



Edge disjoint caterpillar realizations

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ABSTRACT

Edge disjoint realization problems have connections for example to discrete tomography. In this paper, we consider the edge disjoint caterpillar realizations of tree degree sequences. We give necessary and sufficient conditions when two tree degree sequences have edge disjoint caterpillar realizations. We conjecture that an arbitrary number of tree degree sequences have edge disjoint realizations if every vertex is a leaf in at most one tree. We prove that the conjecture is true if the number of tree degree sequences is at most four. We also prove that the conjecture is true if $n \geq \max\{22k - 11, 396\}$, where n is the number of vertices and k is the number of tree degree sequences.

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1. Introduction

A degree sequence $D = d_1, d_2, \dots, d_n$ is a sequence of non-negative integers. A degree sequence is graphical if there is a vertex-labeled graph G in which the degrees of the vertices are exactly D . Such a graph G is called a realization of D . The color degree matrix problem, also known as an edge disjoint realization, edge packing or graph factorization problem, is the following: Given a $k \times n$ degree matrix $\mathbf{D} = \{d_{1,1}, d_{1,2}, \dots, d_{1,n}\}, \{d_{2,1}, d_{2,2}, \dots, d_{2,n}\}, \dots, \{d_{k,1}, d_{k,2}, \dots, d_{k,n}\}$, in which each row of the matrix is a degree sequence, decide if there is an ensemble of edge disjoint realizations of the degree sequences. Such a set of edge disjoint graphs is called a realization of the degree matrix. A realization can also be presented as an edge colored simple graph, in which the edges with a given color form a realization of the degree sequence in a given row of the color degree matrix. This problem is related to discrete tomography [9], which has many applications in industry [7,8].

The existence problem in general is a hard computational problem for any $k \geq 2$ [3]. However, it is easy for some special cases. One special case is when the degree matrix is very sparse, the total sum of the degrees is at most $2n - 1$, where n is the number of vertices. In that case, necessary and sufficient conditions exist for realizing a colored degree matrix with a colored forest [9]. Another interesting case is when each degree sequence is a degree sequence of a tree. We will call these tree degree sequences. Kundu proved that two tree degree sequences have edge disjoint tree realizations if and only if the sum of the degree sequences is graphical [10]. He also proved that a similar statement is not true for three degree sequences. He gave an example of three tree degree sequences such that the sum of any two of them is graphical and the sum of all three degree sequences is graphical, but the degree sequences do not have edge disjoint tree realizations [11]. Besides, he proved that three tree degree sequences always have edge disjoint tree realizations if the

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minimum sum of the degrees on each vertex is at least 5 [11]. This condition includes the case when each vertex is a leaf in at most one of the trees. We conjecture that a degree matrix always has edge disjoint caterpillar realizations if each row is a tree degree sequence and each vertex is a leaf in at most one of the trees.

In this paper we prove that this conjecture holds when the number of degree sequences is at most four or the number of vertices is at least $\max\{22k - 11, 396\}$, where k is the number of rows in the tree degree matrix. Furthermore, we give a necessary and sufficient condition when two tree degree sequences have edge disjoint caterpillar realizations.

2. Preliminaries

In this section, we give some formal definitions and lemmas we use throughout the paper. First, we formally define degree sequences and degree matrices, along with the different types of realizations we consider in this paper.

Definition 1. A degree sequence $D = d_1, d_2, \dots, d_n$ is a list of non-negative integers. A degree sequence is *graphical* if there exists a simple graph G whose degrees are exactly D . Such a graph is a *realization* of D . A degree sequence D is a *tree degree sequence* if each degree is positive and $\sum_{i=1}^n d_i = 2n - 2$.

A degree 1 vertex is called a *leaf*. A degree sequence is a *path degree sequence* if it has exactly two leaves. A realization of a tree degree sequence is called a *caterpillar* if its non-leaf vertices form a path. This path of non-leaf vertices is called the *backbone*.

Definition 2. A matrix $\mathbf{D} = \{\{d_{1,1}, d_{1,2}, \dots, d_{1,n}\}, \{d_{2,1}, d_{2,2}, \dots, d_{2,n}\}, \dots, \{d_{k,1}, d_{k,2}, \dots, d_{k,n}\}\}$ of non-negative integers is called a *degree sequence matrix*.

A degree sequence matrix is a *tree degree sequence matrix* if each row is a tree degree sequence. A tree degree matrix has no common leaves if for every triple (i, j, l) , $d_{i,j} = 1$ implies $d_{l,j} \neq 1$.

An edge colored simple graph G is called a *realization* of a degree matrix $\mathbf{D} \in \mathbb{N}^{k \times n}$, if it is colored with k colors, and for each color c_i , the subgraph containing the edges with color c_i is a realization of the i th row of \mathbf{D} . A realization is called *caterpillar realization* if for each color, the corresponding subgraph is a caterpillar.

The Erdős–Gallai theorem gives necessary and sufficient conditions when a degree sequence is graphical.

Theorem 2.1 ([5]). *A degree sequence $f_1 \geq f_2 \geq \dots \geq f_n$ is graphical if and only if the sum of the degrees is even, and for each $1 \leq s \leq n$ the inequality*

$$\sum_{i=1}^s f_i \leq s(s-1) + \sum_{j=s+1}^n \min\{s, f_j\} \tag{1}$$

holds.

We refer to the inequalities in Eq. (1) as Erdős–Gallai inequalities, or E–G inequalities for short.

When a degree sequence is a sum of tree degree sequences, then only the first few Erdős–Gallai inequalities must be checked, as the following lemma states.

Lemma 2.2 ([6]). *Let $F = f_1 \geq f_2 \geq \dots \geq f_n$ be the sum of k arbitrary tree degree sequences. Then the Erdős–Gallai inequalities in Eq. (1) hold for all $s \geq 2k$.*

In this paper, we will need a stronger statement summarized in the following lemma.

Lemma 2.3. *Let \mathbf{D} be a $2 \times n$ tree degree matrix, in which the second row is a path degree sequence. If $n \geq 6$, then the E–G inequalities for the summed degree sequence $f_j := d_{1,j} + d_{2,j}$ hold for all $s \geq 2$.*

Proof. Notice that the sum of a tree degree sequence is $2n - 2$ and $\sum_{j=3}^n d_{1,j} \geq n - 2$. Also, a path degree sequence does not have a degree larger than two. Therefore, when $s = 2$, the left-hand side of the E–G inequality is bounded above by

$$f_1 + f_2 \leq 2n - 2 - (n - 2) + 2 \times 2 = n + 4,$$

The right-hand side is precisely

$$2 + \sum_{j=3}^n \min\{2, f_j\} = 2 + 2(n - 2) = 2n - 2,$$

since each f_j is at least two. Then it is sufficient to show that

$$n + 4 \leq 2n - 2$$

which holds when $6 \leq n$.

When $s \geq 3$, we have on the left-hand side of the E–G inequality that

$$\sum_{i=1}^s f_i \leq 4n - 4 - 3(n - s) + 2 = n + 3s - 2,$$

since the total sum of the degrees is $4n - 4$, and every column sum is at least 3, except at most two of them. For similar reasons, we have the lower bound of the right-hand side of the E–G inequality:

$$s(s - 1) + 3(n - s) - 2 \leq s(s - 1) + \sum_{j=s+1}^n \min\{s, f_j\}.$$

Therefore, the inequality holds as long as

$$n + 3s - 2 \leq s(s - 1) + 3(n - s) - 2,$$

that is,

$$0 \leq s^2 - 7s + 2n = s^2 - 7s + 12 + 2(n - 6) = (s - 3)(s - 4) + 2(n - 6),$$

where $2(n - 6) \geq 0$. Since $s \geq 3$, the inequality holds. \square

In this paper, we are interested in the caterpillar realizations of tree degree matrices. Our main conjecture is the following:

Conjecture 1. Let $\mathbf{D} = \{\{d_{1,1}, d_{1,2}, \dots, d_{1,n}\}, \{d_{2,1}, d_{2,2}, \dots, d_{2,n}\}, \dots, \{d_{k,1}, d_{k,2}, \dots, d_{k,n}\}\}$ be a tree degree matrix without common leaves. Then \mathbf{D} has a caterpillar realization.

A special case is when the degree matrix contains path degree sequences without common leaves. It is well known that such degree matrices have caterpillar realizations, formally stated and proved in the following lemma:

Lemma 2.4. Let $\mathbf{D} = \{\{d_{1,1}, d_{1,2}, \dots, d_{1,n}\}, \{d_{2,1}, d_{2,2}, \dots, d_{2,n}\}, \dots, \{d_{k,1}, d_{k,2}, \dots, d_{k,n}\}\}$ be a tree degree matrix without common leaves. Furthermore, assume each row is a path degree sequence. Then, \mathbf{D} has edge disjoint path realizations.

Proof. We are going to explicitly construct these realizations. This construction is known as the Waleczki construction [2].

First observe that $n \geq 2k$, otherwise \mathbf{D} cannot accommodate the $2k$ leaves with at most one leaf in each column. Without loss of generality (since we can rearrange the rows and columns), we can say that $d_{1,1} = 1, d_{1, \lceil \frac{n+2}{2} \rceil} = 1, d_{2,2} = 1, d_{2, \lceil \frac{n+2}{2} \rceil + 1} = 1, \dots, d_{k,k} = 1, d_{k, \lceil \frac{n+2}{2} \rceil + k - 1} = 1$. Then the i th path contains the edges $(v_i, v_{i+1}), (v_{i+1}, v_{n+i-1}), (v_{n+i-1}, v_{i+2}), (v_{i+2}, v_{n+i-2}),$ etc., where $n + i - j$ is considered modulo n , taking a value from the set $\{1, 2, \dots, n\}$. \square

Some of our proofs are based on induction using the existence of rainbow matchings. We define them below.

Definition 3. Let G be an edge-colored simple graph. A *rainbow matching* of size k of G is a matching of size k in G such that no two edges have the same color.

3. Sufficient and necessary condition for two edge-disjoint caterpillar realizations

Bérczi et al. [4] gave the following example of a tree degree matrix:

$$\mathbf{D} = \begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 5 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2}$$

The degree matrix in Eq. (2) has edge disjoint tree realizations, but does not have edge disjoint caterpillar realizations. For \mathbf{D} to have a caterpillar realization, each vertex can have at most two adjacent non-leaf edges per caterpillar. Notice that the first vertex has degree 10. At most 2 · 2 of these can be non-leaf edges. So, this vertex is adjacent to at least 6 vertices which are leaves. However, there are only five vertices which are leaves in any of the trees. As one can see, it is a naturally necessary condition that the maximum summed degree cannot be larger than 4 more than the number of vertices which are leaves in any of the trees. In this section, we show that this together with the condition that the summed degree sequence is graphical, are necessary and sufficient conditions to produce edge disjoint caterpillar realizations.

Theorem 3.1. Let \mathbf{D} be a $2 \times n$ degree sequence matrix. Then \mathbf{D} has a caterpillar realization if and only if the following conditions hold:

1. For both $i = 1$ and $i = 2$,

$$\sum_{j=1}^n d_{i,j} = 2n - 2.$$

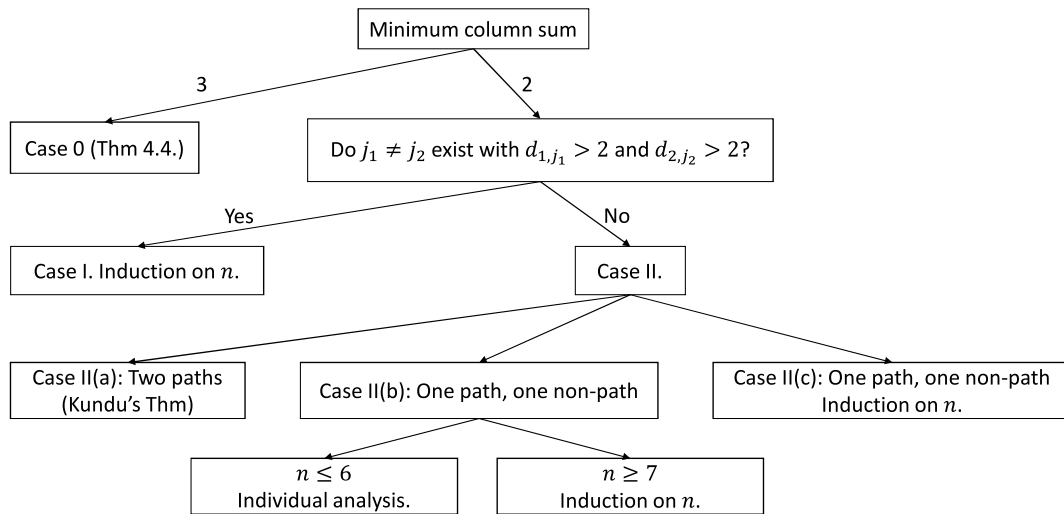


Fig. 1. The case distinctions in Theorem 3.1. See text for details.

2. The degree sequence

$$d_{1,1} + d_{2,1}, d_{1,2} + d_{2,2}, \dots, d_{1,n} + d_{2,n}$$

is graphical.

3. It holds that

$$d_{\max} \leq |S| + 4$$

where

$$d_{\max} := \max_j \{d_{1,j} + d_{2,j}\}$$

and

$$S := \{j \mid \min\{d_{1,j}, d_{2,j}\} = 1\}.$$

Proof. Conditions 1 and 2 are clearly necessary. Condition 3 is also necessary, since any non-leaf vertex will have at most two non-leaf neighbors in a caterpillar realization. If two caterpillar realizations are edge-disjoint, at least $d_{1,i} + d_{2,i} - 4$ leaves must be a neighbor of v_i in one of the caterpillar realizations.

Now we show that the conditions are also sufficient. This part of the proof involves a lengthy case distinction, see also Fig. 1 for an overview.

Let \mathbf{D} be a $2 \times n$ degree matrix that satisfies the conditions in the given theorem. Then the minimum column sum in \mathbf{D} is either 3 or 2. If the minimum sum is 3, then there is a caterpillar realization, according to Theorem 4.4 (Case 0 in Fig. 1). Observe the non-trivial corollary that the necessary conditions hold if the minimum column sum is 3.

If the minimum column sum is 2, then either there exists $j_1 \neq j_2$, such that $d_{1,j_1} > 2$ and $d_{2,j_2} > 2$, or there do not exist two such distinct numbers j_1 and j_2 . We prove that the conditions are sufficient for both cases, they will be denoted by Case I (distinct j_1 and j_2 exist) and Case II (distinct j_1 and j_2 do not exist).

Case I.

In this case, we exhibit a matrix \mathbf{D}' of $n - 1$ columns and show that \mathbf{D}' also satisfies the conditions. By induction, we assume that \mathbf{D}' has a realization G' , and from G' we construct a realization of \mathbf{D} .

Order the columns in decreasing order by their column sums, and w.l.o.g. let $d_{1,1} > 2$ (we can reorder the degree sequences if not). If there exist j_1 and j_2 such that $j_1 \neq j_2$, both of them are in $\{1, 2\}$, $d_{1,j_1} > 2$ and $d_{2,j_2} > 2$, then fix such j_1 and j_2 . Otherwise, let j_1 be 1 and let j_2 be the smallest index for which $d_{2,j_2} > 2$.

Let \mathbf{D}' denote the degree matrix we get from \mathbf{D} by removing a column with sum 2 and subtracting 1 both from d_{1,j_1} and d_{2,j_2} . We are going to prove that \mathbf{D}' also satisfies the conditions given in the theorem.

Clearly, \mathbf{D}' is a tree degree matrix. Also, we remove a vertex that is a leaf in both caterpillars (namely, it is in set S defined in Theorem 3.1), and we also subtract 1 from the largest degree. If the first vertex has the unique largest summed degree in \mathbf{D} , then it will still be largest in G' (though it may not be unique). Thus, condition 3 from the theorem also holds for \mathbf{D}' . If the first vertex does not have the unique largest summed degree, then the inequality in condition 3 cannot be sharp for \mathbf{D} , and thus will also hold for \mathbf{D}' . Indeed, either $d_{1,1} + d_{1,2} \geq d_{\max}$ or $d_{2,1} + d_{2,2} \geq d_{\max}$, due to the pigeonhole

principle (it is possible that both degree sums are exactly d_{\max}). Any tree with two vertex degrees d_1 and d_2 has at least $d_1 + d_2 - 2$ leaves, thus we get that $|S| \geq d_{\max} - 2$.

Therefore, we only have to prove that the column sums of \mathbf{D}' , $f'_j := d'_{1,j} + d'_{2,j}$, form a graphical degree sequence. To prove it, it is sufficient to show that the first three E–G inequalities hold, according to Lemma 2.2.

The first E–G inequality is violated if equality holds in the first E–G inequality regarding \mathbf{D} and f_1 is not the unique maximum. If $f_1 := d_{1,1} + d_{2,1}$ is the unique largest degree, then the first E–G inequality will also hold for f' . Indeed, both sides of the E–G inequality are decreased by 1 (compared to the first E–G inequality for f).

If $f_1 = f_2$ and $j_2 \notin \{1, 2\}$ then $d_{2,1} + d_{2,2} \leq 4$, so we get that

$$f_1 + f_2 = [d_{1,1} + d_{1,2}] + [d_{2,1} + d_{2,2}] \leq [2n - 2 - (n - 2)] + [4] = n + 4$$

therefore, f_1 is at most $\lfloor \frac{n}{2} \rfloor + 2$. We need that

$$\lfloor \frac{n}{2} \rfloor + 2 \leq n - 2$$

which holds if $n \geq 7$. If $n = 6$, then the only possible tree degree matrix in which neither of the rows is a path degree sequence, both f_1 and f_2 are 5, the smallest column sum is 2 and $j_2 \notin \{1, 2\}$ is

$$\begin{pmatrix} 3 & 3 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 1 & 1 & 1 \end{pmatrix},$$

however, in this case the column sums are not graphical. If $n = 5$, then the only possible degree matrix in which neither of the rows is a path degree sequence, both f_1 and f_2 are 4, the smallest column sum is 2 and $j_2 \notin \{1, 2\}$ is

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{pmatrix},$$

however, in this case the column sums are not graphical. Therefore, whenever the column sums of \mathbf{D} are graphical, the column sums of \mathbf{D}' satisfy the first E–G inequality.

We now focus on the second E–G inequality. Assume that f' violates the second E–G inequality. Then

$$f'_1 + f'_2 > 2 + 2(n - 3) = 2n - 4.$$

(Note that f' has only $n - 1$ entries.) However, then f'_1 would be at least $n - 1$, so then f' would violate the first E–G inequality. We proved that f' satisfies the first E–G inequality, a contradiction.

Finally, we focus on the third E–G inequality. Assume that f' violates the third E–G inequality. Then

$$f'_1 + f'_2 + f'_3 > 6 + 2(n - 4) = 2n - 2.$$

(Note that f' has only $n - 1$ entries.) Then $f'_1 + f'_2 + f'_3$ is at least $2n - 1$, and then the sum of the degrees of the remaining $n - 4$ terms is at most $2n - 7$. If $f'_1 + f'_2 + f'_3 = 2n - 1$, then $f'_4 = 3$, and the third E–G inequality is not violated. If $f'_1 + f'_2 + f'_3 = 2n$, then $f'_4 = 2$. However, then $f_1 + f_2 + f_3 = f'_1 + f'_2 + f'_3 + 2$, then f also violates the third E–G inequality, a contradiction. It is not possible that $f'_1 + f'_2 + f'_3 > 2n$ for f'_{n-1} would be at most 1. Therefore, the third E–G inequality holds for f' if it holds for f .

Let G' be a caterpillar realization of \mathbf{D}' , by induction on the number of vertices, we can assume that such a realization exists. Then we can get a caterpillar realization of \mathbf{D} from G' by adding a new vertex v to G' and connecting v to v_{j_1} with an edge of the first color and to v_{j_2} with an edge of the second color.

Case II.

If there do not exist distinct j_i and j_2 , where $d_{1,j_1} > 2$ and $d_{2,j_2} > 2$, then there are three cases:

- (a) Both degree sequences are paths.
- (b) Only one of the degree sequences is a path.
- (c) There is only one vertex, v_1 , such that $d_{1,1} > 2$ and $d_{2,1} > 2$.

We will denote these sub-cases by II(a), II(b) and II(c).

Case II(a)

If both degree sequences are paths, then any tree realization is also a caterpillar realization. Kundu’s theorem says there is a tree realization if the sum of the degree sequences is graphical [10].

Case II(b)

If one of the degree sequences is a path, then without loss of generality, the second degree sequence is a path. When $n \leq 6$, there are five possible tree degree matrices satisfying that the first row is not a path degree sequence, the second row is a path degree sequence, there is at least one column with sum 2 and the column sums form a graphical degree sequence:

$$1. \begin{pmatrix} 3 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{pmatrix}$$

2. $\begin{pmatrix} \underline{3} & 2 & 2 & 1 & 1 & 1 \\ 1 & \underline{2} & 2 & 2 & 2 & 1 \end{pmatrix}$
3. $\begin{pmatrix} \underline{3} & 2 & 2 & 1 & 1 & 1 \\ 2 & \underline{2} & 1 & 2 & 2 & 1 \end{pmatrix}$
4. $\begin{pmatrix} \underline{3} & 2 & 2 & 1 & 1 & 1 \\ 2 & \underline{2} & 2 & 2 & 1 & 1 \end{pmatrix}$
5. $\begin{pmatrix} \underline{4} & 2 & 1 & 1 & 1 & 1 \\ 1 & \underline{2} & 2 & 2 & 2 & 1 \end{pmatrix}$

In each case, we obtain a tree degree matrix \mathbf{D}' by subtracting 1 from the underlined entries and removing a column with sum 2. These \mathbf{D}' matrices have caterpillar realizations since either they are path degree sequences with graphical column sum or the minimum degree is 3 (or both). In each case, the caterpillar realization G' can be extended to a caterpillar realization of \mathbf{D} by adding one more vertex v and connecting v to the vertices where 1 was subtracted from the degree using the appropriate color.

Now we consider the case when $n \geq 7$. We prove the theorem for this sub-sub-case by induction on the number of vertices. Assume that the columns of \mathbf{D} are in decreasing order by their column sum, and amongst the same column sums, order the vertices lexicographically based on the two entries in the column. Since the second row is a path degree sequence, and there is a column with sum 2, at least one of the entries $d_{2,1}$ and $d_{2,2}$ is 2. If $d_{2,2} = 1$, then let \mathbf{D}' be the degree matrix we obtain by removing 1 from $d_{2,1}$ and $d_{1,2}$ and removing a column with sum 2. Otherwise, let \mathbf{D}' be the degree matrix we obtain by removing 1 from $d_{1,1}$ and $d_{2,2}$ and removing a column with sum 2. We show that the degree sequence $f' = f'_1, f'_2, \dots, f'_n$ where $f'_j := d'_{1,j} + d'_{2,j}$ is graphical. Observe that the second degree sequence in \mathbf{D}' is a path, and the number of columns in \mathbf{D}' is at least 6. Therefore, it is sufficient to show that the first E-G inequality holds, due to Lemma 2.3.

If at least the first three columns have the same column sum in \mathbf{D} , then the largest degree is at most $\frac{4n-4-3(n-3)+2}{3} = \frac{n+7}{3}$. This is because the sum of the degrees in the two trees is $4n - 4$, and on the remaining $n - 3$ columns, all column sums are at least 3 except at most two of them (recall that the second degree sequence is a path degree sequence). We need that

$$\frac{n+7}{3} \leq n-2,$$

that is,

$$6.5 \leq n,$$

which holds. Then the first E-G inequality will also hold for F' . If $f_2 > f_3$, then $f'_1 = f_1 - 1$. Since in this case 1 is subtracted from both sides of the first E-G inequality (compared to the first E-G inequality for f), the first E-G inequality holds for f' .

Therefore, the column sums of \mathbf{D}' form a graphical degree sequence. By induction hypothesis, \mathbf{D}' has a caterpillar realization, G' . Then \mathbf{D} also has a caterpillar realization by extending G' with a vertex and connecting it to the vertices where 1 was subtracted from the degree using the appropriate color.

Case II(c)

Finally, if there is only one vertex such that $d_{1,j} > 2$ and $d_{2,j} > 2$, then this is the vertex with the largest summed degree. We prove the following two observations:

1. The number of columns with degree sum 2 is at most four. Indeed, observe that the first tree has $d_{1,1}$ leaves while the second tree has $d_{2,1}$ leaves. Since the number of vertices which are leaves in at least one of the trees must be at least $d_{1,1} + d_{2,1} - 4$, at most four vertices might be leaves in both trees.
2. The number of columns with degree sum 4 is at least the number of columns with degree sum 2. This is because the summed degree sequence is graphical, therefore the E-G inequality holds with $s = 1$. That is, the number of vertices is at least $d_{1,1} + d_{2,1} + 1$. Also observe that the number of vertices with degree sum smaller than 4 is $d_{1,1} + d_{2,1}$ minus the number of vertices with degree sum 2. Therefore, if n_k denotes the number of vertices with degree sum k , we get from the first E-G inequality that

$$n_4 + n_3 + n_2 \geq d_{1,1} + d_{1,2}$$

and we also have the observation that

$$n_3 + n_2 = d_{1,1} + d_{1,2} - n_2.$$

From this, it is easy to see that n_4 is indeed at least n_2 .

Therefore, we have the following four possible sub-sub-cases:

1. $\begin{pmatrix} d_{1,1} & 2 & \dots & 2 & 2 & \dots & 2 & 1 & \dots & 1 & 1 \\ d_{2,1} & \underline{2} & \dots & 2 & 1 & \dots & 1 & 2 & \dots & 2 & 1 \end{pmatrix}$

2. $\left(\begin{array}{cccccccccccc} \underline{d_{1,1}} & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{1} & \dots & \underline{1} & \underline{1} & \underline{1} \\ \underline{d_{2,1}} & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{1} & \dots & \underline{1} & \underline{2} & \dots & \underline{2} & \underline{1} & \underline{1} \end{array} \right)$
3. $\left(\begin{array}{cccccccccccc} \underline{d_{1,1}} & \underline{2} & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{1} & \dots & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ \underline{d_{2,1}} & \underline{2} & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{1} & \dots & \underline{1} & \underline{2} & \dots & \underline{2} & \underline{1} & \underline{1} & \underline{1} \end{array} \right)$
4. $\left(\begin{array}{cccccccccccc} \underline{d_{1,1}} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{1} & \dots & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ \underline{d_{2,1}} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \dots & \underline{2} & \underline{1} & \dots & \underline{1} & \underline{2} & \dots & \underline{2} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \end{array} \right)$

In each case, let \mathbf{D}' be a tree degree matrix we obtain by removing all columns with degree sum 2, removing 1 from each underlined degree and removing 2 from each double underlined degree. The so-obtained matrices will be tree degree matrices without common leaves. Therefore, \mathbf{D}' has a caterpillar realization G' . W.l.o.g., we assume that the vertices that have degree 1 in one of the degree sequences after removing 1 or 2 are leaves adjacent to an end vertex of the backbone of the caterpillar. We can construct a caterpillar realization of \mathbf{D} by adding the appropriate number of vertices to G' and connecting these to vertices where 1 or 2 was subtracted from the degree using the appropriate color. It is easy to see that in each case, we can add these edges without introducing parallel edges. Since we added leaves to backbone vertices or to leaves that were adjacent to end vertices of the backbone, the so-obtained edge disjoint tree realization will also be a caterpillar realization. \square

4. Proving Conjecture 1. for $k \leq 4$

In this section we are going to prove Conjecture 1. for all $k \leq 4$. The proof is based on induction. The base case is the case when each degree sequence is a path degree sequence. Those degree matrices have edge disjoint path realizations, according to Lemma 2.4. In the inductive step, we will find rainbow matchings in sufficiently long paths. The following two lemmas state that such paths exist.

Lemma 4.1. *Let $\mathbf{D} \in \mathbb{N}^{k \times n}$ be a tree degree matrix without common leaves. Then in any caterpillar realization of \mathbf{D} , each caterpillar has a path of length at least $2k - 1$.*

Proof. We will show this by contradiction. Assume there exists a degree sequence that does not have a path of length $2k - 1$. Then, it has at most $2k - 3$ internal nodes and at least $n - 2k + 3$ leaves. Each of the other tree degree sequences has at least two leaves. So altogether, there are at least $n - 2k + 3 + 2(k - 1) = n + 1$ leaves. However, there are only n vertices. So, there must exist one vertex that is a leaf in two of the caterpillars, producing a contradiction. \square

Lemma 4.2. *Let $\mathbf{D} = \{D_1, D_2, \dots, D_k\} \in \mathbb{N}^{k \times n}$ be a tree degree matrix without common leaves. If $n \geq 2k + 2$ and $k \geq 4$, then within $(k - 1)$ arbitrary caterpillars of any caterpillar realization of \mathbf{D} , there exists a path of length at least $2k + 1$.*

Proof. Assume the contrary. Then, there exists a set of $k - 1$ tree degree sequences in \mathbf{D} such that every D_i does not have a path of length $2k + 1$. In other words, each of them must have at most $(2k + 1) - 2 = 2k - 1$ internal nodes, and thus, must have at least $n - 2k + 1$ leaves. Hence, there are at least

$$(k - 1)(n - 2k + 1) + 2 = kn - n - 2k^2 + 3k + 1$$

leaves, which can be at most n . From this, we get that

$$n \leq \frac{2k^2 - 3k - 1}{k - 2}.$$

However, since $n \geq 2k + 2$, we must have that

$$\frac{2k^2 - 3k - 1}{k - 2} \leq 2k + 2,$$

implying that $k \leq 3$, a contradiction. \square

The following lemma is on the existence of a certain vertex in a tree degree matrix without common leaves.

Lemma 4.3. *Let $\mathbf{D} \in \mathbb{N}^{k \times n}$ be a tree degree matrix without common leaves. Assume that not all rows are path degree sequences. Then there exists a column with the following properties:*

1. The sum of the column is $2k - 1$.
2. The column contains a 1 in a row which is not a path degree sequence.

The proof is given in [6].

Now, we are ready to prove the conjecture for $k \leq 4$. With only one tree degree sequence, it is clear that we have a disjoint caterpillar realization. For $k = 2$, the conjecture was proved in [4], however, here is a simplified proof.

Theorem 4.4. *Let \mathbf{D} be a $2 \times n$ tree degree matrix without common leaves. Then \mathbf{D} has a caterpillar realization.*

Proof. The proof is constructive, using an induction on the number of vertices. If both sequences are path sequences, then they have edge disjoint realizations, according to Lemma 2.4. Assume that at least one of the degree sequences is not a path; w.l.o.g., we can assume that the first degree sequence is not a path. According to Lemma 4.3, there is a vertex v which is a leaf in the first degree sequence, and has degree 2 in the second degree sequence (the two rows in \mathbf{D} might have to be swapped). Let v_j be a vertex with degree at least 3 in the non-path degree sequence. Then removing the column representing vertex v and subtracting 1 from $d_{1,j}$ yields a tree degree matrix \mathbf{D}' without common leaves. By our induction hypothesis, it has a caterpillar realization.

Let G' be a realization of \mathbf{D}' . Its caterpillar realization of the second row of \mathbf{D}' contains at least one edge in its backbone. At either end, there is one edge connecting an endpoint to the backbone. Altogether, they form a path of at least three edges. At most two of them can be incident to v_j . Consider an edge not incident to v_j ; let it be (u, w) . We can construct a caterpillar realization of \mathbf{D} from G' in the following way. Add vertex v to G' . Connect v and v_j with an edge of the first color. Remove edge (u, w) and connect v to both u and w with an edge of the second color. The subgraph of each color is a caterpillar realization of the appropriate row of \mathbf{D} , and they are edge-disjoint. Indeed, the caterpillar of the first color in G' is extended with a leaf, and v_j is not a leaf in this caterpillar. Vertex v is a degree 2 vertex in the second caterpillar, either inserted into the backbone or inserted between a leaf and the adjacent last vertex of the backbone. In both cases, the resulting tree is a caterpillar. \square

The proof is very similar for three and four caterpillars. Just instead of a single edge (u, w) avoiding vertex v_j , we have to find a rainbow matching avoiding v_j in appropriate paths. Since we will use this technique multiple times throughout the paper, we introduce it in a separate lemma.

Lemma 4.5. *Let $\mathbf{D} \in \mathbb{N}^{k \times n}$ be a tree degree matrix without common leaves. Let $\mathbf{D}' \in \mathbb{N}^{k \times (n-1)}$ be a tree degree matrix without common leaves that we obtain from \mathbf{D} by deleting a column containing all 2's except a 1 in row i , and subtracting 1 from an entry $d_{i,j} > 2$. Let G' be an arbitrary caterpillar realization of \mathbf{D}' . For the realized caterpillar of row i , let P^i be the path containing the backbone of the caterpillar and two additional edges connecting arbitrary leaves to the end vertices of the backbone. If $\bigcup_{i \neq i} P^i$ contains a rainbow matching of size $k - 1$ avoiding v_j , then \mathbf{D} has a caterpillar realization.*

Proof. We are going to explicitly construct the caterpillar realization of \mathbf{D} from G' . We add a vertex v to G' . Vertex v_j is connected to v with an edge of color i . For each edge (u, w) in the rainbow matching, the edge is removed and both u and w are connected to v with an edge of the color of the removed edge.

We claim this is a caterpillar realization of \mathbf{D} . Indeed, for each color, we got a caterpillar realization of the appropriate row. In case of color i , the caterpillar in \mathbf{D}' is extended with a leaf, and the leaf is connected to a backbone vertex. For all other colors l , a degree 2 vertex is inserted into P^l . The so-obtained tree is a caterpillar. No parallel edges are introduced, because the new edges are formed from v and vertices incident to edges in a rainbow matching which specifically avoids v_j . \square

Theorem 4.6. *Let $\mathbf{D} \in \mathbb{N}^{3 \times n}$ be a tree degree matrix without common leaves. Then \mathbf{D} has a caterpillar realization.*

Proof. The proof is again constructive, using an induction on the number of vertices. The base case is the tree degree matrix in which each row is a path degree sequence. In this case, Lemma 2.4 provides edge disjoint path realizations.

Assume not all the rows are path degree sequences. According to Lemma 4.3, there exists a column l which contains, w.l.o.g., 1 in the first row and 2 in the other two rows. We can also assume that the first row is not a path degree sequence, implying there is a vertex v_j such that $d_{1,j} \geq 3$. Consider \mathbf{D}' obtained from \mathbf{D} by removing column l and subtracting 1 from $d_{1,j}$. Matrix \mathbf{D}' is a tree degree matrix without common leaves, and based on the inductive assumption, it has a caterpillar realization. Let the union of these caterpillars be denoted by G' .

We want to find a rainbow matching in the remaining two rows avoiding v_j . The realized caterpillars of the second and third rows both contain a path of length at least 5, according to Lemma 4.1. In both paths, at most two edges are incident to v_j , so there are at least three edges in each caterpillar not incident to v_j . These three edges form a path of length 3 or a path of length 2 with a separated edge. Suppose all three of these edges from one caterpillar are blocked by the other caterpillar. In both configurations, at most two of the three edges in one of the caterpillars can block all the three edges in the other caterpillar, as shown in Fig. 2. Therefore, there exists at least one of the three edges, call it e_1 , not incident to v_j and not adjacent to some other edge e_2 in the other caterpillar. Furthermore, e_2 is not incident to v_j . Therefore, e_1 and e_2 form a rainbow matching with two prescribed colors and avoid v_j . By Lemma 4.5, \mathbf{D} has a caterpillar realization. \square

The proof for $k = 4$ uses similar ideas, however, we need base cases where $n \leq 10$. Also, finding an appropriate rainbow matching is not easy. So, we separately present it in the following lemma.

Lemma 4.7. *Let $\mathbf{D} = \{D_1, D_2, D_3, D_4\} \in \mathbb{N}^{4 \times n}$ be a tree degree matrix without common leaves. Let G be a caterpillar realization of \mathbf{D} . Assume v_j is an arbitrary vertex, and $G' \subset G$ is a caterpillar realization of an arbitrary three of the four tree degree sequences. If $n \geq 10$, there exists a rainbow matching of size three in $G' \setminus \{v_j\}$.*

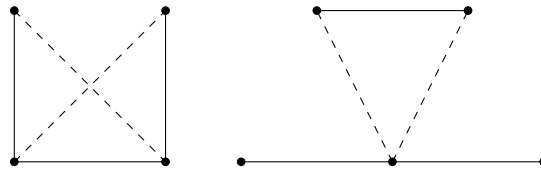


Fig. 2. Only the two dashed edges can both block all three solid edges. See the text for details.

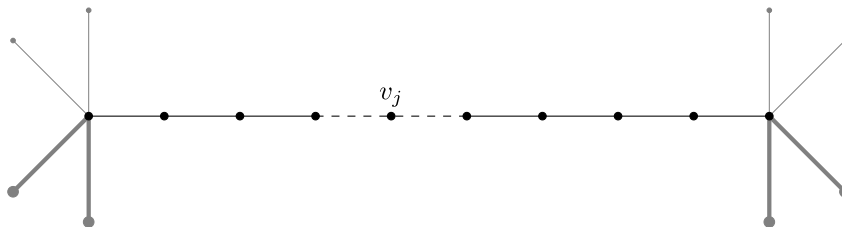


Fig. 3. The graph shows the situation when v_j is an internal node of the green path (longest path drawn in black). The thick gray path represents the blue path, while the thin gray edges represent the red edges. The dashed edges represent the blocked edges. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. By applying Lemma 4.2 and choosing $k = 4$, we derive a special case for four tree degree sequences. Within any three out of four caterpillar realizations of the tree degree sequences, there exists a path of length at least 9. Let the degree sequence whose realization contains the longest path be colored green, and the other two be colored blue and red. We have three cases. Case 1: v_j is an internal node of the green degree sequence. Case 2: v_j is a leaf of the green degree sequence. Case 3: The green path does not contain v_j . We will illustrate these three cases separately.

Fig. 3 illustrates the first general case when v_j is an internal node of the green degree sequence. It only includes the longest path in the realization of the green degree sequence. As we are considering the graph $G \setminus \{v_j\}$, the edges connected to v_j are not considered. Two endpoints of the green path are leaves. So, they cannot be leaves of the blue and red degree sequences. Hence, these endpoints must each be adjacent to two red and two blue edges. Call those eight edges endpoint edges. We have three possible scenarios.

Scenario 1: Less than two of the endpoint edges are incident to v_j .

At most one endpoint edge is blocked by v_j . Assume the color of this edge is blue. Consider the two red endpoint edges at this end of the green path, and the two blue endpoint edges at the opposite end of the green path. If none of these endpoint edges is incident to the two endpoints of the green path, then choose one of the red edges. It blocks at most one of the blue edges, so we have a pair of red and blue edges which is not adjacent. If one of these endpoint edges is incident to the two endpoints of the green path, then w.l.o.g. we can assume that it is a blue edge. Select the other blue endpoint edge, it blocks at most one of the red edges, therefore we again have a red and a blue edge that are not adjacent. For the green edges, we know that each blue and red edge in our rainbow matching set will block one leaf in green and at most another two edges in the green path. Also, v_j blocks two green edges. Altogether, at most eight edges in the green path are blocked and there must exist one green edge that we can select. In this way, we find a rainbow matching of size three.

Scenario 2: Two endpoint edges of the same color are incident to v_j .

W.l.o.g., we can assume that the two endpoint edges incident to v_j are blue. Select any of the red endpoint edges, call it e . It is adjacent to at most four blue edges, one of these blue edges is also incident to v_j , and there is another blue edge incident to v_j . However, there are at least seven blue edges in the path of the blue caterpillar. So, there must be at least two blue edges which are neither adjacent to e nor incident to v_j , call them f_1 and f_2 . The vertex v_j blocks two green edges from the green path. Edge e blocks at most three green edges from the green path. There are at least four remaining green edges. It is impossible that both f_1 and f_2 block all these four edges. Select a blue edge from $\{f_1, f_2\}$ that does not block the green edges incident to e or v_j . Also, select e and the green edge that is not adjacent to the selected blue edge, e , or v_j . These three edges form the appropriate rainbow matching.

Scenario 3: Two endpoint edges of different color are adjacent to v_j .

In this scenario, there are two blue edges and a red edge adjacent to one end of the green path. There are two red edges and a blue edge adjacent to the other end of the green path. None of these blue or red edges are incident to v_j . Even if this ensemble of edges are only five edges because the two ends of the green path are connected with a red or a blue edge, there is an edge at one of the ends of the green path, w.l.o.g., we can say it is a blue edge, and there are two red edges at the other end. The blue edge can block at most one of the red edges. We have disjoint red and blue edges at the two ends of the green path. They block at most six of the green edges, and v_j blocks two of the green edges. So,

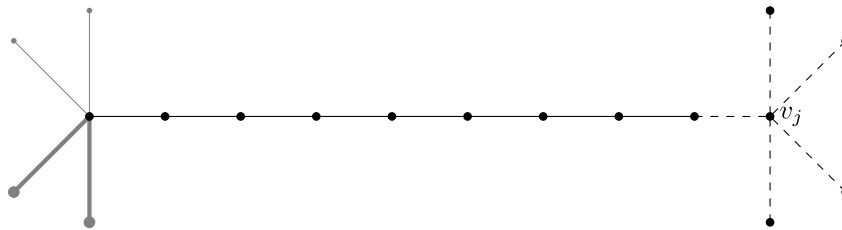


Fig. 4. The graph shows the situation when v_j is a leaf of the green path (longest path drawn in black). The thick gray path represents the blue path, while the thin gray edges represent the red edges. The dashed edges represent the blocked edges.

there is a green edge not adjacent to the selected red and blue edges and not incident to v_j . A pair of these disjoint red and blue edges, along with the green edge, forms the appropriate rainbow matching.

Fig. 4 illustrates the second general case. In this case, the edges adjacent to one end of the green path are all blocked. One of the blocked edges is a green edge in the path, and at least four of the blocked edges are from the remaining two colors.

Consider the other end of the green path. At most one color has an edge that connects to the other end of the path. Assume that edge has color blue. If no edge connects to the other end of the path, choose one arbitrary edge as blue. Select a blue edge that is not adjacent to the other end of the green path as our first edge for the rainbow matching set. Assume the other end of the blue edge is v_i . Then, we need to find a red edge that is not adjacent to either ends of the green path or v_i . By Lemma 4.1, the red path must contain at least 7 edges. Each of the three vertices will block at most two red edges in the red path so there must exist one red edge left over. Select that red edge to be in the rainbow matching set.

Now we find the green edge. The blue edge blocks one green leaf and another two green edges. The red edge will block four green edges. Also, v_j blocks one green edge. Altogether, at most eight green edges are blocked. Since there are nine green edges, we can always select one green edge to put in the rainbow matching set. We have constructed the appropriate rainbow matching set of size three.

Finally, in Case 3, v_j is not on the green path. In that case, we can find disjoint red and blue edges not incident to v_j , see the proof of Theorem 4.6. These two edges block at most eight edges from the green path, so there is a green edge which is not adjacent to the selected red and blue edges and also not incident to v_j . □

We are now ready to prove Conjecture 1 for $k = 4$.

Theorem 4.8. Let $\mathbf{D} \in \mathbb{N}^{4 \times n}$ be a tree degree matrix without common leaves. Then \mathbf{D} has a caterpillar realization.

Proof. The proof is constructive and based on induction. The base cases of the induction are those tree degree matrices that contain only path degree sequences and the matrices with at most 10 vertices. If all rows are path degree sequences, then there exists a caterpillar realization by Lemma 2.4. Up to permuting rows and columns, there are only 14 tree degree matrices without common leaves. In the Appendix, we list these matrices and give a realization for each of them.

Now assume that $\mathbf{D} \in \mathbb{N}^{4 \times n}$ is a tree degree matrix without common leaves, where $n \geq 11$ and there is at least one row which is not a path degree sequence. Then there exists a column l that contains a 1 in a row not containing a path degree sequence, and all other entries in the column are 2, according to Lemma 4.3. Let i be the row such that $d_{i,l} = 1$, and let j be a column for which $d_{i,j} > 2$. Construct \mathbf{D}' in the following way: remove column l , and subtract 1 from $d_{i,j}$. Then \mathbf{D}' is a tree degree matrix without common leaves, and based on the inductive assumption, it has a caterpillar realization. Let G' be such a realization. According to Lemma 4.7, the paths in the caterpillar realizations with color other than i contain a rainbow matching avoiding v_j . By Lemma 4.5, \mathbf{D} has a caterpillar realization. □

5. Degree sequences on many vertices

For more than four tree degree sequences on a small number of vertices, it is hard to prove the existence of a rainbow matching of size $k - 1$ within an arbitrary $k - 1$ of the caterpillar realizations, while avoiding a prescribed vertex. It has been proved that edge disjoint tree realizations exist for any $\mathbf{D} \in \mathbb{N}^{k \times n}$ tree degree matrix without common leaves with $n \geq 4k - 1$ if edge disjoint tree realizations exist for any $\mathbf{D} \in \mathbb{N}^{k \times (4k-2)}$ tree degree matrix without common leaves [6]. We can prove a similar theorem with caterpillar realizations. For this, we need one more lemma on the lower bound of the length of the paths in caterpillar realizations.

Lemma 5.1. Let G be a caterpillar realization of $\mathbf{D} \in \mathbb{N}^{k \times n}$. Consider any $k - 1$ of its caterpillars, and arrange them in increasing order based on the lengths of their paths containing their backbones, and the edges connecting leaves to the ends of the backbone. Then the l th longest path contains at least $\binom{l-1}{l}n + 2$ edges.

Proof. The proof is based on contradiction. Assume that the l th longest path has at most $(\frac{l-1}{l})n + 1$ edges. Then this caterpillar has at most $(\frac{l-1}{l})n$ internal nodes, and thus, at least $\frac{n}{l}$ leaves. Since the lengths of the paths are in increasing order, there are at least l caterpillars with at least $\frac{n}{l}$ leaves. The other $k - l$ caterpillars have at least two leaves. Then there are at least

$$l \times \frac{n}{l} + (k - l) \times 2 = n + 2(k - l) > n$$

leaves altogether, which produces a contradiction for \mathbf{D} is a tree degree matrix without common leaves. \square

Theorem 5.2. *Let k be an arbitrary positive integer. If any $\mathbf{D}' \in \mathbb{N}^{k \times (4k-2)}$ tree degree matrix without common leaves has a caterpillar realization, then any $\mathbf{D} \in \mathbb{N}^{k \times n}$ tree degree matrix without common leaves and $n \geq 4k - 1$ has a caterpillar realization.*

Proof. The proof is still based on induction. The base cases are the tree degree matrices in which each row is a path degree sequence and the tree degree matrices have dimension $k \times (4k - 2)$. For any other tree degree matrix \mathbf{D} , we can construct the corresponding \mathbf{D}' matrix (as we did in the proofs of the previous theorems), which has a realization G' . Next, we need to find a rainbow matching in the paths of $k - 1$ selected caterpillars that avoids a prescribed vertex v_j . We claim we can find this rainbow matching in a greedy way. Arrange the caterpillars in increasing order based on the length of their longest path. We know that the l th caterpillar has a path of length at least $(\frac{l-1}{l})n + 2$, according to Lemma 5.1.

Therefore, it has a matching of size at least $\lceil \frac{(\frac{l-1}{l})n+2}{2} \rceil = \lceil \frac{(l-1)n}{2l} \rceil + 1$. We know that n is at least $4k - 2$ and k is at least $l + 1$, thus the matching has a size at least

$$\lceil \frac{2(l-1)(2l+1)}{2l} \rceil + 1 \geq 2(l-1) + 2.$$

We already selected $l - 1$ edges in the rainbow matching that block at most $2(l - 1)$ edges in the matching of the l th color. Vertex v_j can block at most one edge. Therefore, we have an edge of the l th color that is not adjacent to any of the so-far selected edges, and is not incident to vertex v_j . We can select this edge for the rainbow matching.

Since we are able to find a rainbow matching avoiding v_j , \mathbf{D} has a caterpillar realization, according to Lemma 4.5. \square

We also introduce a theorem that unconditionally claims the existence of caterpillar realizations with large number of vertices. For this theorem, we have a new strategy to directly construct the caterpillar realization for \mathbf{D} . Treat a caterpillar as the union of leaves and a backbone. We define *legs* as the edges that are incident to a leaf. We also define *backbone edges* as all the other edges. The construction strategy is to first construct all the legs and three backbones and then the remaining backbones. The key point is that we will be able to find backbones as Hamiltonian paths in appropriate subgraphs that we obtain by removing the so-far used edges from the complete graph on the given backbone vertices. In general, we will denote this subgraph as F . The existence of these Hamiltonian paths is proved by a theorem similar to Ore's theorem, which states that for a finite and simple graph G , if $d_i + d_j \geq n$ for every pair of distinct non-adjacent vertices i and j of G , then G must contain a Hamiltonian cycle [1,12].

In our case, not all vertices satisfy the conditions of Ore's theorem. However, we will still be able to find Hamiltonian paths in a given graph F . Our strategy is based on the following observations:

1. All backbones are sufficiently long except the shortest three backbones. Actually, only the shortest backbone might have $o(n)$ length, where n is the number of vertices; however, we can easily construct the three shortest backbones. This allows for a better lower bound on the number of vertices necessary to construct the remaining caterpillars.
2. In any tree degree matrix without common leaves, there is at most one vertex (that is, column) with total degree at least $\frac{2n}{3}$ if n is sufficiently large. Actually, for any $c > \frac{1}{2}$, there is at most one vertex whose degree is at least cn if n is sufficiently large.
3. There are at most 11 vertices whose degrees are larger than $\frac{n}{6}$ if n is sufficiently large. Actually, for any $c' > 0$, there are at most a constant number of vertices whose degrees are larger than $c'n$ if n is sufficiently large.

We are going to precisely state and prove these statements below. These observations provide us the following construction strategy.

1. We first construct the legs of the caterpillars and three shortest backbones. Then we construct all other backbones.
2. To construct the other backbones, we “cap” all the backbone vertices whose degree is small in F , with vertices whose degree is large in F . The vertices with small degree in F are the same vertices with large degree in \mathbf{D} . There is a constant number of these vertices. Furthermore, at most one of them might have degree larger than $\frac{2n}{3}$ in \mathbf{D} . Therefore, at most one of them might have too small degree in F . We cap this vertex for each backbone at the end of the first phase. Since all other small degree vertices have degree at least $\frac{n}{6}$ in F , we can easily find high degree vertices to cap them.
3. We fix the edges used for capping the small degree vertices in F , and extend them to a Hamiltonian path. The algorithm to find such a Hamiltonian path is very similar to Palmer's algorithm [1] to find a Hamiltonian cycle in a graph satisfying the degree conditions in the hypothesis of Ore's theorem [12].

Below we state and prove the lemmas concerning the degree properties. The first is a simple observation.

Lemma 5.3. *Let $\mathbf{D} \in \mathbb{N}^{k \times n}$ be a tree degree matrix without common leaves. Assume that $n \geq 6k - 5$. Then there exists at most one vertex whose degree is larger than or equal to $\frac{2n}{3}$.*

Proof. Assume to the contrary there exist at least two vertices with degree at least $\frac{2n}{3}$. Then the total sum of degrees is at least

$$\frac{4n}{3} + (n - 2)(2k - 1) = 2kn + \frac{n}{3} - 4k + 2.$$

However, the sum of degrees is $k(2n - 2)$. If

$$2kn + \frac{n}{3} - 4k + 2 \leq k(2n - 2),$$

then

$$n \leq 6k - 6,$$

a contradiction. \square

The number of relatively high degree vertices is also small.

Lemma 5.4. *Let $\mathbf{D} \in \mathbb{N}^{k \times n}$ be a tree degree matrix without common leaves. If $n \geq 22k - 11$, then there are at most 11 vertices with degree at least $\frac{n}{6}$.*

Proof. Assume the contrary. Then the sum of the degrees is at least

$$12 \cdot \frac{n}{6} + (n - 12)(2k - 1) = 2nk + n - 24k + 12.$$

The sum of degrees is $k(2n - 2)$. Then it holds that

$$2nk + n - 24k + 12 \leq k(2n - 2)$$

from which

$$n \leq 22k - 12,$$

a contradiction. \square

We are now ready to state and prove the following theorem.

Theorem 5.5. *Let $\mathbf{D} \in \mathbb{N}^{k \times n}$ be a tree degree sequence without common leaves. Assume that $k \geq 5$ and $n \geq \max\{22k - 11, 396\}$. Then \mathbf{D} has a caterpillar realization.*

Proof. We explicitly construct a realization in two phases. In the first phase, we construct the legs of the caterpillars, the backbones of the three shortest backbones and all the remaining edges of the largest degree vertex. In the second phase, we construct the remaining backbones.

Phase I

Let $\mathbf{D} = \mathbf{D}_0, \mathbf{D}_1, \dots, \mathbf{D}_m$ be a series of tree degree matrices, such that \mathbf{D}_m contains only path degree sequences, and for any $l = 0, \dots, m - 1$, \mathbf{D}_{l+1} is obtained from \mathbf{D}_l by removing a column containing all 2's except in row i , where the entry is 1, and then subtracting 1 from a $d_{i,j} > 2$. According to Lemma 4.3, we can always find a column with column sum $2k - 1$ and entry $d_{i,j}$. Matrix \mathbf{D}_m has edge disjoint path realizations, according to Lemma 2.4.

Let v denote the vertex with the largest column sum in \mathbf{D} . Furthermore, let the vertices adjacent to leaves in the paths be called “end vertices”. Let G be a subset of the above-mentioned edge disjoint path realizations of \mathbf{D}_m containing the three paths corresponding to the caterpillars with shortest backbones in \mathbf{D} , the legs of the other paths and all edges incident to v . Observe that for each color, v has at most two backbone edges. When it is incident to exactly two backbone edges, then at most one of these edges is incident to an end vertex. Furthermore, when v has one backbone edge, that is, when v is an end vertex, then this edge is not incident to another end vertex. We call the backbone edges of v “capping edges”.

Then going from \mathbf{D}_m to \mathbf{D} , we construct the three aforementioned caterpillars and all the legs of other caterpillars, adding one vertex v to G in each step. For each \mathbf{D}_l to \mathbf{D}_{l-1} transition, if the removed column contains a 1 in a row not corresponding to the three caterpillars, then add a leg between v and v_j with color i and find a rainbow matching avoiding v_j in the backbones of the three caterpillars and extend these caterpillars by pulling these edges onto v . Such a rainbow matching exists, according to Lemma 4.7. If row i contains one of the caterpillars constructed in this phase, then connect v_j

to v with color i , find a rainbow matching in the backbones of the other caterpillars, and pull them onto v . Such a rainbow matching exists, according to Lemma 4.7.

In this way, we construct all legs, the three caterpillars with the shortest backbones and all edges incident to v . Furthermore, we put all of these (appropriately colored) edges in G . For other backbones, the remaining degree of each vertex (that is, the difference of its degree with color i in G and the corresponding entry in row i of \mathbf{D}) is either 1 or 2 (except for v). The vertices with remaining degree 1 are exactly the end vertices of the backbones, as all their other degrees are used for legs (and v is an end vertex if it has one capping edge). We are ready to enter Phase II.

Phase II

We construct the backbones in increasing order according to their length. We add these backbones to G , and before each backbone, let F denote the complement of G restricted to the current backbone vertices. Since the legs of the current caterpillar are added, the two end vertices of the backbone are prescribed. Our task is to find a Hamiltonian path between these two vertices in F . Although the majority of the degrees is large in F , we cannot directly apply Ore’s theorem, because there might exist a few small degree vertices. We are going to cap the small degree vertices with high degree vertices, and then extend them into a Hamiltonian path.

Let m denote the size of F . From Lemma 5.1, we know that m is larger than $\frac{3n}{4}$. We know only v can have a degree reaching $\frac{2n}{3}$, and at most 11 vertices of G can have degrees reaching $\frac{n}{6}$. If a degree in G is less than $\frac{2n}{3}$, then its degree in F is at least

$$\frac{3n}{4} - \frac{2n}{3} = \frac{n}{12}.$$

All other vertices have degree less than $\frac{n}{6}$. Thus, their degree in F is at least

$$\frac{3n}{4} - \frac{n}{6} = \frac{7n}{12}.$$

Therefore, the sum of any two of these high degrees is at least $\frac{7n}{6}$.

If v is a backbone vertex in the current caterpillar, put its one or two capping edges into the set E . Then for each vertex w in the backbone that has degree at least $\frac{n}{6}$ (but less than $\frac{2n}{3}$) in G , we distinguish four cases:

1. Vertex w is incident to a capping edge of v , and it is an end vertex. In this case, we do not have to find further capping edges of w .
2. Vertex w is incident to a capping edge of v , and it is not an end vertex. Then we will find one more capping edge of w .
3. Vertex w is not incident to a capping edge of v , and it is an end vertex. Then we will find one capping edge of w .
4. Vertex w is not incident to a capping edge of v , and it is not an end vertex. Then we will find two capping edges of w .

We claim that we can find the necessary one or two neighbor vertices of w , denoted by u_1 and u_2 in F that have high degree in F (at least $\frac{7n}{12}$), not incident to any edge in E , and where at most one of them is an endpoint of the backbone. Such neighbors must exist, because there are at most 30 forbidden points (at most 10 other low degree vertices and for each of them, at most two neighbors incident to their capping edges). However, these low degree vertices have degree at least $\frac{n}{12}$ in F , and n is at least 396. Therefore, there are at least three neighbors which are not forbidden. We can select two of the three such that at most one of them is an endpoint of the backbone. We add edges (w, u_1) and (w, u_2) to E .

Now we construct the backbone. Arrange the backbone vertices in a cycle, starting and ending with the endpoints of the backbone such that vertices incident to the same edge in E are neighbors. We set up such a permutation, since the endpoints do not have a common neighbor in edge set E . Then we apply an algorithm similar to Palmer’s algorithm to construct a Hamiltonian path [1]. While there remain two neighbor vertices u_1 and u_2 around the cycle in a clockwise direction not having an edge in F , we find a vertex pair w_1 and w_2 such that they are neighbors around the cycle, not both of them are endpoints of the backbone, and $(u_1, w_1) \in E, (u_2, w_2) \in E$, but $(w_1, w_2) \notin E$.

By the pigeonhole principle, such a pair of vertices exists. Both u_1 and u_2 have high degree. The sum of their degrees is at least $\frac{7n}{6}$. So there must exist at least $\frac{n}{6}$ pairs of neighbor vertices such that $(u_1, w_1) \in E$ and $(u_2, w_2) \in E$. There are at most 23 forbidden pairs from the at most 22 pairs forming the edges in E and the pair of endpoint vertices. However, $\frac{n}{6} > 23$.

We swap the appropriate arc of the cycle to make u_1 be a neighbor of w_1 , and u_2 be a neighbor of w_2 . With this operation, we decrease the amount of neighbor pairs u_1, u_2 around the cycle that do not have an edge between them in F . After applying this operation a finite number of times, the number of such neighbors will reach 0. That is, there is a Hamiltonian path in F between the two endpoint vertices of the backbone.

Since for each degree sequence we can find a Hamiltonian path in F between the two endpoints of the backbone, we can construct a caterpillar realization of \mathbf{D} . □

6. Discussion

In this paper, we considered the caterpillar realizations of tree degree matrices. We presented necessary and sufficient conditions when a $2 \times n$ tree degree matrix has an edge disjoint caterpillar realization. Starting from $k = 3$, it seems extremely hard to find necessary and sufficient conditions for a caterpillar realization of a $k \times n$ tree degree matrix. However, the vertices having no common leaves seems to be a sufficient condition, that is, each vertex has degree 1 in at most one of the degree sequences. We were able to prove that this condition is sufficient when $k \leq 4$, or when $n \geq \max\{22k - 11, 396\}$. Naturally, n should be at least $2k$, and we also proved that the conjecture is true if it is true for any $n \leq 4k - 2$. However, it seems difficult to close the gap between $n = 2k$ and $n = 4k - 2$, though it is well known that the conjecture is true for $n = 2k$ [2].

Since any caterpillar is a tree, our conjecture is also a conjecture for edge disjoint tree realizations. The “no common leaves” condition forces the column sums to be more or less evenly distributed, that is, most of the column sums are $o(n)$. It is an open question if other conditions forcing evenly distributed column sums are sufficient for caterpillar (or edge disjoint tree) realizations. It is also an open question how many common leaves are necessary to find a counterexample of a tree degree matrix that has no caterpillar realizations.

CRedit authorship contribution statement

István Miklós: Supervised the research of these undergraduate students, Wrote the introduction and conclusions, Revised the manuscript. **Geneva Schlafly:** Worked on sections 4 and 5, Revised the manuscript. **Yuheng Wang:** Worked on sections 4 and 5, Revised the manuscript. **Zhangyang Wei:** Worked on Section 3, Revised the manuscript.

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Appendix

Up to permutations of degree sequences and vertices, there are 14 tree degree matrices on at most 10 vertices without common leaves. This appendix gives an example caterpillar realization for all of them.

If the number of vertices is eight, there is only one possible tree degree matrix, each degree sequence is a path degree sequence (case 1).

If the number of vertices is eight, there are two possible cases: either all degree sequences are path degree sequences (case 2) or there is a degree 3 (case 3).

If the number of vertices is 10, there are 11 possible cases: all degree sequences are path degree sequences (case 4), there is a degree 3 which might be on a vertex with a leaf (case 5) or without a leaf (case 6), there is a degree 4 (case 7) or there are two degree 3’s in the degree sequences (cases 8–14).

The two 3’s might be in the same degree sequence. The leaves on these two vertices might be in the same degree sequence (case 8) or in different degree sequences (case 9).

If the two degree 3’s are in different degree sequences, they might be on the same vertex (case 10) or on different vertices.

If the two degree 3’s are in different sequences, D_i and D_j , and on different vertices u and v , consider the degrees of u and v in D_i and D_j which are not 3. They might be both 1 (case 11), or maybe one of them is 1 and the other is 2 (case 12), or both of them are 2. In this latter case, the degree 1’s on u and v might be in the same degree sequence (case 13) or in different degree sequences (case 14).

The realizations are represented with an adjacency matrix, in which 0 denotes the absence of edges, and for each degree sequence D_i , i denotes the edges in the realization of D_i .

1.

$$D = \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 \\ 2 & 2 & 0 & 3 & 4 & 4 & 1 & 1 \\ 2 & 3 & 3 & 0 & 4 & 1 & 1 & 2 \\ 3 & 3 & 4 & 4 & 0 & 1 & 2 & 2 \\ 3 & 4 & 4 & 1 & 1 & 0 & 2 & 3 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 3 \\ 4 & 1 & 1 & 2 & 2 & 3 & 3 & 0 \end{pmatrix}$$

2.

$$\mathbf{D} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 1 \\ 2 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 1 \\ 2 & 3 & 3 & 0 & 4 & 0 & 1 & 1 & 2 \\ 3 & 3 & 4 & 4 & 0 & 1 & 1 & 2 & 2 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 2 & 3 \\ 4 & 4 & 0 & 1 & 1 & 2 & 0 & 3 & 3 \\ 4 & 0 & 1 & 1 & 2 & 2 & 3 & 0 & 4 \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 0 \end{pmatrix}$$

3.

$$\mathbf{D} = \begin{pmatrix} 1 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 2 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 \\ 0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 \\ 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4 \\ 3 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4 \\ 3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 \\ 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 \\ 2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 \end{pmatrix}$$

4.

$$\mathbf{D} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0 & 0 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 0 & 1 \\ 2 & 2 & 0 & 3 & 4 & 4 & 0 & 0 & 1 & 1 \\ 2 & 3 & 3 & 0 & 4 & 0 & 0 & 1 & 1 & 2 \\ 3 & 3 & 4 & 4 & 0 & 0 & 1 & 1 & 2 & 2 \\ 3 & 4 & 4 & 0 & 0 & 0 & 1 & 2 & 2 & 3 \\ 4 & 4 & 0 & 0 & 1 & 1 & 0 & 2 & 3 & 3 \\ 4 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 4 \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0 \end{pmatrix}$$

5.

$$\mathbf{D} = \begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 2 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 1 & 1 \\ 0 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 0 & 0 & 0 & 1 & 1 & 2 & 4 \\ 3 & 3 & 4 & 0 & 0 & 1 & 1 & 2 & 2 & 4 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 2 & 3 & 0 \\ 4 & 4 & 0 & 1 & 1 & 2 & 0 & 0 & 3 & 3 \\ 4 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 4 & 3 \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 0 & 0 \\ 2 & 1 & 2 & 4 & 4 & 0 & 3 & 3 & 0 & 0 \end{pmatrix}$$

6.

$$D = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 1 & 2 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 0 & 2 \\ 1 & 0 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 3 \\ 0 & 2 & 0 & 3 & 4 & 4 & 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 0 & 4 & 0 & 1 & 1 & 2 & 3 \\ 3 & 3 & 4 & 4 & 0 & 1 & 1 & 2 & 2 & 1 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 2 & 3 & 0 \\ 4 & 4 & 0 & 1 & 1 & 2 & 0 & 3 & 3 & 0 \\ 4 & 0 & 1 & 1 & 2 & 2 & 3 & 0 & 0 & 4 \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 4 \\ 2 & 3 & 2 & 3 & 1 & 0 & 0 & 4 & 4 & 0 \end{pmatrix}$$

7.

$$D = \begin{pmatrix} 1 & 4 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 3 & 3 & 4 & 4 & 2 & 2 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 0 & 4 & 1 & 1 & 2 & 4 \\ 0 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4 & 2 \\ 3 & 3 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 4 \\ 3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 \\ 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3 \\ 2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0 \\ 2 & 1 & 4 & 2 & 4 & 0 & 3 & 3 & 0 & 0 \end{pmatrix}$$

8.

$$D = \begin{pmatrix} 1 & 3 & 2 & 2 & 1 & 3 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 3 & 4 & 4 & 2 & 3 \\ 1 & 0 & 0 & 3 & 3 & 4 & 4 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 0 & 4 \\ 0 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4 & 3 \\ 3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 1 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 0 \\ 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 0 \\ 2 & 1 & 2 & 0 & 4 & 3 & 0 & 3 & 0 & 4 \\ 3 & 2 & 2 & 4 & 3 & 1 & 0 & 0 & 4 & 0 \end{pmatrix}$$

9.

$$D = \begin{pmatrix} 1 & 3 & 3 & 2 & 1 & 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 3 & 3 & 4 & 4 & 2 & 2 \\ 1 & 0 & 2 & 3 & 3 & 0 & 4 & 1 & 1 & 4 \\ 0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 1 \\ 0 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4 & 2 \\ 3 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4 & 0 \\ 3 & 0 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 4 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 \\ 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3 \\ 2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0 \\ 2 & 4 & 1 & 2 & 0 & 4 & 3 & 3 & 0 & 0 \end{pmatrix}$$

10.

$$D = \begin{pmatrix} 1 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 3 & 4 & 0 & 2 & 4 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 3 \\ 0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 0 \\ 2 & 3 & 3 & 0 & 0 & 0 & 1 & 2 & 4 & 1 \\ 3 & 3 & 4 & 0 & 0 & 1 & 0 & 2 & 4 & 2 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 0 & 3 & 1 \\ 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 4 \\ 2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0 \\ 4 & 3 & 0 & 1 & 2 & 1 & 2 & 4 & 0 & 0 \end{pmatrix}$$

11.

$$D = \begin{pmatrix} 1 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 3 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 2 \\ 1 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 & 4 & 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 0 & 0 & 0 & 1 & 2 & 4 & 1 \\ 3 & 3 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 4 \\ 3 & 4 & 4 & 0 & 1 & 0 & 2 & 0 & 3 & 1 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 \\ 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3 \\ 2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0 \\ 2 & 0 & 4 & 1 & 4 & 1 & 3 & 3 & 0 & 0 \end{pmatrix}$$

12.

$$D = \begin{pmatrix} 1 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 1 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 1 \\ 0 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 0 & 4 \\ 3 & 3 & 4 & 0 & 0 & 1 & 2 & 2 & 4 & 0 \\ 3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 \\ 4 & 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 \\ 4 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 3 & 3 \\ 2 & 1 & 2 & 0 & 4 & 3 & 0 & 3 & 0 & 4 \\ 1 & 1 & 2 & 4 & 0 & 0 & 3 & 3 & 4 & 0 \end{pmatrix}$$

13.

$$D = \begin{pmatrix} 1 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 1 \\ 0 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 4 & 4 & 1 & 1 & 2 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 0 & 4 \\ 3 & 3 & 4 & 0 & 0 & 1 & 0 & 2 & 4 & 2 \\ 3 & 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 3 \\ 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 & 2 \\ 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 0 \\ 2 & 1 & 2 & 0 & 4 & 3 & 0 & 3 & 0 & 4 \\ 1 & 1 & 0 & 4 & 2 & 3 & 2 & 0 & 4 & 0 \end{pmatrix}$$

14.

$$D = \begin{pmatrix} 1 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 3 & 2 & 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 & 3 & 4 & 4 & 2 & 1 \\ 0 & 0 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 4 & 0 & 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 0 & 0 & 1 & 1 & 2 & 4 & 3 \\ 3 & 3 & 4 & 0 & 0 & 1 & 0 & 2 & 4 & 2 \\ 3 & 4 & 0 & 1 & 1 & 0 & 2 & 0 & 3 & 4 \\ 4 & 4 & 1 & 1 & 0 & 2 & 0 & 3 & 0 & 2 \\ 4 & 1 & 1 & 2 & 2 & 0 & 3 & 0 & 3 & 0 \\ 2 & 1 & 2 & 4 & 4 & 3 & 0 & 3 & 0 & 0 \\ 1 & 1 & 4 & 3 & 2 & 4 & 2 & 0 & 0 & 0 \end{pmatrix}$$

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