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Generalised system level approach

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Abstract: Recently, system level approach has been proposed as a novel design tool for large-scale cyber physical systems in the discrete time LTI framework. The main idea of the new paradigm is a shift from constructing and parametrizing the desired optimal controller to the parametrization of the achievable closed loop behaviours (closed loop shape). Then, the controller is synthesised from the system level data in a way that keep the imposed structural constraints intact.
In this paper we lift the main idea of the system level parametrization to a general level, that also includes the class of LPV systems. We also provide a general scheme for the realization related to different Möbius transforms which are formulated in terms of the original data and also keep intact the structural information.

Keywords: system level approach, Youla parametrization, decentralized controller

1. INTRODUCTION AND MOTIVATION

Many optimal controller synthesis procedures rely on the Youla-Kucera parametrization of all internally stabilizing controllers, Kucera [1975], Youla et al. [1976], Kevickzy and Banyasz [2015], and the responses that they achieve, over which relevant performance measures can be easily optimized. Since Youla parameterization defines an isomorphism between a stabilizing controller and the resulting closed loop system response from sensors to actuators, in a design process rather than synthesizing the controller itself, the Youla parameter could be directly optimized. Thus, customized design specifications on the closed loop system can be applied into the controller design method via convex optimization.

However, this approach is not directly applicable in the distributed optimal control setting. Cyberphysical control systems (CPS) often have sparse, uncertain, and distributed communication and computing structure in addition to sensing and actuation. In contrast to centralized systems, cyber-physical systems are large-scale, physically distributed, and interconnected. These systems are composed of several sub-controllers, each equipped with their own sensors and actuators, exchanging locally the available information via a communication network. Often, the corresponding plants and performance requirements are also sparse and structured, and this fact must be exploited to make controller design feasible and tractable. These information sharing constraints make the distributed optimal controller synthesis problem challenging to solve.

The introduction of quadratic invariance (QI) has shown that for a large class of LTI systems such internal structure can be integrated with the Youla parameterization preserving the convexity of the optimal controller synthesis, see Rotkowitz and Lall [2003, 2006]. A set of controllers $\mathcal{K}$ is quadratically invariant if $KPK \in \mathcal{K}$ for any $K \in \mathcal{K}$. Moreover, this condition is tight, in the sense that QI is a necessary and sufficient condition for subspace constraints on the controller to be enforceable via convex constraints on the Youla parameter, see Lessard and Lall [2016]. A major limitation of the QI framework is that, for strongly connected systems, it cannot provide a convex characterization of localized controllers, in which local sub-controllers only access a subset of system-wide measurements. The need for global exchange of information between sub-controllers is a limiting factor in the scalability of the synthesis and implementation process. It turns out that a QI distributed optimal controller is at least as expensive to compute as its centralized counterpart and can be more difficult to implement.

Providing an alternative to the QI framework, in the context of discrete time LTI systems, Anderson et al. [2019] have introduced a new, system level approach (SLA) as an optimal controller synthesis and parameterization framework for constrained optimal controller design in large-scale applications. The approach rather than directly designing only the feedback loop between sensors and actuators, as in the Youla framework, proposes to directly design the entire closed loop response of the system, as captured by the maps from process and measurement disturbances to control actions and states. The method involves three complementary elements: system level parameterizations (SLPs) provide an alternative to the Youla parameterization of all stabilizing controllers and the responses they achieve. SLPs allow to constrain the closed loop response of the system to lie in arbitrary sets, called system level

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constraints (SLCs), and parameterizes the largest class of constrained stabilizing controllers that admit a convex characterization. Finally, the system level synthesis (SLS) problem is formulated and solved: any SLC imposed on the system response imposes a corresponding SLC on the internal structure of the resulting controller. Furieri et al. [2019], Wang et al. [2018, 2019]. In contrast to the QI framework, which imposes structure on the input/output map defined by the controller, the SLA imposes structural constraints on the system response itself. Moreover, this structure carries over to the internal realization of the corresponding controller: there is a conceptual shift from the structure on the input/output map to the internal realization of the controller that allows to expand the class of structured controllers that admit a convex characterization, and vastly increase the scalability of distributed optimal control methods, see also Chen et al. [2020], Tseng and Anderson [2020].

Klein proposed group theory as a mean of formulating and understanding geometrical constructions. In Szabó et al. [2014] the authors emphasise Klein's approach to geometry and demonstrate that a natural framework to formulate various control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms. The main concern of our efforts is to highlight the deep relation that exists between the seemingly different fields of geometry, algebra and control.

In Szabó and Bokor [2015, 2016], Szabó et al. [2017] we have shown that in contrast to the classical Youla-Kucera approach, there is a parametrisation of the entire controller set which can be described entirely in a coordinate free way, i.e., just by using the knowledge of the plant G and of the given stabilizing controller K_0. The corresponding parameter set is given in geometric terms, i.e., by providing an associated algebraic (semigroup, group) structure. Moreover, it turns out that the geometry of stable controllers is surprisingly simple. In the context of this framework QI also gains a natural motivation and interpretation.

Based on an abstract algebraic setting in Szabó and Bokor [2020] we provide an elementary derivation of the Youla-Kucera parametrization of stabilizing controllers and also an alternative, coordinate free, approach of the problem. In contrast to the Youla-Kucera framework, the parameter set is not universal but its elements can be generated by a universal algorithm. Extending the framework to the LFT loops we show by elementary tools that every controller which stabilizes the interior loop of the generalized plant also stabilizes the LFT loop.

We would like to stress that it is a very fruitful strategy to try to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction; finally the solution of the original control problem can be formulated in an algorithmic way by transposing the geometric ideas into the proper algebraic terms. Accordingly, we suppose only that our objects (systems), plants and controllers, are elements of a suitable ring while stability is a property, which is inherited by addition and multiplication of the systems. Among others, a motivation background behind this approach is the LPV framework: developing efficient robust analysis and synthesis algorithms for linear parameter varying (LPV) systems leads to a renewed interest in certain input-output techniques that conveniently manipulates the control loop, Szabó and Bokor [2018].

The main goal of this paper is to lift the main results concerning system level parametrization to this geometric, coordinate and representation free, level. Moreover, by showing a realization related to different Möbius transforms formulated in terms of the original data we also provide the general background that leads to a suitable and potentially efficient implementation of the designed controllers.

Section 2 gives the basic notions related to feedback and LFT stability. Section 3 recalls the fundamental SLP result and provides its generalized counterpart for the basic feedback and performance (LFT) loop. Section 4 is dedicated to the realization (implementation) of the controllers that are formulated in terms of Möbius transforms and relates them to the SLA. Finally some conclusions are formulated.

2. BASIC SETTINGS

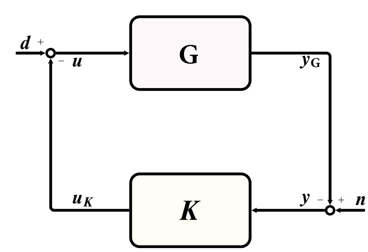


Fig. 1. Feedback connection

To fix the ideas let us consider the feedback-connection depicted on Figure 1. It is convenient to consider the signals

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, g = \begin{pmatrix} u \\ y_G \end{pmatrix}, k = \begin{pmatrix} u_K \\ y \end{pmatrix}, z = \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{H},$$

where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and we suppose that the signals are elements of the Hilbert space $\mathcal{H}_1, \mathcal{H}_2$ (e.g., $\mathcal{H}_i = \mathcal{L}^{n_i}[0,\infty)$) endowed by a resolution structure which determines the causality concept on these spaces. In this model the plant G and the controller K are linear causal maps. For more details on this general setting, see Feintuch [1998].

Recall, that a feedback connection is called well-posed if for every $w \in \mathcal{H}$ there is a unique g and k such that $w = g + k$ (causal invertibility) and the pair (G, K) is called stable if the map $w \to z$ is a bounded causal map, i.e., the pair (G, K) is called well-posed if the inverse

$$\mathcal{H}(G,K) = \begin{pmatrix} I & K \\ G & I \end{pmatrix}^{-1} = \begin{pmatrix} S_u & S_c \\ S_g & S_y \end{pmatrix} = \\ = \begin{pmatrix} (I-KG)^{-1} & -K(I-GK)^{-1} \\ -G(I-KG)^{-1} & (I-GK)^{-1} \end{pmatrix} \quad (1)$$

exists (causal invertibility), and it is called stable if all the block elements, gang of four, are stable.

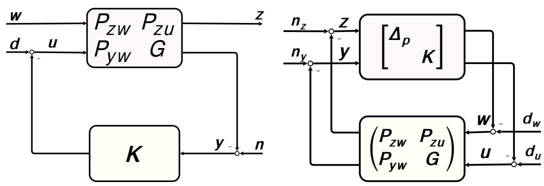
The lower and an upper LFT is defined as

$$\mathfrak{F}_l(P,K) = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}$$

and

$$\mathfrak{F}_u(P,\Delta) = P_{yu} + P_{yw}\Delta(I - P_{zw}\Delta)^{-1}P_{zu}.$$

Stability of the LFT loop means that the causal map $\mathcal{L}(P,K)$ that relates the signals (z,u,y) to (w,d,n) is invertible and the inverse map is stable, see Figure 2(a). It turns out that this is equivalent to the stability of the extended feedback loop for $\Delta_p = 0_p$, see Figure 2(b).



(a) Performance (b) Stability

Fig. 2. Stability of LFTs

Starting from the basic relation between the relevant signals we have

$$\begin{pmatrix} d_u \\ n_y \\ d_w \\ n_z \end{pmatrix} = \begin{pmatrix} I_u & K & 0 & 0 \\ G & I_y & P_{yw} & 0 \\ 0 & 0 & I_w & \Delta_p \\ P_{zu} & 0 & P_{zw} & I_z \end{pmatrix} \begin{pmatrix} u \\ y \\ w \\ z \end{pmatrix}.$$

Invertibility of the operator is equivalent to the non-singularity of the corresponding Schur complement, i.e, nonsingularity of

$$\begin{pmatrix} I_w & \Delta_p \\ P_{zw} & I_z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ P_{zu} & 0 \end{pmatrix} \mathcal{H}(G,K) \begin{pmatrix} 0 & 0 \\ P_{yw} & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} I_w & \Delta_p \\ \mathfrak{F}_l(P,K) & I_z \end{pmatrix},$$

which is always fulfilled for $\Delta_p = 0_p$, when the inverse is

$$\begin{pmatrix} \mathcal{H}(G,K) & -\mathcal{H}(G,K)\begin{pmatrix} 0 & 0 \\ P_{yw} & 0 \end{pmatrix} \\ -\begin{pmatrix} 0 & 0 \\ P_{zu} & 0 \end{pmatrix}\mathcal{H}(G,K) & \begin{pmatrix} I_w & 0 \\ -\mathfrak{F}_l(P,K) & I_z \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{H}(G,K) & \begin{pmatrix} -S_c P_{yw} & 0 \\ -S_y P_{yw} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -P_{zu}S_u & -P_{zu}S_c \end{pmatrix} & \begin{pmatrix} I_w & 0 \\ -\mathfrak{F}_l(P,K) & I_z \end{pmatrix} \end{pmatrix}. \quad (2)$$

Recall that the "gang of nine" is

$$\mathcal{L}(P,K) = \begin{bmatrix} \mathcal{H}(G,K) & -\mathcal{H}(G,K)\begin{pmatrix} 0 \\ P_{yw} \end{pmatrix} \\ -\begin{pmatrix} P_{zu} & 0 \end{pmatrix}\mathcal{H}(G,K) & -\mathfrak{F}_l(P,K) \end{bmatrix}.$$

It is obvious that the LFT loop is well–defined if and only if (G,K) is well defined. However, it is less obvious that this claim remains true for stability, too.

3. GENERALISED SYSTEM LEVEL PARAMETRIZATION

For reference we summarize first the output feedback result from Anderson et al. [2019]: consider a strictly proper plant

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

Letting $\delta_x[t] = B_1 w[t]$ and $\delta_y[t] = D_{21} w[t]$ denote the disturbance on the state and on the measurement, respectively, the dynamics defined by plant can be written as

$$\begin{aligned} x[t+1] &= Ax[t] + B_2 u[t] + \delta_x[t] \\ y[t] &= C_2 x[t] + \delta_y[t]. \end{aligned}$$

Substituting the output feedback control law $u = K(z)y$, define a system response $\{R,M,N,L\}$ from perturbations (δ_x, δ_y) to (x,u), i.e.,

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} R & N \\ M & L \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}.$$

Then, the affine subspace described by:

$$[zI - A \ \ -B_2] \begin{bmatrix} R & N \\ M & L \end{bmatrix} = [I \ \ 0]$$
$$\begin{bmatrix} R & N \\ M & L \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$R, M, N \in \frac{1}{z}\mathcal{RH}_\infty, \quad L \in \mathcal{RH}_\infty$$

parameterizes all system responses achievable by an internally stabilizing controller K. Moreover, given transfer matrices $\{R,M,N,L\}$ satisfying the constraints, the controller $K = L - MR^{-1}N$ achieves the desired response. The controller can be implemented as

$$\begin{aligned} z\beta &= \tilde{R}^{+}\beta + \tilde{N}y \\ u &= \tilde{M}\beta + Ly, \end{aligned}$$

with $\tilde{R}^{+} = z(I - zR)$, $\tilde{M} = zM$, and $\tilde{N} = -zN$.

In this Section we provide an analogous result to the parametrization result for the basic feedback loop and then for the performance loop. The next Section deals with the realization question.

Note, that we do not give any specific assumptions on the nature of the systems. In particular, the result holds in any sound context, e.g., LTI or LPV plants and controllers.

Theorem 1. Let us consider a plant G and a stable matrix with the following block decomposition:

$$\Phi = \begin{bmatrix} \Phi_{uu} & \Phi_{uy} \\ \Phi_{yu} & \Phi_{yy} \end{bmatrix}, \quad \Phi_{uu}, \Phi_{yy} \text{ invertible.}$$

Then there exists a stabilizing controller K such that

$$\Phi = \begin{bmatrix} I & K \\ G & I \end{bmatrix}^{-1}, \quad (3)$$

if and only if the following conditions hold:

$$\begin{bmatrix} \Phi_{uu} & \Phi_{uy} \\ \Phi_{yu} & \Phi_{yy} \end{bmatrix} \begin{bmatrix} I \\ G \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad (4)$$

$$[G \ \ I] \begin{bmatrix} \Phi_{uu} & \Phi_{uy} \\ \Phi_{yu} & \Phi_{yy} \end{bmatrix} = [0 \ \ I]. \quad (5)$$

Proof: since necessity is obvious, it remains to prove sufficiency. Conditions of the theorem reads as

$$\begin{aligned} \Phi_{uu} &= I - \Phi_{uy}G \\ \Phi_{yy} &= I - G\Phi_{uy} \\ \Phi_{yu} &= -\Phi_{yy}G = -G + G\Phi_{uy}G \end{aligned}$$

Thus, we have

$$\begin{bmatrix} \Phi_{uu} & \Phi_{uy} \\ \Phi_{yu} & \Phi_{yy} \end{bmatrix} = \begin{bmatrix} I - \Phi_{uy}G & \Phi_{uy} \\ -G(I - \Phi_{uy}G) & I - G\Phi_{uy} \end{bmatrix}.$$

By the conditions of the theorem $I - \Phi_{uy}G$ is invertible, and computing the Schur complement

$$I - G\Phi_{uy} + G(I - \Phi_{uy}G)(I - \Phi_{uy}G)^{-1}\Phi_{uy} = I,$$

we get that Φ is invertible. By the uniqueness of the inverse it follows the existence of K, which is stabilizing, since Φ is stable. Moreover, we have

$$G = -\Phi_{yu}\Phi_{uu}^{-1} = -\Phi_{yy}^{-1}\Phi_{yu}, \tag{6}$$

$$K = -\Phi_{uy}\Phi_{yy}^{-1} = -\Phi_{uu}^{-1}\Phi_{uy}. \tag{7}$$

The same conclusion can be obtained by an application of the matrix inversion lemma:

$$\begin{bmatrix}\Phi_{uu} & \Phi_{uy}\\ \Phi_{yu} & \Phi_{yy}\end{bmatrix}^{-1} = \begin{bmatrix} W & -W\Phi_{uy}\Phi_{yy}^{-1}\\ -Z\Phi_{yu}\Phi_{uu}^{-1} & Z\end{bmatrix} = \begin{bmatrix} W & WK\\ ZG & Z\end{bmatrix},$$

where, by (4) and (5), we have

$$W = (\Phi_{uu} - \Phi_{uy}\Phi_{yy}^{-1}\Phi_{yu})^{-1} = I$$
$$Z = (\Phi_{yy} - \Phi_{yu}\Phi_{uu}^{-1}\Phi_{uy})^{-1} = I.$$

Note, that for practical applications, where FIR, LPV FIR or state space models are considered, the invertibility condition required by the Theorem is not an issue, and it can be trivially satisfied.

Remark 1. This Theorem also appears in Furieri et al. [2019] where the relation of SLA with Youla parametrization is detailed, which chain of ideas is completed in Zheng et al. [2020]. Here we place the result in a more general, coordinate free context, and provide alternative proofs.

While originally SLA has been introduced with strong links to the Youla parametrization and QI, it should be emphasized that the paradigm goes well beyond that framework. SLA provides an overparametrization of the controller: it is important to realize that, in contrast to the Youla parameter Q, in the optimization step this parametrization remains implicit (does not play any explicit role). In the design phase the technique leads to a linearly constrained (quadratic) optimization problem.

We also note that the previous results do not observe the necessity to extend Theorem 1, i.e., the SLA, to performance loops. In what follows it is shown that for LFT loops we can indeed state an analogous result:

Theorem 2. Let us suppose that the LFT loop is stabilizable and let us consider a stable matrix

$$\Psi = \begin{bmatrix}\Psi_{uu} & \Psi_{uy} & \Psi_{uw}\\ \Psi_{yu} & \Psi_{yy} & \Psi_{yw}\\ \Psi_{zu} & \Psi_{zy} & \Psi_{zw}\end{bmatrix}, \quad \Psi_{uu}, \Psi_{yy} \text{ invertible.}$$

Then there exists a stabilizing controller K such that

$$\Psi = \begin{bmatrix} \mathcal{H}(G,K) & -\mathcal{H}(G,K)\begin{pmatrix}0\\ P_{yw}\end{pmatrix}\\ -\left(P_{zu}\ 0\right)\mathcal{H}(G,K) & -\mathfrak{F}_l(P,K)\end{bmatrix}, \tag{8}$$

if and only if the following conditions hold:

$$\begin{bmatrix}\Psi_{uu} & \Psi_{uy} & \Psi_{uw} & 0\\ \Psi_{yu} & \Psi_{yy} & \Psi_{yw} & 0\\ \Psi_{zu} & \Psi_{zy} & \Psi_{zw} & I\end{bmatrix}\begin{bmatrix} I_u & 0\\ G & P_{yw}\\ 0 & I_w\\ P_{zu} & P_{zw}\end{bmatrix} = \begin{bmatrix} I & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} \tag{9}$$

$$\begin{bmatrix} G & I_y & 0 & P_{yw}\\ P_{zu} & 0 & I_z & P_{zw}\end{bmatrix}\begin{bmatrix}\Psi_{uu} & \Psi_{uy} & \Psi_{uw}\\ \Psi_{yu} & \Psi_{yy} & \Psi_{yw}\\ \Psi_{zu} & \Psi_{zy} & \Psi_{zw}\\ 0 & 0 & I\end{bmatrix} = \begin{bmatrix} 0 & I & 0\\ 0 & 0 & 0\end{bmatrix} \tag{10}$$

Proof: For necessity consider the relations that define the stability of the loop (gang of nine):

$$\begin{bmatrix} I_u & K & 0 & 0\\ G & I_y & P_{yw} & 0\\ 0 & 0 & I_w & 0\\ P_{zu} & 0 & P_{zw} & I_z\end{bmatrix}\begin{bmatrix}\mathcal{H}(G,K) & -\mathcal{H}(G,K)\begin{pmatrix}0 & 0\\ P_{yw} & 0\end{pmatrix}\\ -\begin{pmatrix}0 & 0\\ P_{zu} & 0\end{pmatrix}\mathcal{H}(G,K) & \begin{pmatrix} I_w & 0\\ -\mathfrak{F}_l(P,K) & I_z\end{pmatrix}\end{bmatrix} =$$

$$\begin{bmatrix}\mathcal{H}(G,K) & -\mathcal{H}(G,K)\begin{pmatrix}0 & 0\\ P_{yw} & 0\end{pmatrix}\\ -\begin{pmatrix}0 & 0\\ P_{zu} & 0\end{pmatrix}\mathcal{H}(G,K) & \begin{pmatrix} I_w & 0\\ -\mathfrak{F}_l(P,K) & I_z\end{pmatrix}\end{bmatrix}\begin{bmatrix} I_u & K & 0 & 0\\ G & I_y & P_{yw} & 0\\ 0 & 0 & I_w & 0\\ P_{zu} & 0 & P_{zw} & I_z\end{bmatrix} =$$

$$= \begin{bmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}.$$

For sufficiency, observe that from (9) and (10) we have

$$\begin{bmatrix}\Psi_{uu} & \Psi_{uy}\\ \Psi_{yu} & \Psi_{yy}\end{bmatrix}\begin{bmatrix} I_u\\ G\end{bmatrix} = \begin{bmatrix} I\\ 0\end{bmatrix}$$

$$[G\ I_y]\begin{bmatrix}\Psi_{uu} & \Psi_{uy}\\ \Psi_{yu} & \Psi_{yy}\end{bmatrix} = [0\ I]$$

Then, from Theorem 1 follows the existence of a stabilizing controller K such that

$$\begin{bmatrix}\Psi_{uu} & \Psi_{uy}\\ \Psi_{yu} & \Psi_{yy}\end{bmatrix} = \mathcal{H}(G,K).$$

Since the LFT loop is stabilizable, every stabilizing controller of G is a stabilizing controller for the LFT, see Szabó and Bokor [2020]. Then, the assertion follows by the uniqueness of the inverse.

Note, that the applicability of these results depend on the proper choice of the underlying model class. In the SLA of Anderson et al. [2019] it is essential the choice of the FIR structure, that make the further optimization steps, e.g., MPC design, tractable. For LPV systems one can also chose an analogous class, e.g., LPV FIR or LPV input-output models, see, e.g., Wollnack et al. [2013], Emedi and Karimi [2016], Abbas et al. [2018], Wollnack et al. [2017]. It is, however, well beyond the scope of this paper to enter into the intricacies of a particular design.

Note, that in Anderson et al. [2019] the results can be formulated in terms of four transfer functions due to the special context, i.e., state space description, in which the actual problem is formulated. Since the state space matrices are memoryless (stable) the gang of nine reduces to the gang of four. There is no immediate counterpart in the general geometric setting for the stabilizabiliy (detectability) interpretation of the conditions provided in Anderson et al. [2019].

4. SYSTEM LEVEL SYNTHESIS: CONTROLLER REALIZATION

Let us consider the linear map $T : \begin{pmatrix} z\\ y\end{pmatrix} \mapsto \begin{pmatrix} w\\ u\end{pmatrix}$, and its inverse (if exists) described by the operator matrices

$$T = \begin{pmatrix} A & B\\ C & D\end{pmatrix}, \quad \text{and} \quad T^{-1} = \begin{pmatrix} E & F\\ G & H\end{pmatrix}, \tag{11}$$

respectively. Upper Möbius transformations

$$Z' = \mathfrak{M}_{\overline{T}}(Z) = (C + DZ)(A + BZ)^{-1},$$

relate two graph subspaces, $\mathcal{G}_Z$ and $\mathcal{G}_{Z'}$, through the invertible linear operator T, i.e., $\mathcal{G}_{Z'} = T\mathcal{G}_Z$ on the domain

$$\mathrm{dom}\mathfrak{M}_{\overline{T}} = \{(A + BZ)^{-1} \text{ exists }\}.$$

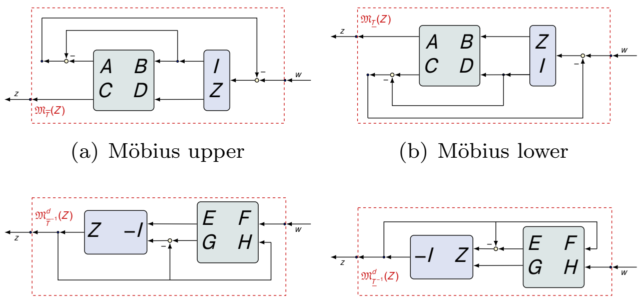
Analogously one can define the lower Möbius transformation as

$$\mathfrak{M}_{\underline{T}}(Z) = (AZ + B)(CZ + D)^{-1},$$

and their dual versions

$$\mathfrak{M}^d_{\overline{T}^{-1}}(Z) = (ZF - H)^{-1}(ZE - G),$$
$$\mathfrak{M}^d_{\underline{T}^{-1}}(Z) = (E - ZG)^{-1}(F - ZH),$$

respectively. Möbius transformations consist the basic tool in expressing the controllers, see, e.g., the Youla parametrization, where T and T^{-1} are provided by the double coprime factorization or the linear fractional transformation, which can expressed as a combination of two different Möbius expressions.



(a) Möbius upper (b) Möbius lower

(c) dual Möbius upper (d) dual Möbius lower

Fig. 3. Realization for Möbius transforms

It turns out that the rational expression defined by the Möbius transformation can be implemented as a suitable feedback connection between the blocks of T (or T^{-1}) and the corresponding graph defined by Z, see Figure 3 for the different possibilities.

Validity of these implementations can be checked by a direct verification, e.g., for the upper Möbius transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$
$$a = w - (x - a), \quad b = Z(w - (x - a)), \quad z = y,$$

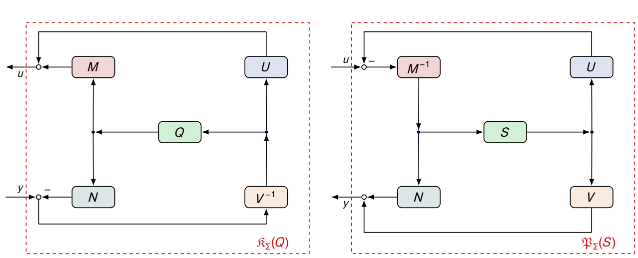
i.e.,

$$w = x, \qquad b = Za,$$

hence

$$w = (A + BZ)a, \quad z = (C + DZ)a.$$

The advantage of the realization is that lifts the structure present in the blocks of T and Z to the level of the implementation. This is in contrast to the traditional schemes which needs to invert explicitly at least one of the blocks, see, e.g., on Figure 4 the traditional schemes used for the Youla parametrization.



(a) Youla (b) dual Youla

Fig. 4. Standard Youla implementation

In the context of Youla parametrization the novel implementation was first proposed in Niemann and Stoustrup [1999]. For structured controllers, however, the structure is not inherited, in general, by the coprime factors. This fact motivates the need for the system level approach.

Finally, we compare our general geometric scheme with the controller realization proposed in Anderson et al. [2019]. As an illustration we consider the state feedback case: the set of all achievable internally stabilizing controllers is parametrized by an affine subspace. Using this property, system level synthesis problem for the LTI state feedback case takes the form:

$$\min g(\Phi_x, \Phi_u)$$
$$\text{such that} \quad \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I$$
$$\Phi_x, \Phi_u \in z^{-1}\mathcal{H}_\infty, \quad \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} \in \mathcal{S}$$

The constraint set $\mathcal{S}$ is used to provide spatial and temporal locality to the closed-loop response. This is typically done by enforcing sparsity on the spectral components of $\{\Phi_x, \Phi_u\}$ and making them finite-impulse-response (FIR) filters. As long as g is a convex functional, e.g., $\mathcal{L}_1$, $\mathcal{H}_\infty$, or $\mathcal{H}_2$ norms, and $\mathcal{S}$ defines a convex set, the SLS problem is convex.

A stabilizing controller is readily obtained from the system response as $K = \Phi_u \Phi_x^{-1} = (z\Phi_u)(z\Phi_x)^{-1}$. Inverting Φ_x, however, is undesirable in most cases as it is heavily dependent on conditioning, and all the structure that Φ_x has will likely be lost. The realization then follows by putting $z\Phi_u$ in the forward path and realizing $(z\Phi_x)^{-1}$ as the feedback path through the $I - z\Phi_x$ block, see Figure 5a. Unlike the solution, which inverts Φ_x, any structure imposed on $\{\Phi_x, \Phi_u\}$ is inherited by the two blocks, i.e., closed-loop constraints are passed on to the controller. Note, that in this particular case the general scheme depicted on Figure 5b, taking $M = z\Phi_x$ and $N = z\Phi_u$, can be further simplified into an equivalent implementation: first consider the full Möbius case, i.e., take

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} M & 0 \\ N & I \end{pmatrix} \begin{pmatrix} x \\ l \end{pmatrix}, \quad Z = 0.$$

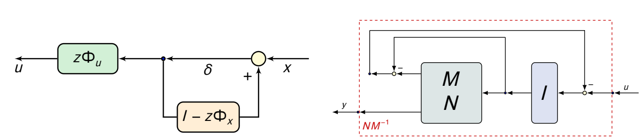
Then, for this particular case simplify Figure 3a

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} x, \quad a = u, \ b = y,$$

to get the connection depicted on Figure 5b. Finally, writing down the signal flow, we get

$$x = u - (I - M)x, \quad y = Nx,$$

which is exactly the relation depicted on Figure 5a.



(a) SLS state feedback (b) General scheme

Fig. 5. SLS Controller Realization

The same considerations hold for the performance loop, i.e., the output feedback case. In Anderson et al. [2019] is already mentioned, that the corresponding controller implementation also admits the following equivalent representation:

$$\begin{bmatrix} R & N \\ M & L \end{bmatrix} \begin{bmatrix} z\beta \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix},$$

which is a Möbius transform. The details are left out for brevity.

5. CONCLUSIONS

System level approach has been proposed as a novel design tool for large-scale cyber physical systems in the discrete time LTI framework as a paradigm shift from constructing and parametrizing the desired controller to the parametrization of the achievable closed loop behaviours (closed loop shape). The controller is synthesised from this system level data in a way that keep the possibly present structural constraints intact.

In this paper we lift the main results concerning system level parametrization to a general (geometric) level, that also includes the class of LPV systems. We also provide a general scheme for the realization related to different Möbius transforms which are formulated in terms of the original data and also keep intact the structural information present in those data.

A further research topic is the extension and analysis of the robust SLA techniques to the general setting. Concerning the LPV framework it is also an important practical question which input-output representations of the LPV systems fit best the tractable and efficient design requirement.

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