis and design*. John Wiley & Sons, 2 edition.
- Steinbuch, M. and Norg, M.L. (1998). Advanced motion control: An industrial perspective. *European Journal of Control*, 4(4), 278–293.
- Tóth, R. (2010). *Modeling and identification of linear parameter-varying systems*, volume 403. Springer, Heidelberg.
- van Solingen, E., van Wingerden, J., and Oomen, T. (2018). Frequency-domain optimization of fixed-structure controllers. *International Journal of Robust and Nonlinear Control*, 28(12), 3784–3805.
- Zhou, K., Doyle, J.C., Glover, K., et al. (1996). *Robust and optimal control*, volume 40. Prentice hall, New Jersey.

Appendix A. PROOF OF THEOREM 1

For a proof of equivalence between 1a) and 1b), see (Doyle et al., 1992, Chapter 3). Regarding the equivalence between 1a) and 1c) for all $p \in \mathbb{P}$, note the following reasoning:

(\Rightarrow) Assume 1a) and let $Q = D_p^{-1}$. This implies that the Bézout identity (7) is satisfied for $X_p = N_{K_p}Q$ and $Y_p = D_{K_p}Q$ because, $\{X_p, Y_p\}$ are coprime iff $Q, Q^{-1} \in \mathcal{RH}_\infty$. Hence, 1c) is satisfied by setting $\alpha_p = Q$ because $\Re\{N_{G_p}X_p + D_{G_p}Y_p\} = 1$ for all $\omega \in \Omega$.

(\Leftarrow) Assume 1c) and let $V = D_p\alpha_p$. Note that $V, V^{-1} \in \mathcal{RH}_\infty$ because 1c) implies that $D_p\alpha_p$ is bi-proper and has no right half-plane (RHP) zeros. Then $D_p = V\alpha_p^{-1}$ satisfies the Bézout identity (7), therefore $D_p^{-1} \in \mathcal{RH}_\infty$. Thus 1c) implies 1a) and consequently 1b). This completes the proof.

Appendix B. PROOF OF THEOREM 3

Requirement R2 can be equivalently stated using Theorem 2, Condition 2b), i.e.,

$$1 - \gamma^{-1}W_T(i\omega)T_{z,w}(G_p, K_p)(i\omega)\hat{\Delta}(i\omega) \neq 0, \quad \forall \omega \in \Omega, \forall p \in \mathbb{P}, \forall \hat{\Delta} \in \mathbf{B}\hat{\Delta}. \quad (\text{B.1})$$

As $D_p \in \mathcal{RH}_\infty$, $D_p(i\omega) \neq 0$, $\forall \omega \in \Omega$ and by multiplying (B.1) with it, the resulting non-singularity condition is:

$$D_p(i\omega) - \gamma^{-1}W_T(i\omega)N_p(i\omega)\hat{\Delta}(i\omega) \neq 0, \quad \forall \omega \in \Omega, \forall p \in \mathbb{P}, \forall \hat{\Delta} \in \mathbf{B}\hat{\Delta}. \quad (\text{B.2})$$

Based on a homotopy argument, (B.2) corresponds to Condition 1b) in Theorem 1, which through 1c) is equivalent with

$$\Re\{(D_p(i\omega) - \gamma^{-1}W_T(i\omega)N_p(i\omega)\hat{\Delta}(i\omega))\alpha_p(i\omega)\} > 0, \quad \forall \omega \in \Omega, \forall p \in \mathbb{P}, \hat{\Delta} \in \mathbf{B}\hat{\Delta}. \quad (\text{B.3})$$

When $\hat{\Delta} = 0 \in \mathbf{B}\hat{\Delta}$, (B.3) reduces to $\Re\{D_p(i\omega)\alpha_p(i\omega)\} > 0$, which is the same as Condition 1c) in Theorem 1, hence (B.3) implies requirement R1.

Let $1 \geq \epsilon > 0$ and consider (B.3) on

$$\mathbf{B}_\epsilon \hat{\Delta} := \left\{ \hat{\Delta} \in \mathcal{RH}_\infty \mid |\hat{\Delta}(i\omega)| \leq 1 - \epsilon, \forall \omega \in \Omega \right\}, \quad (\text{B.4})$$

which is the scaled closed uncertainty ball contained in $\mathbf{B}\hat{\Delta}$. Since any $\hat{\Delta} \in \mathbf{B}_\epsilon \hat{\Delta}$ represents a rotation and contraction in the complex plane, it is necessary and sufficient to check (B.3) on the boundary only, i.e., for $\hat{\Delta} \in \partial \mathbf{B}_\epsilon \hat{\Delta}$, with $|\hat{\Delta}(i\omega)| = 1 - \epsilon$, $\forall \omega \in \Omega$. Note that, in (B.3), $W_T(i\omega)N_p(i\omega)$ only represents complex scaling of this ball which is centered at $D_p(i\omega)$. Hence, (B.3) restricted on $\mathbf{B}_\epsilon \hat{\Delta}$ is equivalent with

$$\Re\{(D_p(i\omega) - \gamma^{-1}(1 - \epsilon)|W_T(i\omega)N_p(i\omega)|)\alpha_p(i\omega)\} > 0, \quad \forall \omega \in \Omega, \forall p \in \mathbb{P}. \quad (\text{B.5})$$

This means that if (B.5) holds, then violation of (B.3) can only happen in $\mathbf{B}\hat{\Delta} \setminus \mathbf{B}_\epsilon \hat{\Delta}$. As (B.5) is continuous in ϵ , by taking the limit $\epsilon \rightarrow 0$, $\mathbf{B}\hat{\Delta} \setminus \mathbf{B}_\epsilon \hat{\Delta} \rightarrow \emptyset$ and we obtain that (13) is equivalent with (B.3). This completes the proof.

Appendix C. PROOF OF THEOREM 4

(\Rightarrow) Assume $K_p = \tilde{N}_{K_p}\tilde{D}_{K_p}^{-1}$ satisfies 4a). Then, by Theorem 3, there exists an $\alpha_p, \alpha_p^{-1} \in \mathcal{RH}_\infty$ such that (13) holds. Choosing $N_{K_p} = \tilde{N}_{K_p}\alpha_p$, $D_{K_p} = \tilde{D}_{K_p}\alpha_p$ results in $K_p = N_{K_p}D_{K_p}^{-1} = \tilde{N}_{K_p}\tilde{D}_{K_p}^{-1}$ and consequently 4b) holds.

(\Leftarrow) Assume 4b) holds. Because $D_p \in \mathcal{RH}_\infty$ and $D_p(i\omega)$ is positive for all $\omega \in \Omega$, $D_p(i\omega)$ cannot encircle the origin when ω traverses the Nyquist contour. Thus $D_p^{-1} \in \mathcal{RH}_\infty$ and K_p internally stabilizes G_p . Furthermore,

$$|D_p(i\omega)| \geq \Re\{D_p(i\omega)\} \quad \forall \omega \in \Omega, \forall p \in \mathbb{P} \quad (\text{C.1})$$

implies

$$|D_p(i\omega)| > \gamma^{-1}|W_T(i\omega)N_p(i\omega)| \quad \forall \omega \in \Omega, \forall p \in \mathbb{P} \quad (\text{C.2})$$

and consequently 4a) holds. This completes the proof.