An upper bound for the size of s-distance sets in real algebraic sets

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Abstract

In a recent paper, Petrov and Pohoata developed a new algebraic method which combines the Croot-Lev-Pach Lemma from additive combinatorics and Sylvester's Law of Inertia for real quadratic forms. As an application, they gave a simple proof of the Bannai-Bannai-Stanton bound on the size of s-distance sets (subsets $\mathcal{A} \subseteq \mathbb{R}^n$ which determine at most s different distances). In this paper we extend their work and prove upper bounds for the size of s-distance sets in various real algebraic sets. This way we obtain a novel and short proof for the bound of Delsarte-Goethals-Seidel on spherical s-distance sets and a generalization of a bound by Bannai-Kawasaki-Nitamizu-Sato on s-distance sets on unions of spheres. In our arguments we use the method of Petrov and Pohoata together with some Gröbner basis techniques.

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1 Introduction

Let $\mathcal{A} \subseteq \mathbb{R}^n$ be an arbitrary set. Denote by $d(\mathcal{A})$ the set of non-zero euclidean distances among the points of $\mathcal{A}$:

$$d(\mathcal{A}) := \{d(\mathbf{p}_1, \mathbf{p}_2);\ \mathbf{p}_1, \mathbf{p}_2 \in \mathcal{A},\ \mathbf{p}_1 \neq \mathbf{p}_2\}.$$

An *s-distance set* is a subset $\mathcal{A} \subseteq \mathbb{R}^n$ such that $|d(\mathcal{A})| \leqslant s$. Here we mention just two theorems from the rich area of sets with few distances, more information can be found for example in [14], [3]. Bannai, Bannai and Stanton proved the following upper bound for the size of an s-distance set in [4, Theorem 1].

Theorem 1. *Let $n, s \geqslant 1$ be integers and suppose that $\mathcal{A} \subseteq \mathbb{R}^n$ is an s-distance set. Then*

$$|\mathcal{A}| \leqslant \binom{n+s}{s}.$$

Delsarte, Goethals and Seidel investigated s-distance sets on the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. These are the *spherical s-distance sets*. They proved a general upper bound for the size of a spherical s-distance set in [11]. In their proof they used Delsarte's method (see [3, Subsection 2.2]).

Theorem 2. *(Delsarte, Goethals, and Seidel) Let $n, s \geqslant 1$ be integers and suppose that $\mathcal{A} \subseteq \mathbb{S}^{n-1}$ is an s-distance set. Then*

$$|\mathcal{A}| \leqslant \binom{n+s-1}{s} + \binom{n+s-2}{s-1}.$$

Before stating our results, we introduce some notation. Let $\mathbb{F}$ be a field. In the following $S = \mathbb{F}[x_1, \ldots, x_n] = \mathbb{F}[\mathbf{x}]$ denotes the ring of polynomials in commuting variables $x_1, \ldots, x_n$ over $\mathbb{F}$. Note that polynomials $f \in S$ can be considered as functions on $\mathbb{F}^n$. For a subset Y of the polynomial ring S and a natural number s we denote by $Y_{\leqslant s}$ the set of polynomials from Y with degree at most s. Let I be an ideal of $S = \mathbb{F}[\mathbf{x}]$. The *(affine) Hilbert function* of the factor algebra S/I is the sequence of non-negative integers $h_{S/I}(0), h_{S/I}(1), \ldots$, where $h_{S/I}(s)$ is the dimension over $\mathbb{F}$ of the factor space $\mathbb{F}[x_1, \ldots, x_n]_{\leqslant s}/I_{\leqslant s}$ (see [8, Section 9.3]). Our main technical result gives an upper bound for the size of an s-distance set, which is contained in a given real algebraic set.

Theorem 3. *Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be an ideal in the polynomial ring, and let $\mathcal{A} \subseteq \mathbb{R}^n$ be an s-distance set such that every polynomial from I vanishes on $\mathcal{A}$. Then*

$$|\mathcal{A}| \leqslant h_{\mathbb{R}[\mathbf{x}]/I}(s).$$

The proof is based on Gröbner basis theory and an improved version of the Croot-Pach-Lev Lemma (see [9] Lemma 1) over the reals. Petrov and Pohoata proved this in [20, Theorem 1.2] and used it to give a new proof of Theorem 1. We generalize their result to give a new upper bound for the size of an s-distance set, which is contained in a given affine algebraic set in the real affine space $\mathbb{R}^n$.

We give several corollaries, where Theorem 3 is applied to specific ideals of the polynomial ring $\mathbb{R}[\mathbf{x}]$, the first ones being the principal ideals $I = (F)$, with $F \in \mathbb{R}[\mathbf{x}]$.

Corollary 4. *Let $F \in \mathbb{R}[\mathbf{x}]$ be a polynomial of degree d. Suppose that $s \geqslant d$. Let $\mathcal{A}$ be an s-distance set such that F vanishes on $\mathcal{A}$. Then*

$$|\mathcal{A}| \leqslant \binom{n+s}{n} - \binom{n+s-d}{n}.$$

For example, when $n = 2$, then F defines a plane curve of degree d. Then for $s \geqslant d$ we obtain

$$|\mathcal{A}| \leqslant \binom{2+s}{2} - \binom{2+s-d}{2} = ds - \frac{d(d-3)}{2}.$$

In particular, when $F(x, y) = y^2 - f(x)$ gives a Weierstrass equation of an elliptic curve, then $|\mathcal{A}| \leqslant 3s$ for $s \geqslant 3$.

Remark 5. We can now easily derive Theorem 2 for $s > 1$. Indeed, consider the real polynomial

$$F(x_1, \ldots, x_n) = 1 - \sum_{i=1}^{n} x_i^2 \in \mathbb{R}[x_1, \ldots, x_n]$$

of degree 2 which vanishes on $\mathbb{S}^{n-1}$. Corollary 4 and the hockey-stick identity gives

$$|\mathcal{A}| \leqslant \binom{n+s}{n} - \binom{n+s-2}{n} = \binom{n+s-1}{s} + \binom{n+s-2}{s-1}.$$

Next, assume that $V = \cup_{i=1}^{p} \mathcal{S}_i$, where the $\mathcal{S}_i$ are spheres in $\mathbb{R}^n$. E. Bannai, K. Kawasaki, Y. Nitamizu, and T. Sato proved the following result in [5, Theorem 1] for the case when the spheres $\mathcal{S}_i$ are *concentric*. We have a much shorter approach to the same bound, in a more general setting, without the assumption on the centers.

Corollary 6. *Let $\mathcal{A}$ be an s-distance set on the union V of p spheres in $\mathbb{R}^n$. Then*

$$|\mathcal{A}| \leqslant \sum_{i=0}^{2p-1} \binom{n+s-i-1}{s-i}.$$

Let $T_i \subseteq \mathbb{R}$ be given finite sets, where $|T_i| = q \geqslant 2$ for each i with $1 \leqslant i \leqslant n$. A *box* is a direct product

$$\mathcal{B} := \prod_{i=1}^{n} T_i \subseteq \mathbb{R}^n.$$

We can easily apply Theorem 3 to obtain an upper bound for the size of s-distance sets in boxes.

Corollary 7. *Let $\mathcal{B} \subseteq \mathbb{R}^n$ be a box as above, and $\mathcal{A} \subseteq \mathcal{B}$ an s-distance set. Then*

$$|\mathcal{A}| \leqslant |\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \ 0 \leqslant \alpha_i \leqslant q-1 \text{ for each } i, \text{ and } \sum_i \alpha_i \leqslant s\}|.$$

Remark 8. In the special case $q = 2$ we have

$$|\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \ 0 \leqslant \alpha_i \leqslant 1 \text{ for each } i, \text{ and } \sum_i \alpha_i \leqslant s\}| = \sum_{j=0}^{s} \binom{n}{j},$$

hence we obtain the upper bound

$$|\mathcal{A}| \leqslant \sum_{j=0}^{s} \binom{n}{j}. \tag{1}$$

In the case when $T_i = T$ for $1 \leqslant i \leqslant n$ and $|T| = 2$, the Euclidean distance is essentially the same as the Hamming distance. For this case (1) was proved by Delsarte [10], see also [2, Theorem 1].

Remark 9. The bound is sharp, when $q = 2$, $n = 2m$ and $s = m$. Then the 0,1 vectors of even Hamming weight give an extremal family $\mathcal{A} \subseteq \mathbb{R}^n$.

Remark 10. The bound of Corollary 7 can be nicely formulated in terms of extended binomial coefficients (see [12, Example 8] or [7, Exercise 16]):

$$|\mathcal{A}| \leqslant \sum_{j=0}^{s} \binom{n}{j}_q.$$

Here $\binom{n}{j}_q$ is an extended binomial coefficient giving the number of restricted compositions of j with n terms (summands), where each term is from the set $\{0, 1, \ldots, q-1\}$. In particular, we have $\binom{n}{j}_2 = \binom{n}{j}$.

Remark 11. In [16] a weaker, but similar upper bound was given for the size of s-distance sets in boxes:

$$|\mathcal{A}| \leqslant 2|\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \ 0 \leqslant \alpha_i \leqslant q-1 \text{ for each } i, \text{ and } \sum_i \alpha_i \leqslant s\}|.$$

The bound appearing in Corollary 7 presents an improvement by a factor of 2.

Let $\alpha_1, \ldots, \alpha_n$ be n different elements of $\mathbb{R}$, and $X_n = X_n(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{R}^n$ be the set of permutations of $\alpha_1, \ldots, \alpha_n$, where each permutation is considered as vector of length n. It was proved in [17, Section 2] that for $s \geqslant 0$

$$h_{X_n}(s) = \sum_{i=0}^{s} I_n(i),$$

where $I_n(i)$ is the number of permutations of n symbols with precisely i inversions. Using this, Theorem 3 implies the following bound:

Corollary 12. *Let $\mathcal{A} \subseteq X_n$ be an s-distance set. Then*

$$|\mathcal{A}| \leqslant \sum_{i=0}^{s} I_n(i).$$

In [19, Section 5.1.1] Knuth gives a generating function for $I_n(i)$ and some explicit formulae for the values $I_n(i)$, $i \leqslant n$.

Let $0 \leqslant d \leqslant n$ be integers and $Y_{n,d} \subseteq \mathbb{R}^n$ denote the set of 0,1-vectors of length n which have exactly d coordinate values of 1. The following (sharp) bound was obtained by Ray-Chaudhuri and Wilson in [21, Theorem 3], formulated in terms of intersections rather than distances.

Corollary 13. *Let $0 \leqslant d \leqslant n$ and s be integers, with $0 \leqslant s \leqslant \min(d, n-d)$. Suppose that $\mathcal{A} \subseteq Y_{n,d}$ is an s-distance set. Then*

$$|\mathcal{A}| \leqslant \binom{n}{s}.$$

In some cases data about the complexification of a real affine algebraic set can be used to give a bound. We give next a statement of this type. For a subset $X \subseteq \mathbb{F}^n$ of the affine space we write $I(X)$ for the ideal of all polynomials $f \in \mathbb{F}[\mathbf{x}]$ which vanish on X.

Corollary 14. *Let $V \subseteq \mathbb{C}^n$ be an affine variety such that the projective closure $\overline{V}$ of V has dimension d and degree k. Suppose also that the ideal $I(V)$ of V is generated by polynomials over $\mathbb{R}$. Let $\mathcal{A} \subseteq V \cap \mathbb{R}^n$ be an s-distance set. Then we have*

$$|\mathcal{A}| \leqslant \frac{k \cdot s^d}{d!} + O(s^{d-1}).$$

For instance, when in Corollary 14 the projective variety $\overline{V}$ is a curve of degree k, then the bound is $ks + b$ for large s, where b is an integer. More specifically, when $\overline{V}$ is an elliptic curve such that $V \subseteq \mathbb{C}^2$ is the set of zeroes of $y^2 - f(x)$, where $f(x) \in \mathbb{R}[x]$ is a cubic polynomial without multiple roots, then in fact, the preceding bound becomes $|\mathcal{A}| \leqslant 3s + b$ for s large (see also the remark after Corollary 4).

The rest of the paper is organized as follows. Section 2 contains some preliminaries on Gröbner bases, Hilbert functions, and related notions. Section 3 contains the proofs of the main theorem and the proof of the corollaries.

2 Preliminaries

A total ordering $\prec$ on the monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ composed from variables $x_1, x_2, \ldots, x_n$ is a *term order*, if 1 is the minimal element of $\prec$, and $uw \prec vw$ holds for any monomials u, v, w with $u \prec v$. Two important term orders are the lexicographic order $\prec_l$ and the deglex order $\prec_{dl}$. We have

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \prec_l x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

iff $i_k < j_k$ holds for the smallest index k such that $i_k \neq j_k$. As for the deglex order, we have $u \prec_{dl} v$ iff either $\deg u < \deg v$, or $\deg(u) = \deg(v)$, and $u \prec_l v$.

Let $\prec$ be a fixed term order. The *leading monomial* $\mathrm{lm}(f)$ of a nonzero polynomial f from the ring $S = \mathbb{F}[\mathbf{x}]$ is the largest (with respect to $\prec$) monomial which occurs with nonzero coefficient in the standard form of f.

Let I be an ideal of S. A finite subset $G \subseteq I$ is a *Gröbner basis* of I if for every $f \in I$ there exists a $g \in G$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$. It can be shown that G is in fact a basis of I. A fundamental result is (cf. [6, Chapter 1, Corollary 3.12] or [1, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal I of S has a Gröbner basis with respect to $\prec$.

A monomial $w \in S$ is a *standard monomial* for I if it is not a leading monomial of any $f \in I$. Let $\mathrm{Sm}(\prec, I, \mathbb{F})$ denote the set of all standard monomials of I with respect to the term-order $\prec$ over $\mathbb{F}$. It is known (see [6, Chapter 1, Section 4]) that for a nonzero ideal I the set $\mathrm{Sm}(\prec, I, \mathbb{F})$ is a basis of the factor space S/I over $\mathbb{F}$. Hence every $g \in S$ can be written uniquely as $g = h + f$ where $f \in I$ and h is a unique $\mathbb{F}$-linear combination of monomials from $\mathrm{Sm}(\prec, I, \mathbb{F})$.

If $X \subseteq \mathbb{F}^n$ is a finite set, then an interpolation argument gives that every function from X to $\mathbb{F}$ is a polynomial function. The latter two facts imply that

$$|\mathrm{Sm}(\prec, I(X), \mathbb{F})| = |X|, \tag{2}$$

where $I(X)$ is the ideal of all polynomials from S which vanish on X, and $\prec$ is an arbitrary term order.

The *initial ideal* $\mathrm{in}(I)$ of I is the ideal in S generated by the set of monomials $\{\mathrm{lm}(f) : \ f \in I\}$.

It is easy to see [8, Propositions 9.3.3 and 9.3.4] that the value at s of the Hilbert function $h_{S/I}$ is the number of standard monomials of degree at most s, where the ordering $\prec$ is deglex:

$$h_{S/I}(s) = |\mathrm{Sm}(\prec_{dl}, I, \mathbb{F}) \cap \mathbb{F}[\mathbf{x}]_{\leqslant s}|. \tag{3}$$

In the case when $I = I(X)$ for some $X \subseteq \mathbb{F}^n$, then $h_X(s) := h_{S/I}(s)$ is the dimension of the space of functions from X to $\mathbb{F}$ which are polynomials of degree at most s.

Next we recall a known fact about the Hilbert function. It concerns the change of the coefficient field. Let $\mathbb{F} \subset \mathbb{K}$ be fields and let $I \subseteq \mathbb{F}[\mathbf{x}]$ be an ideal, and consider the corresponding ideal $J = I \cdot \mathbb{K}[\mathbf{x}]$ generated by I in $\mathbb{K}[\mathbf{x}]$.

Lemma 15. *For the respective affine Hilbert functions for $s \geqslant 0$ we have*

$$h_{\mathbb{F}[\mathbf{x}]/I}(s) = h_{\mathbb{K}[\mathbf{x}]/J}(s).$$

For the convenience of the reader we outline a simple proof.

Proof. It follows from Buchberger's criterion [8, Theorem 2.6.6] that a deglex Gröbner basis of I in $\mathbb{F}[\mathbf{x}]$ will be a deglex Gröbner basis of J in $\mathbb{K}[\mathbf{x}]$, implying that the initial ideals $\mathrm{in}(I)$ and $\mathrm{in}(J)$ contain exactly the same set of monomials, hence their respective

factors have the same Hilbert function $h_{\mathbb{F}[\mathbf{x}]/\mathrm{in}(I)}(s) = h_{\mathbb{K}[\mathbf{x}]/\mathrm{in}(J)}(s)$, see [8, Proposition 9.3.3]. Then by [8, Proposition 9.3.4] we have

$$h_{\mathbb{F}[\mathbf{x}]/I}(s) = h_{\mathbb{F}[\mathbf{x}]/\mathrm{in}(I)}(s) = h_{\mathbb{K}[\mathbf{x}]/\mathrm{in}(J)}(s) = h_{\mathbb{K}[\mathbf{x}]/J}(s),$$

for every integer $s \geqslant 0$. □

The projective (homogenized) version of the next statement is discussed in [13, Example 6.10].

Proposition 16. *Let $F \in \mathbb{F}[\mathbf{x}]$ be a polynomial of degree d. Then for $s \geqslant d$ we have*

$$h_{\mathbb{F}[\mathbf{x}]/(F)}(s) = \binom{n+s}{n} - \binom{n+s-d}{n}.$$

If $0 \leqslant s < d$, then

$$h_{\mathbb{F}[\mathbf{x}]/(F)}(s) = \binom{n+s}{n}.$$

Proof. By definition

$$h_{\mathbb{F}[\mathbf{x}]/(F)}(s) = \dim \mathbb{F}[\mathbf{x}]_{\leqslant s}/(F)_{\leqslant s} = \\ = \dim \mathbb{F}[\mathbf{x}]_{\leqslant s} - \dim(F)_{\leqslant s}.$$

Clearly

$$\dim \mathbb{F}[\mathbf{x}]_{\leqslant s} = \binom{n+s}{n}.$$

Moreover

$$(F)_{\leqslant s} = \{G \in \mathbb{F}[\mathbf{x}]_{\leqslant s} : \quad \text{there exists an } H \in \mathbb{F}[\mathbf{x}] \text{ such that } FH = G\}.$$

Using the fact that $\mathbb{F}[\mathbf{x}]$ is a domain, we see that the dimension of the latter subspace is

$$\dim\{H \in \mathbb{R}[\mathbf{x}] : \quad \deg(H) \leqslant s-d\} = \dim \mathbb{F}[\mathbf{x}]_{\leqslant (s-d)}.$$

The statement now follows from the fact that if $s \geqslant d$, then

$$\dim \mathbb{F}[\mathbf{x}]_{\leqslant (s-d)} = \binom{n+s-d}{n},$$

while for $s < d$ we have

$$\dim \mathbb{F}[\mathbf{x}]_{\leqslant (s-d)} = 0.$$

□

3 Proofs

3.1 Proof of the main result

Petrov and Pohoata proved the following result [20, Theorem 1.2]. They used it to give a short proof of Theorem 1. This improved version of the Croot-Lev-Pach Lemma has a crucial role in the proof of our results.

Theorem 17. *Let W be an n-dimensional vector space over a field $\mathbb{F}$ and let $\mathcal{A} \subseteq W$ be a finite set. Let $s \geqslant 0$ be an integer an let $p(\mathbf{x}, \mathbf{y}) \in \mathbb{F}[\mathbf{x}, \mathbf{y}]$ be a $2n$-variate polynomial of degree at most $2s+1$. Consider the matrix $M(\mathcal{A}, p)_{\mathbf{a},\mathbf{b}\in A}$, where*

$$M(\mathcal{A}, p)(\mathbf{a}, \mathbf{b}) = p(\mathbf{a}, \mathbf{b}).$$

This matrix corresponds to a bilinear form on $\mathbb{F}^{\mathcal{A}}$ by the formula

$$\Phi_{\mathcal{A},p}(f, g) = \sum_{\mathbf{a},\mathbf{b}\in\mathcal{A}} p(\mathbf{a}, \mathbf{b}) f(\mathbf{a}) g(\mathbf{b}),$$

for each $f, g : \mathcal{A} \to \mathbb{F}$. This $\Phi_{\mathcal{A},p}$ defines a quadratic form $\Phi_{\mathcal{A},p}(f, f)$. In the case $\mathbb{F} = \mathbb{R}$ denote by $r_+(\mathcal{A}, p)$ and $r_-(\mathcal{A}, p)$ the inertia indices of the quadratic form $\Phi_{\mathcal{A},p}(f, f)$. Then

(i) $\operatorname{rank}(M(\mathcal{A}, p)) \leqslant 2h_{\mathcal{A}}(s)$,

(ii) if $\mathbb{F} = \mathbb{R}$, then $\max(r_+(\mathcal{A}, p), r_-(\mathcal{A}, p)) \leqslant h_{\mathcal{A}}(s)$.

By combining Theorem 17 with facts about standard monomials, we have the following simple and elegant upper bound for the degree of deglex standard monomials of an s-distance set.

Theorem 18. *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be an s-distance set. Then*

$$Sm(\prec_{dl}, I(\mathcal{A}), \mathbb{F}) \subseteq \mathbb{R}[\mathbf{x}]_{\leqslant s}.$$

Proof. We follow the argument of [20, Theorem 1.1]. Let $\mathcal{A} \subseteq \mathbb{R}^n$ denote an s-distance set. Recall that $d(\mathcal{A})$ denotes the set of (non-zero) distances among points of $\mathcal{A}$. Define the $2n$–variate polynomial by:

$$p(\mathbf{x}, \mathbf{y}) = \prod_{t\in d(\mathcal{A})} \left(t^2 - \|\mathbf{x} - \mathbf{y}\|^2\right) \in \mathbb{R}[\mathbf{x}, \mathbf{y}].$$

Then we can apply Theorem 17 for $p(\mathbf{x}, \mathbf{y})$ whose degree is $2s$. The matrix $M(\mathcal{A}, p)$ is a positive diagonal matrix, giving that

$$r_+(\mathcal{A}, p) = |\mathcal{A}|.$$

It follows from Theorem 17 (ii) that

$$|\mathcal{A}| = r_+(\mathcal{A}, p) \leqslant h_{\mathcal{A}}(s).$$

But equations (3), (2) and the finiteness of $\mathcal{A}$ imply that

$$|\mathcal{A}| \leqslant h_{\mathcal{A}}(s) = |\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R}) \cap \mathbb{R}[\mathbf{x}]_{\leqslant s}| \leqslant |\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R})| = |\mathcal{A}|.$$

We infer that

$$|\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R}) \cap \mathbb{R}[\mathbf{x}]_{\leqslant s}| = |\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R})|,$$

and hence

$$\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R}) \subseteq \mathbb{R}[\mathbf{x}]_{\leqslant s}. \qquad \square$$

Proof of Theorem 3. Theorem 18 gives that

$$\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R}) \subseteq \mathbb{R}[\mathbf{x}]_{\leqslant s}.$$

Since I vanishes on $\mathcal{A}$, we have $I \subseteq I(\mathcal{A})$, hence

$$\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R}) \subseteq \mathrm{Sm}(\prec_{dl}, I, \mathbb{R}).$$

The preceding two relations imply that

$$\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R}) \subseteq \mathrm{Sm}(\prec_{dl}, I, \mathbb{R}) \cap \mathbb{R}[\mathbf{x}]_{\leqslant s}.$$

Now it follows from (3) and (2) that

$$|\mathcal{A}| = |\mathrm{Sm}(\prec_{dl}, I(\mathcal{A}), \mathbb{R})| \leqslant |\mathrm{Sm}(\prec_{dl}, I, \mathbb{R}) \cap \mathbb{R}[\mathbf{x}]_{\leqslant s}| = h_{\mathbb{R}[\mathbf{x}]/I}(s). \qquad \square$$

3.2 Proofs of the Corollaries

Proof of Corollary 4. From Theorem 3 we obtain the bound $|\mathcal{A}| \leqslant h_{\mathbb{R}[\mathbf{x}]/(F)}(s)$, therefore for $s \geqslant d$ we have

$$|\mathcal{A}| \leqslant h_{\mathbb{R}[\mathbf{x}]/(F)}(s) = \binom{n+s}{n} - \binom{n+s-d}{n},$$

by Proposition 16. $\square$

Proof of Corollary 6. It is easy to verify that

$$\sum_{i=0}^{2p-1} \binom{n+s-i-1}{s-i} = \binom{n+s}{s} - \binom{n+s-2p}{n}.$$

Let $V = \cup_{i=1}^{p} \mathcal{S}_i$, and assume, that the center of the sphere $\mathcal{S}_i$ is the point $(a_{1,i}, \ldots, a_{n,i}) \in \mathbb{R}^n$, and the radius of $\mathcal{S}_i$ is $r_i \in \mathbb{R}$ for $i = 1, \ldots, p$. Next consider the polynomials

$$F_i(x_1, \ldots, x_n) = (\sum_{m=1}^{n} (x_m - a_{m,i})^2) - r_i^2 \in \mathbb{R}[x_1, \ldots, x_n]$$

for each i and put $F := \prod_i F_i$. Then $\deg(F) = 2p$ and F vanishes on V. We may apply Corollary 4 for the polynomial F. Then for $s \geqslant 2p$ we obtain the desired bound

$$|\mathcal{A}| \leqslant \binom{n+s}{n} - \binom{n+s-2p}{n}.$$

When $s < 2p$, the bound follows from the Bannai-Bannai-Stanton theorem. $\square$

Proof of Corollary 7. It is well-known and easily proved that the following set of polynomials is a (reduced) Gröbner basis of the ideal $I(\mathcal{B})$ (with respect to any term order):

$$\left\{ \prod_{t \in T_i} (x_i - t) : \ 1 \leqslant i \leqslant n \right\}.$$

This readily gives the (deglex) standard monomials for $I(\mathcal{B})$:

$$\mathrm{Sm}(\prec_{dl}, I(\mathcal{B}), \mathbb{R}) = \{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \ 0 \leqslant \alpha_i \leqslant q-1 \text{ for each } i\}.$$

It follows from Theorem 3 and equation (3) that

$$|\mathcal{A}| \leqslant h_{\mathcal{B}}(s) = |\mathrm{Sm}(\prec_{dl}, I(\mathcal{B}), \mathbb{R}) \cap \mathbb{R}[\mathbf{x}]_{\leqslant s}| =$$

$$= |\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : \ 0 \leqslant \alpha_i \leqslant q-1 \text{ for each } i, \ \text{ and } \sum_i \alpha_i \leqslant s\}|. \qquad \square$$

Proof of Corollary 13. The statement follows at once from the result

$$h_{Y_{n,d}}(s) = \binom{n}{s} \tag{4}$$

proved by Wilson in [22] (formulated there in the language of inclusion matrices, see also [18, Corollary 3.1]), and Theorem 3. $\square$

Proof of Corollary 14. Write $I = I(V) \cap \mathbb{R}[\mathbf{x}]$ and $J = I(V) \subseteq \mathbb{C}[\mathbf{x}]$. It follows from Theorem 3 and Proposition 15 that

$$|\mathcal{A}| \leqslant h_{\mathbb{R}[\mathbf{x}]/I}(s) = h_{\mathbb{C}[\mathbf{x}]/J}(s).$$

From [8, Theorem 9.3.12] we obtain that the affine Hilbert function $h_{\mathbb{C}[\mathbf{x}]/J}(s)$ is the same as the projective Hilbert function $h_{\overline{V}}(s)$ of the projective variety $\overline{V}$. Now [15, Proposition 13.2] and the subsequent remark imply that for s large the Hilbert function will be the same as the Hilbert polynomial: $h_{\overline{V}}(s) = p_{\overline{V}}(s)$, moreover

$$p_{\overline{V}}(s) = \frac{k}{d!} \cdot s^d + \text{ terms of degree at most } d-1 \text{ in } s.$$

This proves the statement. $\square$

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