Algorithms based on $*$-algebras, and their applications to isomorphism of polynomials with one secret, group isomorphism, and polynomial identity testing*

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February 8, 2019

Abstract

We consider two basic algorithmic problems concerning tuples of (skew-)symmetric matrices. The first problem asks to decide, given two tuples of (skew-)symmetric matrices $(B_1, \ldots, B_m)$ and $(C_1, \ldots, C_m)$, whether there exists an invertible matrix A such that for every $i \in \{1, \ldots, m\}$, $A^t B_i A = C_i$. We show that this problem can be solved in randomized polynomial time over finite fields of odd size, the reals, and the complex numbers. The second problem asks to decide, given a tuple of square matrices $(B_1, \ldots, B_m)$, whether there exist invertible matrices A and D, such that for every $i \in \{1, \ldots, m\}$, AB_iD is (skew-)symmetric. We show that this problem can be solved in deterministic polynomial time over fields of characteristic not 2. For both problems we exploit the structure of the underlying $*$-algebras (algebras with an involutive anti-automorphism), and utilize results and methods from the module isomorphism problem.

Applications of our results range from multivariate cryptography, group isomorphism, to polynomial identity testing. Specifically, these results imply efficient algorithms for the following problems. (1) Test isomorphism of quadratic forms with one secret over a finite field of odd size. This problem belongs to a family of problems that serves as the security basis of certain authentication schemes proposed by Patarin (Eurocrypt 1996). (2) Test isomorphism of p-groups of class 2 and exponent p (p odd) with order p^ℓ in time polynomial in the group order, when the commutator subgroup is of order $p^{O(\sqrt{\ell})}$. (3) Deterministically reveal two families of singularity witnesses caused by the skew-symmetric structure. This represents a natural next step for the polynomial identity testing problem, in the direction set up by the recent resolution of the non-commutative rank problem (Garg-Gurvits-Oliveira-Wigderson, FOCS 2016; Ivanyos-Qiao-Subrahmanyam, ITCS 2017).

1 Introduction

We consider two basic algorithmic problems concerning tuples of (skew-)symmetric matrices. For convenience, for $\epsilon \in \{1, -1\}$, we say an $n \times n$ matrix B is ϵ-symmetric, if $B^t = \epsilon B$. Clearly, when $\epsilon = 1$ (resp. $\epsilon = -1$), B is symmetric (resp. skew-symmetric).

*A preliminary version of this paper appeared in SODA 2018 as [IQ18].

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The first problem asks to decide, given two tuples of $n \times n$ ϵ-symmetric matrices $(B_1, \ldots, B_m)$ and $(C_1, \ldots, C_m)$, whether there exists an invertible $n \times n$ matrix A, such that $\forall i \in [m]$, $A^t B_i A = C_i$. We call this problem *the isometry problem for ϵ-symmetric matrix tuples*. We show that this problem can be solved in randomized polynomial time when the underlying field is a finite field of odd size, the field of real numbers, or the field of complex numbers.

The second problem asks to decide, given a tuple of $n \times n$ matrices $(B_1, \ldots, B_m)$, whether there exist invertible $n \times n$ matrices A and D, such that $\forall i \in [m]$, AB_iD is ϵ-symmetric. We call this problem the *ϵ-symmetrization problem for matrix tuples*. We show that this problem can be solved in deterministic polynomial time, as long as the underlying field is not of characteristic 2.

At first sight, these two problems seem to be of interest mostly in computer algebra. However, as we explain below, these results are motivated by, and therefore have applications to, three seemingly unrelated research topics. These are multivariate cryptography, group isomorphism problem, and polynomial identity testing problem, which are traditionally studied in cryptography, computational group theory, and algebraic complexity theory, respectively. The algorithm for isometry testing of ϵ-symmetric matrix tuples leads to substantial improvements over recent algorithms from multivariate cryptography and group isomorphism [BFP15, BMW17]. In particular, the algorithm for isometry testing of symmetric matrix tuples completely settles the so-called Isomorphism of Quadratic Polynomials with One Secret problem over finite fields of odd size [Pat96]. The algorithm for the ϵ-symmetrization problem represents a natural next step for the polynomial identity testing problem in the direction set up by the recent resolution of the non-commutative rank problem [GGOW16, IQS17b, IQS17a].

The algorithms for the isometry problem and the ϵ-symmetrization problem share two key ingredients in common. The first one is to utilize the structure of $*$-algebras, that is algebras with an involutive anti-automorphism, underlying these problems. More specifically, given a field $\mathbb{F}$, an $\mathbb{F}$-algebra $\mathfrak{A}$ with anti-automorphism $* : \mathfrak{A} \to \mathfrak{A}$ of order at most 2 is termed as a $*$-algebra. We refer the reader to Section 2 for more details on the structure of $*$-algebras. Our use of $*$-algebras is inspired by the works of J. B. Wilson, who pioneered the use of $*$-algebras in computing with p-groups [Wil09a, Wil09b, BW12]. The second one is the results and methods from the module isomorphism problem, which asks to decide, given two tuples of matrices $(B_1, \ldots, B_m)$, $(C_1, \ldots, C_m)$, whether there exists an invertible matrix A, such that $\forall i \in [m]$, $AB_i = C_iA$. This problem admits two deterministic efficient algorithms by [CIK97, IKS10] and [BL08]. These results and the techniques are used frequently in both algorithms.

In this introduction, we first elaborate on the applications, from Section 1.1 to 1.3. Since the applications span across three different areas, in order to provide the contexts for readers with different backgrounds, we shall not refrain from including certain background information, despite that it is well-known for researchers in the respective area. In Section 1.4, we formally present the results, explain more on the two key ingredients shared by both algorithms, and describe some open problems.

We now set up some notation. $\mathbb{F}$, $\mathbb{E}$, and $\mathbb{K}$ are used to denote fields. $\mathbb{F}_q$ denotes the finite field of size q, $\mathbb{R}$ the real field, and $\mathbb{C}$ the complex field. Unless otherwise stated, we work with fields of characteristic not 2. $M(n, \mathbb{F})$ denotes the linear space of $n \times n$ matrices over $\mathbb{F}$, and $\mathrm{GL}(n, \mathbb{F})$ the group of invertible matrices in $M(n, \mathbb{F})$. $S^\epsilon(n, \mathbb{F})$ denotes the linear space of $n \times n$ ϵ-symmetric matrices over $\mathbb{F}$. We may write $M(n, q)$, $\mathrm{GL}(n, q)$, and $S^\epsilon(n, q)$ for $M(n, \mathbb{F}_q)$, $\mathrm{GL}(n, \mathbb{F}_q)$, and $S^\epsilon(n, \mathbb{F}_q)$, respectively. A *matrix space* is a linear subspace of $M(n, \mathbb{F})$, and $\langle \cdot \rangle$ denotes linear span. Let $\mathbf{B} = (B_1, \ldots, B_m) \in M(n, \mathbb{F})^m$ be a matrix tuple. For $A, D \in M(n, \mathbb{F})$, $A\mathbf{B}D :=$

$(AB_1D, \ldots, AB_mD)$ and $\mathbf{B}^t := (B_1^t, \ldots, B_n^t)$.

1.1 Multivariate cryptography

In 1996, Patarin proposed a family of asymmetric cryptography schemes based on equivalence of polynomials in [Pat96], which can be used for identification and signature schemes. One scheme in this family is based on the assumed hardness of the following problem.

Problem 1. (Isomorphism of Quadratic Forms with One Secret (IQF1S)) Let $\mathbf{f} = (f_1, \ldots, f_m)$ and $\mathbf{g} = (g_1, \ldots, g_m)$ be two tuples of homogeneous quadratic polynomials in n variables $\{x_1, \ldots, x_n\}$ over a finite field $\mathbb{F}$. Decide if there exists $A \in \mathrm{GL}(n, \mathbb{F})$ such that $\forall k \in [m]$, $f_k^A = g_k$, where $A = (a_{i,j})_{i,j\in[n]}$ acts on $\{x_1, \ldots, x_n\}$ by sending x_i to $\sum_{j\in[n]} a_{i,j}x_j$.

For readers familiar with Patarin's work [Pat96], IQF1S is Patarin's Isomorphism of Polynomials with One Secret (IP1S) restricting to quadratic polynomials, which asks the same question but for possibly inhomogeneous quadratic polynomials and affine transformations.[1] Such a restriction is well justified from the practical viewpoint, as it minimizes the public-key storage and improves the actual performance, so this has been studied most in the literature. Since Patarin's introduction of these problems, IQF1S and several related problems have been intensively studied [PGC98,GMS03, Per05,FP06,Kay11,BFFP11,MPG13,BFV13,PFM14,BFP15].

Most notably, in [BFP15], Berthomieu et al. presented an efficient randomized algorithm for IQF1S under the conditions that (1) $\mathbf{f}$ satisfies a regularity condition, namely that there exists a non-degenerate form in the linear span of f_i's, (2) the underlying field is large enough and of characteristic not 2, and (3) the desired solution may be from an extension field [BFP15, Theorem 2]. They further observed that, it seems that most known algorithms on IQF1S would fail on the irregular instances, and proposed the complexity of such instances as an open question [BFP15, Sec. 1, Open Question].

By the classical correspondence between quadratic forms and symmetric matrices, it is easy to see the equivalence between IQF1S and the isometry problem of tuples of symmetric matrices. Our algorithm for the latter problem then translates to a complete solution of IQF1S over finite fields of odd size, answering [BFP15, Sec. 1, Open Question] for such fields.

Theorem 2. *Let $\mathbb{F}$ be a finite field of odd size. There exists a randomized polynomial-time algorithm that solves the Isomorphism of Quadratic Forms with One Secret problem over $\mathbb{F}$.*

Furthermore, there has been a large body of works which aim to build public key cryptography schemes based on the hardness of solving systems of quadratic polynomials over finite fields. This approach is regarded as one candidate for post-quantum cryptography, in particular as a signature scheme [CJL+16]. We refer the reader to the thesis of Wolf [Wol05] for an overview, and the recent article [PCDY17] and references therein for recent advances in this area. IQF1S and related problems play an important role in such schemes. As pointed out in [Wol05, Sec. 2.6.1], though often not explicitly stated, it seems crucial to assume that IQF1S and related problems are difficult to ensure the security of these schemes. Theorem 2 then suggests that the "one-secret" versions of such schemes based on quadratic polynomials may not be secure.

[1] Patarin's formulation is known to reduce to the formulation here [BFP15, Proposition 5].

1.2 Group isomorphism problem

Group isomorphism problem (GpI) asks to decide whether two finite groups of order n are isomorphic. It has been studied for several decades in both Computational Group Theory (CGT) and Theoretical Computer Science. The difficulty of this problem depends crucially on how we represent the groups in the algorithms. If the goal is to obtain an algorithm running in time $\mathrm{poly}(n)$, then we may assume that we have at our disposal the Cayley (multiplication) table of the group, as the Cayley table can be recovered from most reasonable models for computing with finite groups in time $\mathrm{poly}(n)$. Therefore, we restrict our discussion mostly to this very redundant model, which is meaningful mainly because we do not know a $\mathrm{poly}(n)$-time or even an $n^{o(\log n)}$-time algorithm [Wil14] (log to the base 2), despite that a simple $n^{\log n+O(1)}$-time algorithm has been known for decades [FN70, Mil78]. The past few years have witnessed a resurgence of activity on algorithms for this problem with worst-case analyses in terms of the group order; we refer the reader to [GQ17] which contains a survey of these algorithms.

It is long believed that p-groups (groups of a prime power order) form the bottleneck case for GpI. In fact, the decades-old quest for a polynomial-time algorithm has focused on class-2 p-groups, with little success. Even if we restrict further to p-groups of class 2 and exponent p, the problem is still difficult. Recently, some impressive progress on such p-groups was made on the CGT side, as seen in the works of Wilson, Brooksbank, and their collaborators [Wil09a, LW12, BMW17].

Most notably, a main result in [BMW17] is a polynomial-time algorithm for p-groups of class 2 and exponent p, when the commutator subgroup is of order p^2, in the model of quotients of permutation groups [KL90]. This of course settles the same case in the Cayley table model. In fact, the same class of groups in the Cayley table model can be handled using one specific technique called the Pfaffian isomorphism test in [BMW17, Sec. 6.2]. Still, despite all the progress, an efficient algorithm for p-groups of class 2 and exponent p, with the commutator subgroup of order even p^3, was not known in the Cayley table model. Since we now have an efficient algorithm to test isometry of tuples of skew-symmetric matrices, the following result can be established.

Theorem 3. *Let p be an odd prime, and let two p-groups of class 2 and exponent p of order p^ℓ, G and H, be given by Cayley tables. If the commutator subgroup of G is of order $p^{O(\sqrt{\ell})}$, then there exists a deterministic[2] polynomial-time algorithm to test whether G and H are isomorphic.*

We explain how to obtain Theorem 3 from our result. While the following reduction is well-known in CGT, we include it here for readers from other areas. Given a class 2 and exponent p p-group G, let $[G, G]$ denote its commutator subgroup. Due to the exponent p and class 2 condition, we have $G/[G, G] \cong \mathbb{Z}_p^n$ and $[G, G] \cong \mathbb{Z}_p^m$ for some n and m such that $n + m = \ell$. Fixing bases of $G/[G, G]$ and $[G, G]$, and taking the commutator bracket, we obtain a skew-symmetric bilinear map $b_G : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p^m$, represented by $\mathbf{B} \in S^{-1}(n, p)^m$. For H to be isomorphic to G, it is necessary that $\dim_{\mathbb{Z}_p}(H/[H, H]) = \dim_{\mathbb{Z}_p}(G/[G, G])$ and $\dim_{\mathbb{Z}_p}([H, H]) = \dim_{\mathbb{Z}_p}([G, G])$, so by the same construction we obtain another $\mathbf{C} \in S^{-1}(n, p)^m$. We then need the following definition.

Definition 4. *Given $\mathbf{B} = (B_1, \ldots, B_m)$ and $\mathbf{C} = (C_1, \ldots, C_m)$ from $S^\epsilon(n, \mathbb{F})$, $\mathbf{B}$ and $\mathbf{C}$ are* pseudo-isometric*, if there exists $X \in \mathrm{GL}(n, \mathbb{F})$ such that $\langle X^t B_1 X, \ldots, X^t B_m X\rangle = \langle C_1, \ldots, C_m\rangle$.*

The key connection then is Baer's correspondence, which, put in this context, gives that G and H are isomorphic if and only if $\mathbf{B}$ and $\mathbf{C}$ are pseudo-isometric [Bae38]. By the condition that

[2]The deterministic here is due to the last statement on derandomization of Theorem 7 (1). That statement applies to the setting here, because the underlying field if $\mathbb{F}_p$ and our target is an algorithm with running time $p^{O(\sqrt{\ell})}$.

$m = O(\sqrt{\ell})$, we can enumerate all bases of $\mathbf{C}$ at a multiplicative cost of $p^{m^2} = p^{O(\ell)}$, and for each fixed basis, apply the algorithm for isometry testing. This gives Theorem 3.

As Brooksbank and Wilson have communicated to us, our algorithm may be useful in some models studied in CGT. Also, in multivariate cryptography, the problem Isomorphism of Quadratic Forms with Two Secrets (IQF2S) just asks to test the pseudo-isometry of tuples of symmetric matrices. Formally, the IQF2S problem asks to decide, given $\mathbf{B}, \mathbf{C} \in S^1(n, \mathbb{F})$, whether they are pseudo-isometric. Therefore a result analogous to Theorem 3 can be obtained for IQF2S.

1.3 Polynomial identity testing

Fix $\epsilon \in \{1, -1\}$. Let us see how to cast the ϵ-symmetrization problem as an instance of the polynomial identity testing problem. Given $\mathbf{B} = (B_1, \ldots, B_m) \in M(n, \mathbb{F})^m$, there exist invertible matrices A, D such that $\forall i \in [m]$, AB_iD is ϵ-symmetric if and only if $\forall i \in [m]$, $D^{-t}AB_i = D^{-t}(AB_iD)D^{-1}$ is ϵ-symmetric. Therefore we can reduce to finding an invertible matrix E such that $\forall i \in [m]$, EB_i is ϵ-symmetric. Suppose for now that E is a matrix of variables. The equations $\forall i \in [m], EB_i = \epsilon B_i^t E^t$ set up a system of linear forms in these variables. Let $C_1, \ldots, C_\ell$ be a linear basis of the solution space, and $\mathcal{C}$ be the matrix space $\langle C_1, \ldots, C_\ell \rangle \leq M(n, \mathbb{F})$. The problem then becomes to decide whether $\mathcal{C}$ contains an invertible matrix. To decide whether a matrix space, given by a linear basis, contains only non-invertible matrix is known as the symbolic determinant identity testing (SDIT) problem, which is equivalent to the polynomial identity testing (PIT) for weakly skew arithmetic circuits [Tod92][3].

When $|\mathbb{F}| = \Omega(n)$, SDIT admits a randomized efficient algorithm via the Schwartz-Zippel lemma. To devise a deterministic efficient algorithm for SDIT is a major problem in algebraic complexity theory due to its implication to arithmetic circuit lower bounds. Specifically, in [CIKK15] (building on [KI04]), Carmosino et al. show that such an algorithm implies the existence of a polynomial family such that its graph is in NE, but it cannot be computed by polynomial-size arithmetic circuits. Such a lower bound is generally considered to be beyond current techniques, and would be recognized as a breakthrough if established. The research into PIT has received quite a lot of attention since early 2000's (see the surveys [Sax09, SY10, Sax13]).

Our algorithm for the ϵ-symmetrization problem then provides a deterministic solution to this specific instance of SDIT. Our motivation to look at this problem at the first place was from the recent resolution of the non-commutative rank problem by Garg et al. [GGOW16] and Ivanyos et al. [IQS17b, IQS17a], and the intricate relation between the non-commutative rank problem and SDIT, which we explain below.

A matrix space $\mathcal{B} \leq M(n, \mathbb{F})$ is non-singular, if $\mathcal{B}$ contains an invertible matrix, and singular otherwise. SDIT then asks to decide whether a matrix space is singular. To obtain an arithmetic circuit lower bound via [CIKK15], it is actually enough to put SDIT in NP, that is, to find a small witness that helps to testify the singularity of singular matrix spaces. One such singularity witness, which is the reminiscent of the "shrunk subset" as in Hall's marriage theorem for bipartite graphs, and closely related to the linear matroid intersection problem [Lov89], is the following. For $\mathcal{B} \leq M(n, \mathbb{F})$, $U \leq \mathbb{F}^n$ is a shrunk subspace of $\mathcal{B}$, if $\dim(U) > \dim(\mathcal{B}(U))$ where $\mathcal{B}(U) = \langle B(U) : B \in \mathcal{B} \rangle$. The decision version of the non-commutative rank problem then asks to decide whether $\mathcal{B}$ has a

[3] An arithmetic circuit is weakly skew if each product gate is of fan-in 2 and has at least one child such that the subcircuit rooted at it is separate from the other parts of the circuit [Tod92, MP08]. The computation power of weakly skew circuit is known to be equivalent to the model of symbolic determinants, and between arithmetic formulas and arithmetic circuits.

shrunk subspace. Deterministic efficient algorithms for the non-commutative rank problem were recently devised in [GGOW16] (over $\mathbb{Q}$) and in [IQS17b, IQS17a] (over any field).

A direct consequence of settling the non-commutative rank problem on SDIT is that we can restrict our attention to those singular matrix spaces without a shrunk subspace, which we call exceptional spaces. As described by Lovász in [Lov89] (see also [Atk83, EH88]), the skew-symmetric structure naturally yields two families of exceptional spaces. To introduce them we need the following definition. Two matrix spaces $\mathcal{B}, \mathcal{C} \leq M(n, \mathbb{F})$ are *equivalent*, if there exist $A, D \in \mathrm{GL}(n, \mathbb{F})$ such that $A\mathcal{B}D = \mathcal{C}$ (equal as subspaces). Note that whether a matrix space is singular is preserved by the equivalence relation. We now list the two families from [Lov89].

(1) If n is odd and $\mathcal{B} \leq M(n, \mathbb{F})$ is equivalent to a subspace in $S^{-1}(n, \mathbb{F})$, then $\mathcal{B}$ is singular, as every skew-symmetric matrix is of even rank.

(2) Given $C_1, \ldots, C_n \in S^{-1}(n, \mathbb{F})$, let $\mathcal{C} \leq M(n, \mathbb{F})$ consist of all the matrices of the form $[C_1 v, C_2 v, \ldots, C_n v]$ over $v \in \mathbb{F}^n$. Since $v^t[C_1 v, C_2 v, \ldots, C_n v] = [v^t C_1 v, v^t C_2 v, \ldots, v^t C_n v] = 0$, $\mathcal{C}$ is singular, and we call such $\mathcal{C}$ a skew-symmetric induced matrix space. If $\mathcal{B}$ is equivalent to a skew-symmetric induced matrix space, then $\mathcal{B}$ is singular as well. Note that w.l.o.g. we can assume that $\mathcal{B}$ is a subspace of $M(n, \mathbb{F})$ of dimension n.

These two families of exceptional matrix spaces can be deterministically recognized as follows.

Theorem 5. *Let $\mathbb{F}$ be a field of characteristic not 2. Given $\mathcal{B} = \langle B_1, \ldots, B_m \rangle \leq M(n, \mathbb{F})^m$, there exists a deterministic polynomial-time algorithm that decides whether $\mathcal{B}$ is equivalent to a subspace in $S^{-1}(n, \mathbb{F})$, or a skew-symmetric induced matrix space.*

We explain how Theorem 5 follows from our ϵ-symmetrization algorithm. The case (1) is straightforward: apply the skew-symmetrization algorithm to the given linear basis of $\mathcal{B}$. In case (2), suppose $B_i = [b_{i,1}, \ldots, b_{i,n}]$ where $b_{i,j} \in \mathbb{F}^n$, $j \in [n]$ are the columns of B_i. Following an observation of Lovász in [Lov89], construct $B'_i = [b_{1,i}, \ldots, b_{n,i}]$ for $i \in [n]$. It can be verified that $\mathcal{B}$ is equivalent to some $\mathcal{C}$ of the form described in (2) if and only if $\mathcal{B}' = \langle B'_1, \ldots, B'_n \rangle$ is equivalent to a subspace in $S^{-1}(n, \mathbb{F})$. We can then apply the skew-symmetrization algorithm to $(B'_1, \ldots, B'_n)$ to conclude.

1.4 Results and techniques

Statement of the results. We first define three equivalence relations for matrix tuples.

Definition 6. *Let $\mathbf{B} = (B_1, \ldots, B_m), \mathbf{C} = (C_1, \ldots, C_m) \in M(n, \mathbb{F})^m$. $\mathbf{B}$ and $\mathbf{C}$ are* conjugate*, if $\exists A \in \mathrm{GL}(n, \mathbb{F})$, such that $A\mathbf{B} = \mathbf{C}A$. They are* equivalent*, if $\exists A, D \in \mathrm{GL}(n, \mathbb{F})$, such that $A\mathbf{B} = \mathbf{C}D$. They are* isometric*, denoted as $\mathbf{B} \sim \mathbf{C}$, if $\exists A \in \mathrm{GL}(n, \mathbb{F})$, such that $A^t \mathbf{B} A = \mathbf{C}$; such an A is called an isometry from $\mathbf{B}$ to $\mathbf{C}$.*

We show that testing whether two ϵ-symmetric matrix tuples are isometric can be solved efficiently over $\mathbb{F}_q$ with q odd, $\mathbb{R}$, and $\mathbb{C}$. Note that the algorithm for $\mathbb{F}_q$ is probabilistic.

Theorem 7. *1. (Finite fields of odd size) Given $\mathbf{B}, \mathbf{C} \in S^{\epsilon}(n, q)^m$ with q odd, there exists a randomized polynomial-time algorithm that decides whether $\mathbf{B}$ and $\mathbf{C}$ are isometric. If $\mathbf{B}$ and $\mathbf{C}$ are isometric, the algorithm also computes an explicit isometry in $\mathrm{GL}(n, q)$. This algorithm can be derandomized at the price of running in time $\mathrm{poly}(n, m, \log q, p)$ where $p = \mathrm{char}(\mathbb{F}_q)$.*

2. *(The real field $\mathbb{R}$) Let $\mathbb{E} \subseteq \mathbb{R}$ be a number field. Given $\mathbf{B}, \mathbf{C} \in S^{\epsilon}(n, \mathbb{E})^m$, there exists a deterministic polynomial-time algorithm that decides whether $\mathbf{B}$ and $\mathbf{C}$ are isometric over some number field $\mathbb{K}$ such that $\mathbb{E} \subseteq \mathbb{K} \subseteq \mathbb{R}$. If $\mathbf{B}$ and $\mathbf{C}$ are indeed isometric, the algorithm also computes an explicit isometry, represented as a product of matrices, where each matrix is over some extension field of $\mathbb{E}$ of extension degree* $\mathrm{poly}(n, m)$.

3. *(The complex field $\mathbb{C}$) Let $\mathbb{E}$ be a number field. Given $\mathbf{B}, \mathbf{C} \in S^{\epsilon}(n, \mathbb{E})^m$, there exists a deterministic polynomial-time algorithm that decides whether $\mathbf{B}$ and $\mathbf{C}$ are isometric over some number field $\mathbb{K}$ such that $\mathbb{E} \subseteq \mathbb{K}$. If $\mathbf{B}$ and $\mathbf{C}$ are indeed isometric, the algorithm also computes an explicit isometry, represented as a product of matrices, where each matrix is over some extension field of $\mathbb{E}$ of extension degree* $\mathrm{poly}(n, m)$.

We call $\mathbf{B} \in M(n, \mathbb{F})^m$ *ϵ-symmetrizable*, if $\mathbf{B}$ is equivalent to a tuple of ϵ-symmetric matrices. Our second main result concerns the problem of testing whether a matrix tuple is ϵ-symmetrizable.

Theorem 8. *Let $\mathbb{F}$ be a field of characteristic not 2. Given $\mathbf{B} \in M(n, \mathbb{F})^m$, there exists a deterministic algorithm that decides whether $\mathbf{B}$ is ϵ-symmetrizable, and if it is, computes $A, D \in \mathrm{GL}(n, \mathbb{F})$ such that $A\mathbf{B}D \in S^{\epsilon}(n, \mathbb{F})^m$. The algorithm uses polynomially many arithmetic operations. Over a number field the final data as well as all the intermediate data have size polynomial in the input data size, hence the algorithm runs in polynomial time.*

Two key ingredients. Let us first review the concept of $*$-algebras, and see how to get a $*$-algebra from a tuple of ϵ-symmetric matrices. Recall that, a $*$-algebra A is an algebra with $* : A \to A$ being an anti-automorphism of order at most 2. $*$-algebras have been studied since 1930's [Alb39] (see [Lew06] for a recent survey). Let $M(n, \mathbb{F})^{op}$ be the opposite full matrix algebra, which is the ring consisting of all matrices in $M(n, \mathbb{F})$ with the multiplication $\circ$ as $A \circ B = BA$. $*$-algebras arise from ϵ-symmetric matrix tuples by considering the *adjoint algebra* of $\mathbf{B} \in S^{\epsilon}(n, \mathbb{F})^m$, which consists of $\{(A, D) \in M(n, \mathbb{F})^{op} \oplus M(n, \mathbb{F}) | A^t \mathbf{B} = \mathbf{B} D\}$, with a natural involution $*$ as $(A, D)^* = (D, A)$.

We then turn to the module isomorphism problem (MI). Given $\mathbf{B}, \mathbf{C} \in M(n, \mathbb{F})^m$, MI asks if $\mathbf{B}$ and $\mathbf{C}$ are conjugate. This problem is termed as module isomorphism, as the matrix tuple $\mathbf{B} = (B_1, \ldots, B_m)$ can be viewed as a linear representation of a finitely generated algebra generated by m elements. Two deterministic polynomial-time algorithms for MI have been devised in [CIK97, IKS10] and [BL08]. Note that MI may also be cast as an instance of the polynomial identity testing problem like the ϵ-symmetrization problem.

More comparison with previous works. Some comparisons with previous works were already stated in Section 1.1 and 1.2. We now add some more details on the technical side. In Section 1.1, we mentioned the work of Berthomieu et al. [BFP15] which solves the IQF1S possibly over an extension field, for regular instances and large enough fields. Here we seek "rational" solutions (i. e. those over the given base field) in the finite case and seek solutions over a real extension field. An interesting observation is that the algorithm of Berthomieu et al. may be cast as working with a $*$-algebra, but in a much restricted setting. We explain this in detail in Appendix A. In Section 1.2, we described how our result, when applied to p-group isomorphism, compares to the result of Brooksbank et al. [BMW17]. The relevant technique there, called the Pfaffian isomorphism test [BMW17, Sec. 6.2], is completely different from ours, and seems quite restricted to pairs of skew-symmetric matrices.

The work [BW12] by Brooksbank and Wilson is the most important precursor to our Theorem 7. In [BW12], the main result, rephrased in our setting, is an efficient algorithm that, given $\mathbf{B} \in S^\epsilon(n,q)^m$ with q odd, computes a generating set for the group $\{X \in \mathrm{GL}(n,q) \mid X^t\mathbf{B}X = \mathbf{B}\}$. This is exactly the "automorphism version" of the isometry problem. However, unlike many other isomorphism problems, the isometry problem is not known to reduce to this automorphism version. This is similar to the module isomorphism problem: the automorphism version of MI asks to compute a generating set of the unit group in a matrix algebra, which was solved in [BO08]. The ideas and the techniques for the unit group computation in [BO08] and for MI in [CIK97, IKS10, BL08] are totally different. So Theorem 7 cannot be easily deduced as a corollary from [BW12].

Generalizations of the main results. Theorem 7 can be generalized to the following setting. Following [BW12], for an linear automorphism $\theta \in \mathrm{GL}(W)$ we call a bilinear map over a field $\mathbb{F}$, $b : V \times V \to W$ *θ-Hermitian*, if for all $u, v \in V$, $b(u,v) = \theta(b(v,u))$. Obviously, nontrivial Hermitian maps exist only if θ^2 is the identity. Hermitian bilinear maps subsume symmetric bilinear maps (θ being the identity matrix) and skew-symmetric bilinear maps (θ being -1 times the identity matrix). It allows for (after fixing bases of V and W) a tuple of mixed symmetric and skew-symmetric matrices. In fact, by a change of basis of W, we may always assume that θ is a diagonal matrix with 1 and -1's on the diagonal and in our arguments and algorithms we only need the replace ϵ by a tuple $(\epsilon_1, \ldots, \epsilon_m)$ and equations of type $B_i^t = \epsilon B_i$ by $B_i^t = \epsilon_i B_i$. Furthermore, the concept captures Hermitian forms by [BW12, Sec. 3.1]: for a Hermitian form $b : V \times V \to \mathbb{F}_{q^2}$ where $V \cong \mathbb{F}_{q^2}^n$, we can represent it as a pair of bilinear forms over $\mathbb{F}_q$, $b_1, b_2 : V' \times V' \to \mathbb{F}_q$ where $V' \cong \mathbb{F}_q^{2n}$, and $\theta \in \mathrm{GL}(2,q)$ corresponds to the field involution $\alpha \to \alpha^q$ for $\alpha \in \mathbb{F}_{q^2}$. Hermitian complex or quaternionic matrices are also included: assume that D is a finite dimensional division algebra over $\mathbb{F}$ with involution $\overline{\cdot} : D \to D$, such that $\mathbb{F}$ coincides with the subfield of the center of D consisting of the elements fixed by $\overline{\cdot}$. Then the map $*$ sending a matrix to the transpose of its elementwise $\overline{\cdot}$-conjugate is an involution on $M(n,D)$, and the matrices invariant under $*$ are called $*$-Hermitian. Indeed, let d be the dimension of D over $\mathbb{F}$. Then we can interpret D and D^n as vector spaces of dimension d resp. dn over $\mathbb{F}$, and a matrix in $M(n,D)$ as an $\mathbb{F}$-bilinear map from $D^n \times D^n$ to D. Then $*$-Hermitian matrices are interpreted as Hermitian bilinear maps for $\overline{\cdot}$. (Naturally, an m-tuple of $*$-Hermitian matrices become a Hermitian map from $D^n \times D^n$ to D^m.)

Interestingly, Theorem 7 allows us to solve the isometry problem for a tuple of arbitrary matrices over $\mathbb{F}_q$ with q odd, $\mathbb{R}$, or $\mathbb{C}$. Given $\mathbf{B}, \mathbf{C} \in M(n,\mathbb{F})^m$, we can construct $\mathbf{B'} = (\frac{1}{2}(B_1 + B_1^t), \ldots, \frac{1}{2}(B_m + B_m^t), \frac{1}{2}(B_1 - B_1^t), \ldots, \frac{1}{2}(B_1 - B_1^t))$, and similarly $\mathbf{C'}$. Here we use the fact that we work over fields of characteristic not 2. Then it is easy to verify that $\mathbf{B} \sim \mathbf{C}$ if and only if $\mathbf{B'} \sim \mathbf{C'}$. Indeed, if $A \in \mathrm{GL}(n,\mathbb{F})$ satisfies that $A^tB_iA = C_i$, then A also satisfies that $A^t(\frac{1}{2}(B_i \pm B_i^t))A = \frac{1}{2}(A^tB_iA \pm A^tB_i^tA) = \frac{1}{2}(C_i \pm C_i^t)$. On the other hand, if $A^t(\frac{1}{2}(B_i + B_i^t))A = \frac{1}{2}(C_i + C_i^t)$ and $A^t(\frac{1}{2}(B_i - B_i^t))A = \frac{1}{2}(C_i - C_i^t)$, summing these two we get that $A^tB_iA = C_i$. Combining with the observation from the last paragraph, we have the following.

Corollary 9. *The statement of Theorem 7 holds for $\mathbf{B}, \mathbf{C} \in M(n,\mathbb{F}_q)^m$, $M(n,\mathbb{E})^m$ with a number field $\mathbb{E} \subseteq \mathbb{R}$, or $M(n,\mathbb{E})^m$ with a number field $\mathbb{E}$.*

Theorem 8 can also be generalized to transforming bilinear maps to θ-Hermitian ones, including the case of tuples of complex and quaternionic matrices.

Some open problems. There are two immediate open problems left.

The first one is to extend both of our results to fields of characteristic 2. While presenting the algorithm for the isometry problem in Section 3, we indicate explicitly in each step whether the characteristic not 2 is required, and one may want to examine those steps where the characteristic not 2 condition is crucial. For the ϵ-symmetrization problem, one may want to start with examining the key lemma, Lemma 31, in the setting of characteristic-2 fields.

The second one is to solve the isometry test problem over a number field without going to extension fields. To extend our current approach to deal with the second problem involves certain number-theoretic obstacles even over $\mathbb{Q}$. Namely, our present method relies on representing a simple algebra explicitly as a full matrix algebra over a division ring, but there is a randomized reduction from factoring squarefree integers to this task for a central simple algebra of dimension 4 over $\mathbb{Q}$ assuming the Generalized Riemann Hypothesis [Rón87]. Even deciding whether a four dimensional non-commutative simple algebra over $\mathbb{Q}$ is isomorphic to $M(2,\mathbb{Q})$ is equivalent to deciding quadratic residuosity modulo composite numbers. This kind of obstacles appears to be inherent: a ternary quadratic form over $\mathbb{Q}$ is isotropic if and only if an associated non-commutative simple algebra of dimension four over $\mathbb{Q}$ is isomorphic to $M(2,\mathbb{Q})$. Now consider an indefinite symmetric 3 by 3 matrix B with rational entries having determinant d. Then the ternary quadratic form with Gram matrix B is either anisotropic or isometric to the form having matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

Thus over $\mathbb{Q}$, the isometry problem a single ternary quadratic form is at least as hard as deciding whether an algebra is isomorphic to $M(2,\mathbb{Q})$. Actually, there is a randomized polynomial time reduction from testing whether a simple algebra over a number field $\mathbb{F}$ is isomorphic with a full matrix algebra over $\mathbb{F}$ to factoring integers, see [Rón92] and [IR93]. However, for the constructive version of isomorphisms with full matrix algebras such a reduction is only known for the case $M(n,K)$ where n is bounded by a constant, and K is from a finite collection of number fields [IRS12]. Therefore, to determine the relation between the complexity of the isometry problem and that of factoring, it might be useful to devise an alternative approach which gets around constructing explicit isomorphisms with full matrix algebras.

Future directions. Given Theorem 7, the next target is of course to study IQF2S and isomorphism testing of p-groups of class 2 and exponent p. For these two problems, the first goal would be to design, for $\mathbf{B} \in S^{\epsilon}(n,q)^m$, an algorithm in time $q^{O(n+m)}$. In the context of p-groups of class 2 and exponent p, this amounts to solve isomorphism testing for this group class in time polynomial in the group order, which seems a difficult problem already. By Theorem 7, this target seems most difficult when m and n are comparable, say $m = n$. One idea may be to reduce to the parameters m' and n' such that $m' = O(n^{1/2})$ and $n' = \mathrm{poly}(n)$, so that we can use Theorem 7 to get an algorithm in time $q^{O(n)}$. It is also noteworthy that recently, Yinan Li and the second author devised an algorithm for $m = \Theta(n)$ in *average-case* time $q^{O(n)}$ [LQ17]; the average-case analysis is done in a random model for linear spaces of skew-symmetric matrices over finite fields, that can be viewed as a linear algebraic analogue of the Erdős-Rényi model for random graphs.

Theorem 5 represents a natural step in the direction for derandomizing SDIT set up by the resolution of the non-commutative rank problem [GGOW16, IQS17b, IQS17a]. While most research activities on PIT and SDIT put constraints on the structural properties of the arithmetic circuits

[Sax09, SY10, Sax13], this direction puts constraints on the singularity witnesses which are inspired by geometric considerations [EH88] and/or combinatorial considerations [Lov89]. At present, we are not aware of an explicit connection between these two different styles of constraints. It is an interesting question as to whether these geometric and/or combinatorial considerations can be made more systematic to yield a formal strategy to attack SDIT.

Organization of the article. In Section 2, we present certain preliminaries, including those structural results of $*$-algebras that are relevant to us. In Sections 3, we give a detailed description of the algorithm for Theorems 7. In Section 4, we show that for the ϵ-symmetrization problem, how to handle the cases when the Jacobson radical is not known to be efficiently computable, or the field is too small, finishing the proof of Theorem 8.

2 Preliminaries

Notation. For $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}$. For a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F})$ denotes the characteristic of $\mathbb{F}$. $\mathbf{0}$ is the zero vector. For $B \in M(n, \mathbb{F})$, $i, j \in [n]$, $S, T \subseteq [n]$, $B(i, j)$ is the (i, j)th entry of B, $B(S, T)$ is the submatrix indexed by row indices in S and column indices in T. We use I_n to denote the $n \times n$ identity matrix, and $\langle \cdot \rangle$ to denote the linear span. The vector space $\mathbb{F}^n$ consists of length-n *column* vectors over $\mathbb{F}$.

Given a quadratic field extension $\mathbb{F}/\mathbb{F}'$, for $\alpha \in \mathbb{F}$, its conjugation $\overline{\alpha}$ is the image of α under the quadratic field involution. When $\mathbb{F} = \mathbb{C}$ and $\mathbb{F}' = \mathbb{R}$ this is simply the complex conjugation. We use $\mathbb{H}$ to denote the quaternion division algebra over $\mathbb{R}$, and i, j, k be the fundamental quaternion units. For $\alpha = a + bi + cj + dk \in \mathbb{H}$, its conjugation, denoted also by $\overline{\alpha}$, is $a - bi - cj + dk$. Given $A \in M(n, \mathbb{F})$ or $M(n, \mathbb{H})$, $\overline{A}$ denotes the matrix obtained by applying conjugation to every entry of A. For $\epsilon \in \{1, -1\}$ and $A \in M(n, \mathbb{F})$ or $M(n, \mathbb{H})$, A is ϵ-Hermitian, if $\overline{A}^t = \epsilon A$.

We will also meet matrices over division rings, and therefore, for a division ring D, the notation $M(n, D)$ (for the full $n \times n$ matrix ring over D) and $\mathrm{GL}(n, D)$ (for the group of units in $M(n, D)$).

Representation of fields and field extensions. For the isometry problem, we assume the input matrices are over a field $\mathbb{E}$ such that $\mathbb{E}$ is a finite extension of its prime field $\mathbb{F}$ (so $\mathbb{F}$ is either a field of prime order or $\mathbb{Q}$). Therefore $\mathbb{E}$ is a finite-dimensional algebra over $\mathbb{F}$. If $\dim_{\mathbb{F}}(\mathbb{E}) = d$, then $\mathbb{E}$ is the extension of $\mathbb{F}$ by a single generating element α, so $\mathbb{E}$ can be represented by the minimal polynomial of α over $\mathbb{F}$, together with an isolating interval for α in the case of $\mathbb{R}$, or an isolating rectangle for α in the case of $\mathbb{C}$. When we say that we work over $\mathbb{R}$ (resp. $\mathbb{C}$), the input is given as over a number field $\mathbb{E} \subseteq \mathbb{R}$ (resp. $\mathbb{E} \subseteq \mathbb{C}$). The algorithm is then allowed to work with extension fields of $\mathbb{E}$ in $\mathbb{R}$ (resp. $\mathbb{C}$), as long as the extension degrees are polynomially bounded. On the other hand, if we say that we work with a number field, we usually assume that we do not need to work with further extensions.

For the ϵ-symmetrization problem, we work with the arithmetic model, namely the fundamental steps are basic field operations, and the complexity is determined by counting the number of such basic operations. Furthermore, over number fields we are also concerned with the bit complexity. So when we say that some procedure works over any field, we mean that the procedure uses polynomially arithmetic operations, and when over number fields, $\mathbb{R}$ or $\mathbb{C}$, the bit complexity is also polynomial.

Tuples of matrices. A matrix tuple is an element in $M(n,\mathbb{F})^m$, and an ϵ-symmetric matrix tuple is an element in $S^\epsilon(n,\mathbb{F})^m$. We will mostly use $\mathbf{B}$, $\mathbf{C}$ to denote matrix tuples. Given $\mathbf{B} = (B_1,\ldots,B_m) \in M(n,\mathbb{F})^m$, define its kernel, $\ker(\mathbf{B})$, as $\cap_{i\in[m]}\ker(B_i)$, and its image, $\mathrm{im}(\mathbf{B})$, as $\langle \cup_{i\in[m]}\mathrm{im}(B_i)\rangle$. $\mathbf{B} \in M(n,\mathbb{F})^m$ is *non-degenerate*, if $\ker(\mathbf{B}) = \mathbf{0}$, and $\mathrm{im}(\mathbf{B}) = \mathbb{F}^n$. For $\mathbf{B} \in S^\epsilon(n,\mathbb{F})^m$, due to the ϵ-symmetric condition, it can be verified easily that $\mathrm{im}(\mathbf{B}) = \{v \in \mathbb{F}^n : \forall u \in \ker(\mathbf{B}), u^t v = 0\}$. So $\mathbf{B} \in S^\epsilon(n,\mathbb{F})^m$ is non-degenerate if and only if $\ker(\mathbf{B}) = \mathbf{0}$.

Given $\mathbf{B} = (B_1,\ldots,B_m) \in M(n,\mathbb{F})^m$, $\mathbf{B}^t = (B_1^t,\ldots,B_m^t)$. Given $\alpha \in \mathbb{F}$, $\alpha\mathbf{B} = (\alpha B_1,\ldots,\alpha B_m)$. So for $\mathbf{B} \in S^\epsilon(n,\mathbb{F})$, $\mathbf{B}^t = \epsilon\mathbf{B}$. Given $A, D \in M(n,\mathbb{F})$, $A\mathbf{B}D = (AB_1D,\ldots,AB_mD)$. Given $\mathbf{B},\mathbf{C} \in M(n,\mathbb{F})^m$, $\mathbf{B}$ and $\mathbf{C}$ are *conjugate*, if there exists $A \in \mathrm{GL}(n,\mathbb{F})$ such that $A\mathbf{B} = \mathbf{C}A$. $\mathbf{B}$ and $\mathbf{C}$ are *equivalent*, if there exists $A, D \in \mathrm{GL}(n,\mathbb{F})$ such that $A\mathbf{B} = \mathbf{C}D$. The classical module isomorphism problem asks to decide whether $\mathbf{B}$ and $\mathbf{C}$ are conjugate.

Theorem 10 ([CIK97,BL08,IKS10])**.** *Let $\mathbf{B}$ and $\mathbf{C}$ be from $M(n,\mathbb{F})^m$. There exists a deterministic algorithm that decide whether $\mathbf{B}$ and $\mathbf{C}$ are conjugate. The algorithm uses polynomially many arithmetic operations. Over number fields the bit complexity of the algorithm is also polynomial.*

Structure of algebras. The proofs in this paper rely heavily on structure of finite dimensional algebras, so we recall in nutshell some of the most important notions and facts from their theory. Classical references include [Pie82], and a concise introduction can be found in [AB95, Sec. 5]. All the algebras we consider are finite dimensional associative algebras over some field $\mathbb{F}$. An ideal is a linear subspace of $\mathfrak{A}$ closed under multiplication by elements of $\mathfrak{A}$, both from the left and from the right. Left ideals are subspaces closed under multiplication by elements of $\mathfrak{A}$ from the left, right ideals are defined analogously. In this context, an ideal or, more generally, a subalgebra S is nilpotent when S^n, the subspace spanned by products of length n of element from S are zero for some n. An algebra $\mathfrak{A}$ has a largest nilpotent ideal $\mathrm{Rad}(\mathfrak{A})$, called the Jacobson radical, also simply referred to as the radical in this paper. We will make use of an alternative characterization of the radical, namely, it is the intersection of the maximal right ideals (or the intersection of maximal left ideals).

An algebra is simple when it contains no proper and non-trivial (two-sided) ideals. A semisimple algebra is isomorphic to a direct sum of simple algebras. The factor algebra $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$ is semisimple. In a finite dimensional algebra over a field, every nonzero element is either a unit (i.,e., has a multiplicative inverse) or a zero divisor (can be multiplied by nonzero elements from both sides to obtain zero). In a division algebra, also known as a skewfield, every nonzero element is a unit. A simple algebra is isomorphic to a full matrix algebra over a division algebra [AB95, Theorem 17 on pp.129]. Over finite fields all the division algebras are actually commutative, or in other words, they are fields; this is known as Wedderburn's little theorem. Over an algebraically closed field there is even only one division algebra, that is the base field itself [AB95, Lemma 14 on pp. 127]. The structural results summarized above are also known as Wedderburn's theory, and a concise introduction can be found in [AB95, Sec. 5].

An idempotent is a nonzero element e with $e^2 = e$. A semisimple algebra necessarily contains at least one idempotent: the identity element. Non-nilpotent algebras also contain idempotents (but not necessarily identity elements). This follows from the following fact.

Fact 11. *Let $\mathfrak{A}$ be a non-nilpotent algebra over a field $\mathbb{F}$. Every basis of $\mathfrak{A}$ contains a non-nilpotent element.*

Proof. Let $\mathbb{K}$ be the algebraic closure of $\mathbb{F}$. Observe that $\mathfrak{A}$, as $= \mathbb{F} \otimes_{\mathbb{F}} \mathfrak{A}$, is embedded into $\mathbb{K} \otimes \mathfrak{A} =: \overline{\mathfrak{A}}$. Then $\overline{\mathfrak{A}}$ is a non-nilpotent $\mathbb{K}$-algebra and hence it has a full matrix algebra as a factor. The image of an $\mathbb{F}$-basis of $\mathfrak{A}$ under the composition of the embedding of $\mathfrak{A}$ into $\overline{\mathfrak{A}}$ with the natural projection to this factor gives a system that spans a full matrix algebra over $\mathbb{K}$. Now observe that a full matrix algebra cannot be spanned by nilpotent matrices: nilpotent matrices have zero traces but there exist matrices with nonzero trace even in positive characteristic. It follows that this $\mathbb{F}$-basis must contain at least one non-nilpotent element. □

The proof of Fact 11 shows that it is easy to find a non-nilpotent element in a non-nilpotent algebra. Back to our task of locating an idempotent, let y be a non-nilpotent element. Then the (commutative) subalgebra generated by $x = y^n$ for sufficiently large n (say $n = \dim \mathfrak{A}$) has an identity element e, which is necessarily idempotent. To see this, note that the action of y by left multiplication on the vector space $\mathfrak{A}$ yields the Fitting decomposition $\mathfrak{A}_0 \oplus \mathfrak{A}_1$, such that $\mathfrak{A}_0 = \ker(y^n)$ and $\mathfrak{A}_1 = \mathrm{im}(y^n)$ for a large enough n. Consider the restriction of y^n on $\mathfrak{A}_1$; its characteristic polynomial f has a nonzero constant term α. Then $(f(y^n) - \alpha)/\alpha$ is an element of the subalgebra generated by y^n that gives an identity e on $\mathfrak{A}_1$. Now observe that $y^n \in \mathfrak{A}_1$, so indeed $ey^{nk} = y^{nk}$ for any $k \in \mathbb{N}$. While the above argument shows the existence of e, a more straightforward way to compute this e would be to express e as a linear combination of powers of x whose coefficients are variables. Then e can be computed in polynomial time, by solving a system of linear equations expressing the condition $ex = x$.

A matrix representation of an algebra $\mathfrak{A}$ is a homomorphism of $\mathfrak{A}$ into a matrix algebra. There is a straightforward linear representation over $\mathbb{F}$ at hand, the so-called left regular representation, as follows. Let $V(\mathfrak{A})$ be the vector space supporting the algebra $\mathfrak{A}$. Then $a \in \mathfrak{A}$ naturally acts as a linear map on $V(\mathfrak{A})$ as ℓ_a by sending $x \in V(\mathfrak{A})$ to ax. The properties of the algebra operations (most notably, though not exclusively, associativity of multiplication) ensure that $\ell : \mathfrak{A} \to \mathrm{Hom}(V(\mathfrak{A}), V(\mathfrak{A}))$ by sending a to ℓ_a is a homomorphism from $\mathfrak{A}$ into the algebra of $\mathbb{F}$-linear transformations of $\mathfrak{A}$. It is an embedding when $\mathfrak{A}$ has an identity element. We remark that image of ℓ_a is the right ideal $a\mathfrak{A}$ generated by a, while its kernel is the right annihilator $\mathrm{Ann}_r(a) = \{x \in \mathfrak{A} : ax = 0\}$ of $\mathfrak{A}$. It is straightforward to check that $\mathrm{Ann}_r(a)$ is also a right ideal.

Structure of $*$-algebras. We collect basic facts about $*$-algebras here. A classical reference for $*$-algebras is Albert's book [Alb39]. Fix a field $\mathbb{F}$, and let $\mathfrak{A}$ be an $\mathbb{F}$-algebra, e.g. an algebra over $\mathbb{F}$. Given an anti-automorphism $* : \mathfrak{A} \to \mathfrak{A}$ of order at most 2, $(\mathfrak{A}, *)$ is termed as a $*$-algebra. We will always assume that for an $\mathbb{F}$-algebra $\mathfrak{A}$, $*$ fixes $\mathbb{F}$, that is $\alpha^* = \alpha$ for $\alpha \in \mathbb{F}$. An element $a \in \mathfrak{A}$ is *$*$-symmetric* if $a^* = a$, and *$*$-unitary* if $a^*a = 1$. A $*$-homomorphism between $(\mathfrak{A}, *)$ and $(\mathfrak{A}', \circ)$ is an algebra homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}'$ such that $\phi(a^*) = \phi(a)^\circ$. An ideal $I \subseteq \mathfrak{A}$ is an $*$-ideal, if $I^* = I$. The Jacobson radical of $\mathfrak{A}$, denoted as $\mathrm{Rad}(\mathfrak{A})$, is the largest nilpotent ideal of $\mathfrak{A}$ as an $\mathbb{F}$-algebra. It is straightforward to verify that $\mathrm{Rad}(\mathfrak{A})$ is a $*$-ideal. A $*$-algebra is $*$-simple, if it does not contain non-trivial $*$-ideals. Note that for a $*$-algebra $(S, *)$, if S is simple, then it must be $*$-simple. The semisimple $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$, with the induced involution (again denoted as $*$), is $*$-isomorphic to $(S_1, *) \oplus (S_2, *) \oplus \cdots \oplus (S_k, *)$, where each $(S_i, *)$ is a $*$-simple algebra.

A $*$-simple algebra $(S, *)$ over $\mathbb{F}$ falls into two categories. Either S is a simple algebra, or S is a direct sum of two anti-isomorphic simple algebras with $*$ interchanging the two summands [Alb39, Chap. X.3]. We shall refer to the latter as *exchange type*, and its structure is easy to describe: an exchange-type $*$-simple algebra $(S, *)$ is $*$-isomorphic to $(M(n, D) \oplus M(n, D)^{op}, \circ)$, where $\circ$ is an involution sending (A, B) to $(\phi^{-1}(B), \phi(A))$ for some algebra automorphism ϕ of $M(n, D)$.

When S is simple, a general result regarding the possible forms of involutions is [Alb39, Chap. X.4, Theorem 11]. We can explicitly list these forms for $\mathbb{F}_q$ with q odd, $\mathbb{R}$, and $\mathbb{C}$ as follows.

Over $\mathbb{F}_q$ with q odd, finite simple $*$-algebras are classified as follows (see also [BW12, Sec. 3.3]). To start with, recall that a finite simple algebra S over $\mathbb{F}_q$ is isomorphic to $M(n, \mathbb{F}_{q'})$ where $\mathbb{F}_{q'}$ is an extension field of $\mathbb{F}_q$. So without loss of generality we may assume $S = M(n, \mathbb{F}_{q'})$. Then any involution $*$ on $M(n, \mathbb{F}_{q'})$ is in one of the following forms.

- *Orthogonal type* For $X \in M(n, \mathbb{F}_{q'})$, $X^* = A^{-1}X^tA$ for some $A \in \mathrm{GL}(n, \mathbb{F}_{q'})$, $A = A^t$.
- *Symplectic type* For $X \in M(n, \mathbb{F}_{q'})$, $X^* = A^{-1}X^tA$ for some $A \in \mathrm{GL}(n, \mathbb{F}_{q'})$, $A = -A^t$.
- *Hermitian type* $\mathbb{F}_{q'}$ is a quadratic extension of a subfield $\mathbb{F}_{q''}$. For $X \in M(n, \mathbb{F}_{q'})$, $X^* = A^{-1}\overline{X}^tA$ for some $A \in \mathrm{GL}(n, \mathbb{F}_{q'})$, $\overline{A}^t = A$.

Over $\mathbb{R}$, finite simple $*$-algebras are classified as follows (see also [Lew77]). To start with, recall that, by a theorem of Frobenius (see e.g. [Pal68]), a finite simple algebra S over $\mathbb{R}$ is isomorphic to either $M(n, \mathbb{R})$, $M(n, \mathbb{C})$, or $M(n, \mathbb{H})$. So without loss of generality we may assume S is one of the above. Then any involution $*$ on S is in one of the following forms. Note that each type corresponds to a classical group as in [Wey97].

- *Orthogonal type* $S = M(n, \mathbb{R})$. For $X \in M(n, \mathbb{R})$, $X^* = A^{-1}X^tA$, $A \in \mathrm{GL}(n, \mathbb{R})$, $A = A^t$.
- *Symplectic type* $S = M(n, \mathbb{R})$. For $X \in M(n, \mathbb{R})$, $X^* = A^{-1}X^tA$, $A \in \mathrm{GL}(n, \mathbb{R})$, $A = -A^t$.
- *Complex orthogonal type* $S = M(n, \mathbb{C})$. For $X \in M(n, \mathbb{C})$, $X^* = A^{-1}X^tA$, $A \in \mathrm{GL}(n, \mathbb{C})$, $A = A^t$.
- *Complex symplectic type* $S = M(n, \mathbb{C})$. For $X \in M(n, \mathbb{C})$, $X^* = A^{-1}X^tA$, $A \in \mathrm{GL}(n, \mathbb{C})$, $A = -A^t$.
- *Unitary type* $S = M(n, \mathbb{C})$. For $X \in M(n, \mathbb{C})$, $X^* = A^{-1}\overline{X}^tA$, $A \in \mathrm{GL}(n, \mathbb{C})$, $A = \overline{A}^t$.
- *Quaternion unitary type* $S = M(n, \mathbb{H})$. For $X \in M(n, \mathbb{H})$, $X^* = A^{-1}\overline{X}^tA$, $A \in \mathrm{GL}(n, \mathbb{H})$, $A = \overline{A}^t$.
- *Quaternion orthogonal type* $S = M(n, \mathbb{H})$. For $X \in M(n, \mathbb{H})$, $X^* = A^{-1}\overline{X}^tA$, $A \in \mathrm{GL}(n, \mathbb{H})$, $A = -\overline{A}^t$.

On $\mathbb{C}$, $\overline{\cdot}$ denotes the standard conjugation $a + bi \mapsto a - bi$, while on $\mathbb{H}$ it is $a + bi + cj + dk \mapsto a - bi - cj - dk$.

Over $\mathbb{C}$, finite simple $*$-algebras are classified as follows. To start with, recall that a finite simple algebra S over $\mathbb{C}$ is isomorphic to $M(n, \mathbb{C})$, because the only finite dimensional division algebra over an algebraically closed field is the field itself. So without loss of generality we may assume S is $M(n, \mathbb{C})$. Then any involution $*$ on S is in one of the following forms.

- *Orthogonal type* For $X \in M(n, \mathbb{C})$, $X^* = A^{-1}X^tA$, $A \in \mathrm{GL}(n, \mathbb{C})$, $A = A^t$.
- *Symplectic type* For $X \in M(n, \mathbb{C})$, $X^* = A^{-1}X^tA$, $A \in \mathrm{GL}(n, \mathbb{C})$, $A = -A^t$.

Adjoint algebras of ϵ-symmetric matrix tuples. We first present the formal definition.

Definition 12. *Let $\mathbb{F}$ be a field and fix $\epsilon \in \{1, -1\}$. For $\mathbf{B} = (B_1, \ldots, B_m) \in S^\epsilon(n, \mathbb{F})^m$, the* adjoint algebra *of $\mathbf{B}$, denoted as* $\mathrm{Adj}(\mathbf{B})$*, is $\{(A, D) \in M(n, \mathbb{F})^{op} \oplus M(n, \mathbb{F}) | \forall i \in [m], A^t B_i = B_i D\}$.* $\mathrm{Adj}(\mathbf{B})$ *is a $*$-algebra over $\mathbb{F}$ with $(A, D)^* = (D, A)$.*

Note that it is a subalgebra of $M(n, \mathbb{F})^{op} \oplus M(n, \mathbb{F})$, $\mathbb{F}$ embeds in as $(\alpha I_n, \alpha I_n)$ for $\alpha \in \mathbb{F}$, and $*$ fixes $\mathbb{F}$. If $\mathbf{B}$ is non-degenerate then the projection of $\mathrm{Adj}(\mathbf{B})$ to either $M(n, \mathbb{F})^{op}$ or $M(n, \mathbb{F})$ is faithful. Therefore, *in the non-degenerate case*, we can identify $(\mathrm{Adj}(\mathbf{B}), *)$ as a subalgebra of $M(n, \mathbb{F})$ consisting of $\{D \in M(n, \mathbb{F}) \mid \exists A \in M(n, \mathbb{F}) \text{ s.t. } \forall i \in [m], A^t B_i = B_i D\}$, and for $D \in \mathrm{Adj}(\mathbf{B})$, D^* is just the (unique) solution of $\forall i \in [m], A^t B_i = B_i D$. In particular we have $A^t \mathbf{B} = \mathbf{B} A^*$.

Note that a linear basis of the adjoint algebra of a tuple of ϵ-symmetric matrices can be computed efficiently by solving a system of linear forms. The $*$-map is also easily implemented.

3 Proof of Theorem 7

3.1 An outline of the main algorithm for Theorem 7.

Let $\mathbb{F}$ be a field. Recall that we have $\mathbf{B} = (B_1, \ldots, B_m)$ and $\mathbf{C} = (C_1, \ldots, C_m) \in S^\epsilon(n, \mathbb{F})^m$. The goal is to decide if there exists $F \in \mathrm{GL}(n, \mathbb{F})$ such that $\forall i \in [m]$, $F^t B_i F = C_i$. The main steps of the algorithm are as follows.

1. *Reduce to the non-degenerate case.* If $\mathbf{B}$ is degenerate, that is $\cap_{i \in [m]} \ker(B_i) \neq \mathbf{0}$, we can reduce to the non-degenerate case by restricting to the non-degenerate part. See Section 3.2.

2. *Solve the twisted equivalence problem.* In this step we test whether $\mathbf{B}$ and $\mathbf{C}$ are "twisted equivalent", that is, whether there exist $A, D \in \mathrm{GL}(n, q)$ such that $A^t \mathbf{B} = \mathbf{C} D$. This problem can be solved efficiently by reducing to the module isomorphism problem. See Section 3.3.

3. *Reduce to decomposing a symmetric element in a $*$-algebra.* At the beginning of this step we know that $\mathbf{B}$ and $\mathbf{C}$ are twisted equivalent under some $A, D \in \mathrm{GL}(n, q)$. Note that if $D = A^{-1}$ then we are done. If not, the hope is to transform A and D appropriately to get an invertible matrix F such that $\mathbf{B}$ and $\mathbf{C}$ are twisted equivalent under F and F^{-1}, if such an F exists. Let $E = A^{-1} D^{-1}$. Since $\mathbf{C}$ is non-degenerate, the adjoint algebra of $\mathbf{C}$ can be defined alternatively as a subalgebra of $M(n, \mathbb{F})$, $\mathfrak{A} = \mathrm{Adj}(\mathbf{C}) := \{D \in M(n, \mathbb{F}) \mid \exists A \in M(n, \mathbb{F}) \text{ s.t. } \forall i \in [m], A^t C_i = C_i D\}$. The involution $*$ sends $D \in \mathrm{Adj}(\mathbf{C})$ to D^*, which is the (unique) solution of $\forall i \in [m], A^t C_i = C_i D$. It can be verified that $E \in \mathfrak{A}$, and $E^* = E$. The important observation then is that, there exists such F if and only if there exists $X \in \mathfrak{A}$ such that $E = X^* X$. See Section 3.4.

4. *Solve the $*$-symmetric decomposition problem.* This is the main technical piece of this algorithm. This step relies on certain results about the structure of $*$-algebras, which has been summarized in Section 2. The basic idea is to utilize the algebra structure of $\mathfrak{A}$, to reduce to the semisimple case, and then further to the simple case. To deal with the simple case turns out to be exactly the isometry problem for a *single* (symmetric, skew-symmetric, or Hermitian...) form, which can be solved using existing algorithms. We now outline the main steps.

4.a. *Compute the algebra structure of* $\mathfrak{A}$. We start with computing the algebra structure of $\mathfrak{A}$, including the Jacobson radical $\mathrm{Rad}(\mathfrak{A})$, the decomposition of the semisimple quotient into simple summands, and for each simple summand, an explicit isomorphism with a matrix ring over a division algebra. This can be achieved by resorting to known algorithms by Rónyai [Rón90] and Eberly [Ebe91a, Ebe91b]. This step is the main bottleneck to extend this algorithm to number fields (without going to extension fields). See Section 3.5.1.

4.b. *Recognize the* $*$*-algebra structure.* We then take into account the $*$-algebra structure. The involution $*$ preserves the Jacobson radical, so it induces an involution on the semisimple quotient, denoted again by $*$. For a particular summand S of the semisimple quotient, $*$ either switches S with another summand, or preserves it. In the the latter case, by the structure theory of $*$-algebras in the simple case, $*$ has to be in a particular form, and this form can be computed explicitly by resorting to the module isomorphism problem. See Section 3.5.2.

4.c. *Reduce to the semisimple case.* In this step, we show that any solution to the $*$-symmetric decomposition problem for $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$ and $E + \mathrm{Rad}(\mathfrak{A})$ can be lifted efficiently to a solution to the $*$-symmetric decomposition problem for $\mathfrak{A}$ and E. This procedure crucially relies on that we work with fields of characteristic not 2, and is the main bottleneck to extend this algorithm to fields of characteristic 2. This means that we can reduce to work with semisimple $*$-algebra $\mathfrak{A}$ in the following. See Section 3.5.3.

4.d. *Reduce to the* $*$*-simple and simple case.* In this step, we want to tackle the $*$-symmetric decomposition problem for a semisimple $*$-algebra $\mathfrak{A}$. Recall that a decomposition of $\mathfrak{A}$ as a sum of simple summands has been computed in Step (4.a). We present a reduction to the same problem for those simple summands that are preserved by $*$. This means that we can reduce to work with a simple $*$-algebra $\mathfrak{A}$. See Section 3.5.4.

4.e. *Tackle the simple case by reducing to the isometry problem for a single form.* In this step, we want to solve the $*$-symmetric decomposition problem for a simple $*$-algebra $\mathfrak{A}$. Recall that an explicit isomorphism of $\mathfrak{A}$ with a matrix ring over a division algebra has been computed in Step (4.a), and a particular form of $*$ on $\mathfrak{A}$ has been computed in Step (4.b). By these two pieces of information, we can reduce the $*$-symmetric decomposition problem for $\mathfrak{A}$ to the isometry problem for a *single* classical (symmetric, skew-symmetric, Hermitian...) form. See Section 3.5.5.

4.f. *Solve the isometry problem for a single form.* To solve the isometry problem for a single classical form is a classical algorithmic problem. One approach is to transform a given form into the standard form, by first block diagonalizing it, and then bringing the diagonal blocks to basic ones. Do this for both forms, compare whether the respective standard forms are the same, and if so, recover the isometry from the changes of bases in the standardizing procedures. See Section 3.5.6.

From Step (4.f) above, we may view the whole procedure as a reduction from isometry testing of an ϵ-symmetric matrix tuple to isometry testing of classical forms. Over $\mathbb{R}$, these classical forms are exactly those ones that define the classical groups in the sense of Weyl [Wey97] (see Section 2). In particular, in principle all possible classical forms – symmetric, skew-symmetric, Hermitian, skew-Hermitian over $\mathbb{R}$, $\mathbb{C}$, and the quaternion algebra $\mathbb{H}$ – can arise, even when we deal with only

a symmetric matrix tuple. It will be interesting to implement our algorithm and examine whether every classical form type indeed arises.

There is a tricky issue if we want to output an isometry over $\mathbb{R}$ and $\mathbb{C}$ as described in Theorem 7 (2) and (3). Over $\mathbb{R}$ and $\mathbb{C}$, the simple summands of a semisimple algebra may be defined over different extension fields, and one needs to be careful not to mix these fields arbitrarily as that may lead to an extension field of exponential degree. To overcome this problem we need an alternative solution to the $*$-symmetric decomposition problem as described in Section 3.6, based on $*$-invariant Wedderburn-Malcev complements of the Jacobson ideal of a $*$-algebra [Taf57].

In the following subsections, from Section 3.2 to 3.5, we give the detailed procedure, which solves completely the case of $\mathbb{F}_q$, as well as the decision version of the isometry problem for $\mathbb{R}$ and $\mathbb{C}$. The main algorithm fails to construct an explicit isometry as described in Theorem 7 (2) and (3). We remedy this by providing an alternative algorithm in Section 3.6, which replaces some steps of the main algorithm.

3.2 Main algorithm I: reduce to the non-degenerate case.

This step works over any field. The procedure is standard but we give details here for completeness.

Recall that $\mathbf{B} \in S^\epsilon(n,\mathbb{F})^m$, as an ϵ-symmetric matrix tuple, is non-degenerate if $\ker(\mathbf{B}) = \mathbf{0}$ (Section 2). Now suppose we are given $\mathbf{B} \in S^\epsilon(n,\mathbb{F})^m$, and let $d = \dim(\ker(\mathbf{B}))$. Form a change of basis matrix $S = [v_1, \ldots, v_n]$, $v_i \in \mathbb{F}^n$, such that $\{v_{n-d+1}, \ldots, v_n\}$ is a basis of $\ker(\mathbf{B})$, and $\langle v_1, \ldots, v_{n-d}\rangle$ is a complement subspace of $\ker(\mathbf{B})$. Then for every $i \in [m]$, $S^t B_i S = \begin{bmatrix} B_i' & 0 \\ 0 & 0 \end{bmatrix}$ where $B_i' \in S^\epsilon(n-d,\mathbb{F})$. We call $\mathbf{B}' = (B_1', \ldots, B_m')$ a non-degenerate tuple extracted from $\mathbf{B}$. It is easy to show the following.

Proposition 13. *Given $\mathbf{B}, \mathbf{C} \in S^\epsilon(n,\mathbb{F})^m$, let $\mathbf{B}' \in S^\epsilon(\ell_1,\mathbb{F})^m$ (resp. $\mathbf{C}' \in S^\epsilon(\ell_2,\mathbb{F})^m$) be a non-degenerate tuple extracted from $\mathbf{B}$ (resp. $\mathbf{C}$). Then $\mathbf{B} \sim \mathbf{C}$ if and only if $\ell_1 = \ell_2$, and $\mathbf{B}' \sim \mathbf{C}'$.*

Since extracting a non-degenerate tuple from $\mathbf{B}$ involves only standard linear algebraic computations, this step can be performed in deterministic polynomial time. So in the following we can assume that $\mathbf{B}$ and $\mathbf{C}$ are both non-degenerate.

3.3 Main algorithm II: solve the twisted equivalence problem.

This step works over any field. $\mathbf{B}, \mathbf{C} \in M(n,\mathbb{F})^m$ are twisted equivalent, if there exist $A, D \in \mathrm{GL}(n,\mathbb{F})$ such that $A^t\mathbf{B} = \mathbf{C}D$. This differs from the usual equivalence as in Definition 6 due to the transpose of A. But any solution (A, D) to the equivalence problem clearly gives a solution to the twisted equivalence problem by (A^t, D). The reason to introduce the twisted equivalence is because we want to be closer to the isometry concept. We now show how to test whether $\mathbf{B}$ and $\mathbf{C}$ are equivalent, by a reduction to the module isomorphism problem.

Proposition 14. *Given $\mathbf{B}, \mathbf{C} \in M(n,\mathbb{F})^m$, there exists a deterministic algorithm that decides whether $\mathbf{B}$ and $\mathbf{C}$ are equivalent (and therefore twisted equivalent). The algorithm uses polynomially many arithmetic operations. Over number fields the bit complexity of the algorithm is also polynomial.*

Proof. From $\mathbf{B} = (B_1, \ldots, B_m)$, construct a tuple of matrices $\mathbf{B}' = (B'_0, B'_1, \ldots, B'_m)$, where $B'_i \in M(2n, \mathbb{F})$, as follows. Every B_i is viewed as a 2×2 block matrix with each block of size $n \times n$. $B'_0 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$, and for $i \in [m]$, $B'_i = \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix}$. Similarly construct $\mathbf{C}'$.

We claim that there exist $A, D \in \mathrm{GL}(n, \mathbb{F})$ satisfying $A\mathbf{B} = \mathbf{C}D$ if and only if there exists an invertible $E \in \mathrm{GL}(2n, \mathbb{F})$ satisfying $E\mathbf{B}' = \mathbf{C}'E$. For the if direction, let $E = \begin{bmatrix} A & G \\ H & D \end{bmatrix}$. By $EB'_0 = C'_0E$, we have $G = H = 0$. Therefore, as $E \in \mathrm{GL}(2n, \mathbb{F})$, $A, D \in \mathrm{GL}(n, \mathbb{F})$. Furthermore, for $i \in [m]$, by $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & C_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, we see that $AB_i = C_iD$. For the only if direction, if $AB_i = C_iD$ for all $i \in [m]$, then it is easy to see that $E = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ satisfies that $E\mathbf{B}' = \mathbf{C}'E$.

Therefore, the above construction gives an efficient reduction from the equivalence problem for $\mathbf{B}$ and $\mathbf{C}$ to the conjugacy problem for $\mathbf{B}'$ and $\mathbf{C}'$. We can then call the procedure in Theorem 10 to conclude. □

Note that if $\mathbf{B} \sim \mathbf{C}$ then $\mathbf{B}$ and $\mathbf{C}$ are indeed twisted equivalent. In other words, if $\mathbf{B}$ and $\mathbf{C}$ are not twisted equivalent we conclude that they are not isometric either. Therefore, in the following we assume that we have computed $A, D \in \mathrm{GL}(n, \mathbb{F})$ such that $A^t\mathbf{B} = \mathbf{C}D$.

3.4 Main algorithm III: reduce to decomposing a $*$-symmetric element in a $*$-algebra.

This step works over any field. From previous steps, for the non-degenerate $\mathbf{B}, \mathbf{C} \in S^\epsilon(n, \mathbb{F})$, we have computed $A, D \in \mathrm{GL}(n, \mathbb{F})$ such that $A^t\mathbf{B} = \mathbf{C}D$.

Let $\mathfrak{A} = \mathrm{Adj}(\mathbf{C})$, with the natural involution $*$. Since $\mathbf{C}$ is non-degenerate, $\mathfrak{A}$ can be embedded as a subalgebra of $M(n, \mathbb{F})$ (see Section 2.) Let $E = A^{-1}D^{-1}$. Note that E is invertible.

Claim 15. *Let E and $\mathfrak{A}$ be as above. E is a $*$-symmetric element in $\mathfrak{A}$.*

Proof. Observe that $A^t\mathbf{B} = \mathbf{C}D \Leftrightarrow \mathbf{B}D^{-1} = A^{-t}\mathbf{C} \Leftrightarrow D^{-t}\mathbf{B}^t = \mathbf{C}^tA^{-1} \Leftrightarrow D^{-t}\mathbf{B} = \mathbf{C}A^{-1}$, where the last $\Leftrightarrow$ uses that $\mathbf{B}$ and $\mathbf{C}$ are from $S^\epsilon(n, \mathbb{F})$. Therefore $(A^{-1}D^{-1})^t\mathbf{C} = D^{-t}A^{-t}\mathbf{C} = D^{-t}\mathbf{B}D^{-1} = \mathbf{C}A^{-1}D^{-1}$. □

The following proposition is a conceptually crucial observation for the algorithm.

Proposition 16. *Let $\mathbf{B}$, $\mathbf{C}$, $\mathfrak{A}$, and E be as above. Then $\mathbf{B} \sim \mathbf{C}$ if and only if there exists $X \in \mathfrak{A}$ such that $X^*X = E$.*

Proof. For the if direction, by $X^*X = A^{-1}D^{-1}$, we have $AX^* = D^{-1}X^{-1}$. Also observe that $D^{-t}\mathbf{B} = \mathbf{C}A^{-1}$, and $(X^*)^t\mathbf{C} = \mathbf{C}X \Leftrightarrow \mathbf{C}X^* = X^t\mathbf{C} \Leftrightarrow X^{-t}\mathbf{C} = \mathbf{C}(X^*)^{-1}$. So $(D^{-1}X^{-1})^t\mathbf{B} = X^{-t}D^{-t}\mathbf{B} = X^{-t}\mathbf{C}A^{-1} = \mathbf{C}(X^*)^{-1}A^{-1}$, which gives $(D^{-1}X^{-1})^t\mathbf{B}(AX^*) = \mathbf{C}$. Now recall that $AX^* = D^{-1}X^{-1}$, so $D^{-1}X^{-1}$ is the desired isometry.

For the only if direction, suppose $Z^t\mathbf{B}Z = \mathbf{C}$. Setting $X = Z^{-1}D^{-1}$ and $Y = A^{-1}Z$, we have $AY = D^{-1}X^{-1} = Z$. So $\mathbf{C} = Z^t\mathbf{B}Z = Y^tA^t\mathbf{B}D^{-1}X^{-1} = Y^t\mathbf{C}X^{-1}$, which gives $Y = X^*$. By $YX = A^{-1}D^{-1}$, $X^*X = A^{-1}D^{-1}$ follows. □

Proposition 16 then leads to the following question.

Problem 17 ($*$-symmetric decomposition problem)**.** Let $\mathfrak{A}$ be a matrix algebra in $M(n, \mathbb{F})$ with an involution $*$, and $E \in \mathfrak{A}$ be an invertible $*$-symmetric element. Compute $X \in \mathfrak{A}$ such that $X^*X = E$, if there exists such an element.

3.5 Main algorithm IV: solve the $*$-symmetric decomposition problem.

This is the main technical piece of this algorithm. The strategy is to utilize the algebra structure of $\mathfrak{A}$, and reduce the problem to the case when $\mathfrak{A}$ is a simple algebra. When $\mathfrak{A}$ is simple and can be explicitly represented as a full matrix ring over division algebras, the problem turns out to be equivalent to solving the isometry problem for a single classical (symmetric, skew-symmetric, Hermitian...) form, which then can be solved using existing algorithms.

3.5.1 Decomposition algorithm I: compute the algebra structure.

By resorting to known results, this step works over finite fields [Rón90, Iva00, EG00], the real field, and the complex field [FR85, Ebe91a, Ebe91b]. We now cite these results as follows.

Theorem 18 ([Rón90]; see also [Iva00, EG00])**.** *Suppose we are given a linear basis of an algebra $\mathfrak{A}$ in $M(n, \mathbb{F}_q)$. There is a Las Vegas algorithm that computes*

1. *a linear basis of the Jacobson radical* $\mathrm{Rad}(\mathfrak{A})$*, and*
2. *an epimorphism $\pi : \mathfrak{A} \to M(n_1, \mathbb{F}_{q_1}) \oplus \cdots \oplus M(n_k, \mathbb{F}_{q_k})$ with kernel* $\mathrm{Rad}(\mathfrak{A})$*, and $\mathbb{F}_{q_i}$ an extension field of $\mathbb{F}_q$. $\mathbb{F}_{q_i}$ is specified by a linear basis over $\mathbb{F}_q$.*

The algorithm runs in time $\mathrm{poly}(n, \log q)$*, and can be derandomized at the price of running in time* $\mathrm{poly}(n, \log q, p)$ *where $p = \mathrm{char}(\mathbb{F}_q)$.*

Furthermore, there are efficient deterministic algorithms that

i. given $a \in \mathfrak{A}$, compute $\pi(a)$, and

ii. given $b \in M(n_1, \mathbb{F}_{q_1}) \oplus \cdots \oplus M(n_k, \mathbb{F}_{q_k})$, compute $a \in \mathfrak{A}$ such that $\pi(a) = b$.

Theorem 19 ([FR85, Ebe91a, Ebe91b, Rón94])**.** *Let $\mathbb{E}$ be a number field, and suppose we are given a linear basis of an algebra $\mathfrak{A}$ in $M(n, \mathbb{E})$. Then there exists a deterministic polynomial-time algorithm that computes*

1. *a linear basis of the Jacobson radical* $\mathrm{Rad}(\mathfrak{A})$ *over $\mathbb{E}$, and*
2. • *Over $\mathbb{R}$: (a) the number k of simple components of $\mathfrak{A} \otimes_{\mathbb{E}} \mathbb{R}$,*
   (b) specifications of extension fields $\mathbb{E} \subseteq \mathbb{E}_1, \ldots, \mathbb{E}_k \subseteq \mathbb{R}$, such that each $\mathbb{E}_i$ is of degree at most $\binom{\dim_{\mathbb{E}} \mathfrak{A}}{2}$ over $\mathbb{E}$,
   (c) bases of simple algebras $B_1 \subseteq A \otimes_{\mathbb{E}} \mathbb{E}_1$, ..., $B_k \subseteq A \otimes_{\mathbb{E}} \mathbb{E}_k$, such that $B_i \otimes_{\mathbb{E}_i} \mathbb{R}$, $i \in [k]$, are all the simple components of $\mathfrak{A} \otimes_{\mathbb{E}} \mathbb{R}$, and
   (d) for each $i \in [k]$, an extension field $\mathbb{K}_i \subseteq \mathbb{R}$ over $\mathbb{E}_i$ with extension degree at most $\dim_{\mathbb{E}_i} B_i$, the linear basis of a division algebra $D_i \subseteq B_i \otimes_{\mathbb{E}_i} \mathbb{K}_i$ over $\mathbb{K}_i$, and the linear basis of a subalgebra $M_i \subseteq B_i \otimes_{\mathbb{E}_i} \mathbb{K}_i$ over $\mathbb{K}_i$, such that $M_i \cong M(n_i, \mathbb{K}_i)$, and $B_i \otimes_{\mathbb{E}_i} \mathbb{K}_i \cong M_i \otimes_{\mathbb{K}_i} D_i \cong M(n_i, D_i)$. $\dim_{\mathbb{K}_i} D_i$ can be 1, 2, or 4, and when $\dim_{\mathbb{K}_i} D_i = 4$, D_i is non-commutative.

- *Over $\mathbb{C}$: (a) the number k of simple components of $\mathfrak{A} \otimes_{\mathbb{E}} \mathbb{C}$,*
  - *(b) specifications of extension fields $\mathbb{E} \subseteq \mathbb{E}_1, \ldots, \mathbb{E}_k$, such that each $\mathbb{E}_i$ is of degree at most $\dim_{\mathbb{E}} \mathfrak{A}$ over $\mathbb{E}$,*
  - *(c) bases of simple algebras $B_1 \subseteq A \otimes_{\mathbb{E}} \mathbb{E}_1$, ..., $B_k \subseteq A \otimes_{\mathbb{E}} \mathbb{E}_k$, such that $B_i \otimes_{\mathbb{E}_i} \mathbb{C}$, $i \in [k]$, are all the simple components of $\mathfrak{A} \otimes_{\mathbb{E}} \mathbb{C}$, and*
  - *(d) for each $i \in [k]$, an extension field $\mathbb{K}_i$ over $\mathbb{E}_i$ with extension degree at most $\sqrt{\dim_{\mathbb{E}_i} B_i}$, the linear basis of a subalgebra $M_i \subseteq B_i \otimes_{\mathbb{E}_i} \mathbb{K}_i$ over $\mathbb{K}_i$, such that $M_i \cong M(n_i, \mathbb{K}_i)$.*

Remark 20. 1. Comparing Theorem 18 and Theorem 19, we see that a statement corresponding to Theorem 18 (ii) was missing in Theorem 19. This is because a preimage of $b \in M(n_1, D_1) \oplus \cdots \oplus M(n_k, D_k)$ may live in $\mathfrak{A} \otimes_{\mathbb{E}} \mathbb{K}$ for some field $\mathbb{K}$ with an exponential extension degree over $\mathbb{E}$. This suggests that representing the isometry in the settings of $\mathbb{R}$ and $\mathbb{C}$ as a *single* matrix would be inefficient.

2. The randomized version of Theorem 19 is shown by Eberly in [Ebe91a, Ebe91b], and is subsequently derandomized by Rónyai in [Rón94]. To completely derandomize Theorem 18 is a difficult problem as this relies on algorithms for polynomial factorization over finite fields.

3.5.2 Decomposition algorithm II: recognize the $*$-algebra structure.

This step works over $\mathbb{F}_q$ with q odd, $\mathbb{R}$, and $\mathbb{C}$. It may be possible to handle fields of even characteristics, but we leave it for further study. The case of finite fields of odd characteristics has been settled by Brooksbank and Wilson in [BW12]. Here we provide a unified and somewhat simpler treatment over those fields just mentioned.

To start with, recall that from previous steps we have computed the algebra structure of $\mathfrak{A} \subseteq M(n, \mathbb{F})$, including a linear basis of $\mathrm{Rad}(\mathfrak{A})$ and an epimorphism $\pi : \mathfrak{A} \to S_1 \oplus \cdots \oplus S_k$ where S_i is a simple algebra over the designated field (after some scalar extension when over $\mathbb{R}$ or $\mathbb{C}$). We have also computed explicit isomorphisms between S_i and matrix rings over division rings. Since $\mathrm{Rad}(\mathfrak{A})$ is a $*$-ideal, the involution $*$ induces an involution, which we denote again by $*$, on $\pi(\mathfrak{A})$. Then for each S_i, either $S_i^* = S_i$, or $S_i^* = S_j$ for some $j \neq i$. The goal is that, in the former case, we want to express the involution $*$ explicitly in the forms presented in Section 2.

Proposition 21. *Let $\mathbb{E}/\mathbb{F}$ be a field extension specified by a linear basis over $\mathbb{F}$. Given an involution $*$ of $M(n, \mathbb{E})$ as an $\mathbb{F}$-algebra, there exists a deterministic polynomial-time algorithm that (1) decides whether $*$ induces a quadratic field involution of $\mathbb{E}$ over a subfield $\mathbb{E}'$, and (2) computes $A \in \mathrm{GL}(n, \mathbb{E})$ such that for every $X \in M(n, \mathbb{E})$, $X^* = A^{-1} X'^t A$, where X' is either X (when $*$ fixes $\mathbb{E}$) or $\overline{X}$ (when $*$ induces a quadratic field involution).*

Proof. For (1), we apply $*$ to every basis element b in the linear basis of $\mathbb{E}$ over $\mathbb{F}$. If $*$ changes none of them, then $\mathbb{E}$ is also invariant under $*$. If $*$ changes some of them, the sums $b + b^*$ linearly span a subfield $\mathbb{E}'$ such that $\mathbb{E}/\mathbb{E}'$ is a quadratic field extension, and $*$ induces the quadratic field involution. For (2), for any $X \in M(n, \mathbb{E})$ let X' be as defined in the statement. We take a linear basis $\{B_1, \ldots, B_{n^2}\}$ of $M(n, \mathbb{E})$ (the standard basis will do), and set up $Y B_i^* = B_i'^t Y$, for $i \in [n^2]$, and Y is an $n \times n$ variable matrix. By [Alb39, Chap. X.4, Theorem 11], there must exist some $A \in \mathrm{GL}(n, \mathbb{E})$ as a valid solution to Y in the above equations. From the algorithmic viewpoint, this is an instance of the module isomorphism problem, and we can apply the procedure in Theorem 10 to conclude. □

Note that Proposition 21 covers all simple types over $\mathbb{F}_q$ with q odd and $\mathbb{C}$, as well as those simple types over $\mathbb{R}$ except the two quaternion types. We now handle the two quaternion types in the real field setting.

Proposition 22. *Let $\mathbb{H}$ be given by a linear basis over $\mathbb{R}$. Given an involution $*$ of $M(n,\mathbb{H})$ as an $\mathbb{R}$-algebra, there exists a deterministic polynomial-time algorithm that computes $A \in \mathrm{GL}(n,\mathbb{H})$ such that for every $X \in M(n,\mathbb{H})$, $X^* = A^{-1}\overline{X}^t A$.*

Proof. Let $f : \mathbb{H} \to M(4,\mathbb{R})$ be the regular representation of $\mathbb{H}$ on $\mathbb{R}^4$. Let $\{C'_1, C'_2, C'_3, C'_4\}$ be a linear basis of the centralizing algebra of $f(\mathbb{H})$ in $M(4,\mathbb{R})$, which is isomorphic to $\mathbb{H}^{op}$. Now think of matrices in $M(4n,\mathbb{R})$ as $n \times n$ block matrices with each block of size 4×4. For $i \in [4]$, let $C_i \in M(4n,\mathbb{R})$ be the diagonal block matrix, with all diagonal blocks being C'_i. f naturally embeds $M(n,\mathbb{H})$ to $M(4n,\mathbb{R})$. By the double centralizer theorem, the centralizing algebra of C_i's is $f(M(n,\mathbb{H}))$.

The above reasoning suggests the following construction. Take a basis $\{B_1, \ldots, B_{n^2}\}$ of $M(n,\mathbb{H})$, and let $B'_i = \overline{B_i}^t$. Set up $Yf(B_i) = f(B'_i)Y$, $i \in [n^2]$, $YC_j = C_jY$, $j \in [4]$, where Y is a $4n \times 4n$ variable matrix. By $YC_j = C_jY$, any valid solution to Y lies in $f(M(n,\mathbb{H}))$. By an analogous argument as in the proof of Proposition 21, there must exist an invertible $A \in \mathrm{GL}(4n,\mathbb{R})$ as a valid solution to Y, and can be solved as as an instance of the module isomorphism problem by Theorem 10. Finally, after getting such an $A \in \mathrm{GL}(4n,\mathbb{R})$, it is straightforward to compute the preimage of A in $M(n,\mathbb{H})$, concluding the proof. □

3.5.3 Decomposition algorithm III: reduce to the semisimple case.

This step works over fields of characteristic $\neq 2$, and is the main bottleneck for handling fields of characteristic 2.

Proposition 23. *Let $\mathfrak{A}$ be a $*$-algebra over $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$. Let $E \in \mathfrak{A}$ be an invertible $*$-symmetric element, and suppose there exists $Y \in \mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$, such that $Y^*Y + \mathrm{Rad}(\mathfrak{A}) = E + \mathrm{Rad}(\mathfrak{A})$. Then there exists $X \in \mathfrak{A}$ such that $X^*X = E$, and there exists a deterministic polynomial-time algorithm that outputs such an X.*

Proof. To recover $X \in \mathfrak{A}$ such that $X^*X = E$, consider the following situation: suppose we have a $*$-ideal J of $\mathfrak{A}$ with $J^2 = 0$, and an invertible $E \in \mathfrak{A}$ with $E^* = E$. Given Y such that $Y^*Y + J = E + J$, the goal is to find $Z \in J$ such that $(Y+Z)^*(Y+Z) = E$. Expanding to $Y^*Y + Y^*Z + YZ^* + Z^*Z = E$, by $Z^*Z = 0$ we need to satisfy $Y^*Z + Z^*Y = E - Y^*Y$. Note that $E - Y^*Y$ is $*$-symmetric. So setting $U = \frac{1}{2}(E - Y^*Y)$, $Z = Y^{-*}U$ is the desired, and $X = Y + Z$ satisfies $X^*X = E$.

We now apply the above procedure to the setting of the proposition. For $i \in \mathbb{N}$, let $J_i = \mathrm{Rad}(\mathfrak{A})^{2^i}$, so that $J_{i+1} = J_i^2$. Since the Jacobson radical $\mathrm{Rad}(\mathfrak{A})$ is nilpotent, we know that for some $k \leq \lceil \log n \rceil$, $J_k = 0$. Given $Y_i \in \mathfrak{A}/J_i$ satisfying $Y_i^*Y_i + J_i = E + J_i$, consider $\mathfrak{A}/J_{i+1}$, in which J_i/J_{i+1} satisfies the assumption on J in the last paragraph. We can then utilize the procedure there to get $Y_{i+1} \in \mathfrak{A}/J_{i+1}$, such that $Y_{i+1}^*Y_{i+1} + J_{i+1} = E + J_{i+1}$. Let the given Y satisfying $Y^*Y + \mathrm{Rad}(\mathfrak{A}) = E + \mathrm{Rad}(\mathfrak{A})$ be Y_0, and perform the above procedure iteratively for at most $k \leq \lceil \log n \rceil$ times. We then obtain the desired $X \in \mathfrak{A}$ such that $X^*X = E$. □

3.5.4 Decomposition algorithm IV: reduce to the $*$-simple and simple case.

This step works for any field. Suppose we have a semi-simple algebra $\mathfrak{A}$ decomposed into a direct sum of simple summands $S_1 \oplus \cdots \oplus S_k$, and let $*$ be an involution on $\mathfrak{A}$. Without loss of generality, we can assume that there exists $j \leq \lfloor k/2 \rfloor$, such that $*$ exchanges S_{2i-1} and S_{2i} for $i \in [j]$, and stabilizes S_i for $i > 2j$. Let $E \in \mathfrak{A}$ be an invertible $*$-symmetric element, and let E_i be the projection of E to S_i. Recall that our goal is to find $X \in \mathfrak{A}$ such that $X^*X = E$, if such an X exists.

Proposition 24. *Let $\mathfrak{A}$, S_i, E, and E_i be as above. There exists $X \in \mathfrak{A}$ such that $X^*X = E$, if and only if for every $i > 2j$, there exists $X_i \in S_i$ such that $X_i^* X_i = E_i$.*

Proof. For the if direction, we claim that $X = E_1 \oplus I \oplus E_3 \oplus I \oplus \cdots \oplus E_{2j-1} \oplus I \oplus X_{2j+1} \oplus \cdots \oplus X_k$ is a solution, where I denotes the identity element in the respective summand. To see this, let us suppose $*$ exchanges S_1 and S_2. Then by $(E_1, E_2)^* = (E_1, E_2)$, we have $(E_1, I)^* = (I, E_2)$. So $X^*X = (I \oplus E_2 \oplus I \oplus E_4 \oplus \cdots \oplus I \oplus E_{2j} \oplus X_{2j+1}^* \oplus \cdots \oplus X_k^*)(E_1 \oplus I \oplus E_3 \oplus I \oplus \cdots \oplus E_{2j-1} \oplus I \oplus X_{2j+1} \oplus \cdots \oplus X_k) = E$.

For the only if direction, suppose $X = X_1 \oplus X_2 \oplus \cdots \oplus X_{2j-1} \oplus X_{2j} \oplus X_{2j+1} \oplus \cdots \oplus X_k$ satisfies $X^*X = E$. Then it is straightforward to verify that for $i > 2j$, $X_i^* X_i = E_i$. □

3.5.5 Decomposition algorithm V: the simple case by reducing to the isometry problem for a single form.

This step works over any field. From previous steps, we now have (1) $M(n, D)$ where D is a field or a division algebra, (2) an involution $*$ on $M(n, D)$, which induces an involution $\bar{\cdot} : D \to D$ (possibly identity), such that $X^* = A^{-1}\overline{X}^t A$ and $\overline{A}^t = \epsilon A$ for some $\epsilon \in \{1, -1\}$, and (3) an invertible $*$-symmetric element E.

Here is the other conceptually crucial observation.

Proposition 25. *Let notation be as above. Let $F = AE$. Then F is a form of the same type as A, and there exists X such that $X^*X = E$, if and only if A and F are isometric.*

Proof. To see that F is a form of the same type as A, we have $E = E^* = A^{-1}\overline{E}^t A$ (by the $*$-symmetry of E) and $\overline{A}^t = \epsilon A$. It follows that $AE = \overline{E}^t A$, from which we get $\overline{AE}^t = \overline{E}^t \overline{A}^t = \epsilon \overline{E}^t A = \epsilon AE$.

For the second statement, we consider the if direction first. If for some $Y \in \mathrm{GL}(n, D)$, $Y^t A \overline{Y} = F = AE$, then $A^{-1} Y^t A \overline{Y} = E$. Setting $X = \overline{Y}$, we have $A^{-1}\overline{X}^t AX = E$. Noting that $A^{-1}\overline{X}^t A = X^*$, we obtain the desired $X^*X = E$. The only if direction can be seen easily by inverting the above reasoning. □

3.5.6 Decomposition algorithm VI: solve the isometry problem for a single form.

To solve the isometry problem for a single form over a division ring, we will in fact compute the canonical form for such a form. The isometry problem can then be solved by comparing the canonical forms. Over $\mathbb{F}_q$ with q odd, a concrete isometry can be obtained by using the transformations to the canonical forms. To recover a concrete isometry (represented in some form) over $\mathbb{R}$ or $\mathbb{C}$ requires more technical machinery and we leave it to Section 3.6. The existence of canonical forms is well-known for $\mathbb{F}_q$ with q odd (see e.g. [Wil09c, Chap. 3.4]), for $\mathbb{R}$ (see e.g. [Lew77]), and for $\mathbb{C}$.

Computing the canonical form involves two steps. Let $E \in M(n, D)$ such that $\overline{E}^t = \epsilon E$, where $\bar{\cdot} : D \to D$ is an involution, and $\epsilon \in \{1, -1\}$.

The first step is to compute an orthogonal basis for E, that is a linear basis of D^n $\{e_1, \ldots, e_n\}$, such that for every $i \in [n]$, $e_i^t E \overline{e_j} \neq 0$ for exactly one e_j. This is known as the Gram-Schmidt procedure, and an efficient algorithm in this general setting has been obtained by Wilson.

Theorem 26 ([Wil13]). *Let E be as above. There exists a deterministic polynomial-time algorithm that computes an orthogonal basis for E.*

After the first step, by transforming to the orthogonal basis, E can be assumed to be a diagonal block matrix, with each block is of size 1 or 2. The second step is to simplify these diagonal blocks as much as possible. We now need to handle each field separately. Recall that E is non-degenerate.

Block diagonal forms over $\mathbb{F}_q$. We distinguish among the three simple types over $\mathbb{F}_q$.

- *Orthogonal type* In this case, each block is of size 1, e.g. E is a diagonal matrix. Fix a non-square ω in $\mathbb{F}_q$, which can be computed efficiently, by either using randomness, or in a deterministic way if we assume the Generalized Riemann Hypothesis or the characteristic of $\mathbb{F}_q$ is small. We can first simplify E as $\mathrm{diag}(1, \ldots, 1, \omega, \ldots, \omega)$, by resorting to square root computations over finite fields. This can be done in randomized polynomial time by e.g. the Tonelli-Shanks algorithm. A deterministic polynomial-time algorithm exists, if we assume the Generalized Riemann hypothesis, or the characteristic of the finite field is small. Then, if the number of ω's is larger than 1, then write ω as a sum of two squares $\alpha^2 + \beta^2$, which is always possible over a finite field. Algorithmically, this can be done by solving the equation $x^2 + y^2 = \omega$ in deterministic polynomial time by an algorithm of van de Woestijne [vdW05, Theorem A.3]. Given such α, β, $\mathrm{diag}(\omega, \omega)$ can be transformed to $\mathrm{diag}(1, 1)$ by $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}^t = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$. Therefore the possible standard forms are $\mathrm{diag}(1, \ldots, 1)$ or $\mathrm{diag}(1, \ldots, 1, \omega)$.

- *Symplectic type* In this case, each block is of size 2, so we examine one block $\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$. Now by expressing α as a sum of squares, similar trick applies to bring it to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Indeed, the standard form for a non-degenerate alternating bilinear form of size $2k \times 2k$ is the block diagonal matrix, with each block on the diagonal being $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (see e.g. [Wil09c, Sec. 3.4.4]).

- *Hermitian type* In this case, each block is of size 1. Let the associated field extension be $\mathbb{F}_q / \mathbb{F}_{q'}$ where $q = q'^2$, and suppose $\mathbb{F}_q = \mathbb{F}_{q'}(\tau)$, where τ is a square root of a non-square $\omega \in \mathbb{F}_{q'}$. Then for $\alpha = a + b\tau$, $\overline{\alpha} = a - b\tau$. For a diagonal entry $\alpha \in \mathbb{F}_q$, $\alpha = \overline{\alpha}$, we need to compute $\beta \in \mathbb{F}_q$ such that $\beta\overline{\beta} = \alpha$, which always exists. Setting $\beta = x + y\tau$, we need to solve the equation $\beta\overline{\beta} = x^2 - y^2\tau^2 = \alpha$. Again this can be solved in deterministic polynomial time by [vdW05, Theorem A.3]. Indeed, there always exists an orthonormal basis for a non-degenerate Hermitian form, so the standard form is just the identity matrix (see e.g. [Wil09c, Sec. 3.4.5]).

Block diagonal forms over $\mathbb{R}$. By [Lew77], for the symplectic, complex orthogonal, complex symplectic, quaternion orthogonal types, we can always bring a given form to the identity matrix or the standard non-degenerate skew-symmetric matrix. For other types, we can bring a given form to $\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$, where the number of 1's and the number of -1's is called the signature of the canonical form.

Block diagonal forms over $\mathbb{C}$. By [Lew77], for the two types here we can always bring a given form to the identity matrix or the standard non-degenerate skew-symmetric matrix.

Remark 27. Starting from two ϵ-symmetric matrix tuples $\mathbf{B}$ and $\mathbf{C}$, suppose they are twisted equivalent. We then perform the operations as above, so that for each simple component of the semisimple quotient of the $*$-algebra $\mathfrak{A} = \mathrm{Adj}(\mathbf{C})$, we have a pair of forms. The question of whether $\mathbf{B}$ and $\mathbf{C}$ are isometric then reduces to test whether these pairs of forms are isometric, or in other words, to compare whether they have the same standard form. In particular, over $\mathbb{C}$, since for each simple type there exists only one standard form, the twisted equivalence of $\mathbf{B}$ and $\mathbf{C}$ already implies that they are isometric.

3.6 An alternative algorithm for the isometry problem

In this section we work over $\mathbb{R}$ and $\mathbb{C}$. We now present an algorithm that, in the $\mathbb{R}$ and $\mathbb{C}$ settings, can output an explicit isometry, which is represented by a product of several matrices, where each matrix is over an extension field of the number field $\mathbb{E}$ with polynomial extension degree, and the entries are of polynomial bit sizes.

We still follow the main algorithm, steps I to III, as described in Sections 3.2 to 3.4, to reduce to solving the decomposition problem, Problem 16, for a $*$-algebra $\mathfrak{A}$ in $M(n, \mathbb{E})$. Then recall that, by the decomposition algorithm steps I to III, as described in Sections 3.5.1 to 3.5.3, we can reduce to the semisimple setting and then further to the simple setting. In this simple setting, however, we need to work with different extension fields for different simple summands, and one cannot mix all those extension fields because that would result in an extension field with exponential extension degree (see Remark 20). Even within each summand, since we need to take square roots to bring the forms into canonical forms, these square roots cannot mix arbitrarily because of the same problem.

3.6.1 Alternative decomposition algorithm III

Compare with with Section 3.5.3.

To tackle these problems, we first devise another reduction to the semisimple case, based on the existence of $*$-invariant Wedderburn-Malcev complements over fields of characteristic $\neq 2$ [Taf57]. The following constructive version of Taft's result [Taf57] is by Brooksbank and Wilson [BW12], in conjunction with the algorithm from [dGIKR97] that computes a Wedderburn-Malcev complement over number fields.

Proposition 28 ([BW12, Proposition 4.3, Remark 4.2])**.** *Let $\mathbb{E}$ be a number field, and $\mathfrak{A} \subseteq M(n, \mathbb{E})$ a $*$-algebra. Then there exists a deterministic polynomial-time algorithm that computes a linear basis of $\mathrm{Rad}(\mathfrak{A})$, and a linear basis of a subalgebra S, such that $\mathfrak{A} = \mathrm{Rad}(\mathfrak{A}) \oplus S$ and $S^* = S$.*

Proof. The statement on computing $\mathrm{Rad}(\mathfrak{A})$ is already in Theorem 19. The procedure to compute S is in [BW12], and for completeness we include a sketch. We then note that the bit complexity is also polynomially bounded.

We first resort to Theorem 19 to compute a linear basis of $\mathrm{Rad}(\mathfrak{A})$. We then use the algorithm in [dGIKR97, Theorem 3.1] to compute a Wedderburn-Malcev complement S' of $\mathfrak{A}$ in deterministic polynomial time. If $\mathrm{Rad}(\mathfrak{A}) = 0$ then $\mathfrak{A}$ itself is what we want. If $\mathrm{Rad}(\mathfrak{A}) \neq 0$, let $\pi : \mathfrak{A} \to S'$ be the natural projection. The involution $*$ induces an involution $\circ$ on S' by sending $s \in S'$ to $\pi(s^*)$. Suppose S' is generated by $\{s_1, \ldots, s_\ell\}$. Let S'' be the algebra generated by $\{1/2(s_1+s_1^{\circ *}), \ldots, 1/2(s_\ell+s_\ell^{\circ *})\}$. We then can reduce to compute a $*$-invariant Wedderburn-Malcev complement in $\mathrm{Rad}(\mathfrak{A})^2 \oplus S''$. The number of iterative calls is at most $\lceil \log n \rceil$.

Finally, note that in each iteration the operations are $*$-maps and projections, which only increase the bit size by an additive factor of polynomial size. Therefore the bit complexity of the above procedure is also polynomial. □

Based on the Taft decomposition, we can reduce to the semisimple case in a more transparent way as follows.

Proposition 29. *Let* $\mathrm{Rad}(\mathfrak{A}) \oplus S$ *be a Taft decomposition of a* $*$*-algebra* $\mathfrak{A} \subseteq M(n, \mathbb{E})$*. Given a* $*$*-symmetric element* $a' \in \mathfrak{A}$*, there is a deterministic polynomial-time algorithm that computes* $u \in 1 + \mathrm{Rad}(\mathfrak{A})$*, such that* $u^* a' u = a \in S$*.*

Proof. Let a' be decomposed as $a + r$ with $a \in S$ and $r \in \mathrm{Rad}(\mathfrak{A})$. We will show how to find $t \in \mathrm{Rad}(\mathfrak{A})$ such that $(1+t)^* a'(1+t) = a + r'$ for $r' \in \mathrm{Rad}(\mathfrak{A})^2$. Then by iterating such a procedure to get $r'' \in \mathrm{Rad}(\mathfrak{A})^4$, ..., we would be done.

To start with, by the $*$-symmetry of a', $\mathrm{Rad}(\mathfrak{A})$, and S, we have that a and r are both $*$-symmetric as well. We expand $(1+t)^* a'(1+t) = a + r + t^* a + at + t^* r + rt + t^* at + t^* rt$. Since $t^* r + rt + t^* at + t^* rt \in \mathrm{Rad}(\mathfrak{A})^2$, we need $r + t^* a + at = 0$. This can be achieved by setting $t = -\frac{1}{2} a^{-1} r$, noting that $t^* a = t^* a^* = (at)^*$. We then have $(1+t)^* a'(1+t) = a + r'$ for $r' \in \mathrm{Rad}(\mathfrak{A})^2$.

To prove that the bit complexity is polynomial, we note that the number of iterations is at most $\log n$, and in the ℓth iteration we get at most 16^ℓ words in the alphabet $\{r, a^{-1}\}$, with each word of length at most 5^ℓ. The latter is because, if we let r_ℓ be the residue in the ℓth step, then $r_{\ell+1} = t^* r_\ell + r_\ell t + t^* at + t^* r_\ell t = -\frac{3}{4} r_\ell a^{-1} r_\ell + \frac{1}{4} r_\ell a^{-1} r_\ell a^{-1} r_\ell$. □

Given $u \in 1 + \mathrm{Rad}(\mathfrak{A})$ such that $u^* a' u = a \in S$, if we can decompose $a = x^* x$, then $x u^{-1}$ is a solution for a'. The advantage over the procedure in Proposition 23 is the following. If x is represented as a product of matrices, each of which is over a different extension field, then the procedure in Proposition 23 may mix these entries over different extension fields and cause an extension degree blow-up. On the other hand, the procedure in Proposition 29 takes x and returns $x u^{-1}$, which is still a product of matrices, as the output, therefore avoiding the extension degree blow-up issue.

3.6.2 Alternative decomposition algorithm IV

Compare with Section 3.5.4.

We now reduce to work with a semisimple $*$-algebra $\mathfrak{A}$ in $M(n, \mathbb{E})$ and a $*$-symmetric element E. By Theorem 19, we have extension fields $\mathbb{E} \subseteq \mathbb{E}_i$ and simple algebras $S_i \subseteq \mathfrak{A} \otimes_{\mathbb{E}} \mathbb{E}_i$, $i \in [k]$, such that the extension degree of $\mathbb{E}_i$ over $\mathbb{E}$ is upper bounded by $\binom{\dim_{\mathbb{E}} \mathfrak{A}}{2}$ in the real case and $\dim_{\mathbb{E}} \mathfrak{A}$ in the complex case. We reduce to the simple case by the following construction. Without loss of generality, we can assume that there exists $j \leq \lfloor k/2 \rfloor$, such that $*$ exchanges S_{2i-1} and S_{2i} for $i \in [j]$, and stabilizes S_i for $i > 2j$. Let E_i be the projection of E to S_i. For $i \leq 2j$, i odd, (E_i, I) is

the solution to the decomposition problem for $(E_i, E_{i+1}) \in S_i \oplus S_{i+1}$. For $i > 2j$, suppose $X_i \in S_i$ satisfies $X_i^* X_i = E_i$. We then embed (E_i, I) into $\mathfrak{A} \otimes_{\mathbb{E}} \mathbb{E}_i$, and X_i into $\mathfrak{A} \otimes_{\mathbb{E}} \mathbb{E}_i$, by adding identities in other summands. Let X be the product of these matrices. It is easy to see that $X^* X = E$ over some extension field $\mathbb{K}$ ($\mathbb{K}$ needs to include all $\mathbb{E}_i$). Note that X is then represented by a product of matrices, with each matrix over a possibly different extension field.

3.6.3 Alternative decomposition algorithm VI

Compare with Section 3.5.6.

We then follow Section 3.5.5 to reduce to the isometry problem for a single form. Note that to solve the isometry problem, we need to take square roots, which, if not handled well, may lead to extension fields of exponential extension degree. Therefore, we also output a product of matrices as an isometry between two single forms, keeping those diagonal matrices with square roots on the diagonal intact. Specifically, when working with two forms A and F over $\mathbb{K}$, the isometry is represented as $T' D' D^{-1} T^{-1}$ where T and T' are the orthogonal transformations, and D' and D are diagonal matrices with entries being various square roots. We can also represent $D' D^{-1}$ as a single diagonal matrix with entries being from an extension field of degree at most 4.

4 Proof of Theorem 8

Recall that in the ϵ-symmetrization problem, we are given a matrix tuple $\mathbf{B} = (B_1, \ldots, B_m) \in M(n, \mathbb{F})^m$, and need to decide whether there exist $A, D \in \mathrm{GL}(n, \mathbb{F})$ such that $\forall i \in [m]$, AB_iD is ϵ-symmetric. In Section 4.1, we present an algorithm when (1) $\mathbb{F}$ is large enough, and (2) the Jacobson radical of a matrix algebra can be computed efficiently in a deterministic way. Note that (2) holds for fields of characteristic 0 [Dic23], finite fields [Rón90], as well as many others of positive characteristic [CIW97]. This algorithm follows the strategy for module isomorphism problem as used in [CIK97], and relies crucially on Lemma 31.

We will deal with the remaining cases (a) $|\mathbb{F}|$ is large enough but we do not assume the ability to compute the Jacobson radical in Section 4.2, and (b) $|\mathbb{F}|$ is small in Section 4.3. The algorithm for (a) is obtained by associating certain projective modules to right ideals, and adapting the algorithm in Section 4.1 to work with that concept. The algorithm for (b) follows the strategy for module isomorphism problem as used in [BL08], and relies crucially on another lemma about $*$-algebra, namely Lemma 35.

To start, note that if $\dim(\cap_{i\in[m]} \ker(B_i)) + \dim(\langle \cup_{i\in[m]} \mathrm{im}(B_i) \rangle) \neq n$, then $\mathbf{B}$ cannot be ϵ-symmetrizable. This is because, if $\mathbf{B}$ is ϵ-symmetric, then $\cap_{i\in[m]} \ker(B_i)$ and $\langle \cup_{i\in[m]} \mathrm{im}(B_i) \rangle$ are orthogonal to each other with respect to the standard inner product of vectors, so their dimensions sum up to n. Then observe that $\dim(\cap_{i\in[m]} \ker(B_i)) = \dim(\cap_{i\in[m]} \ker(AB_iD))$, and $\dim(\langle \cup_{i\in[m]} \mathrm{im}(B_i) \rangle) = \dim(\langle \cup_{i\in[m]} \mathrm{im}(AB_iD) \rangle)$. If $\dim(\cap_{i\in[m]} \ker(B_i)) + \dim(\langle \cup_{i\in[m]} \mathrm{im}(B_i) \rangle) = n$ but $\cap_{i\in[m]} \ker(B_i) \neq \mathbf{0}$ then we can reduce to the $\cap_{i\in[m]} \ker(B_i) = \mathbf{0}$ analogously as it is done in Step (1) for the isometry problem (Section 3.2). So in the following we assume $\cap_{i\in[m]} \ker(B_i) = \mathbf{0}$ and $\langle \cup_{i\in[m]} \mathrm{im}(B_i) \rangle = \mathbb{F}^n$.

4.1 An algorithm for Theorem 8 under certain technical conditions

In this section we present an algorithm for Theorem 8 when (1) $\mathbb{F}$ is large enough, and (2) the Jacobson radical of a matrix algebra can be computed efficiently in a deterministic way.

Recall that, as explained at the beginning of Section 1.3, the ϵ-symmetrization problem is equivalent to ask whether there exists $E \in \mathrm{GL}(n,\mathbb{F})$ such that $E\mathbf{B} \in S^\epsilon(n,\mathbb{F})^m$. That is, whether the matrix space $L^\epsilon(\mathbf{B}) := \{Z \in M(n,\mathbb{F}) : \forall i \in [m], ZB_i = \epsilon B_i^t Z^t\}$ contains a full-rank matrix. A linear basis $Z_1, \ldots, Z_\ell$ of $L^\epsilon(\mathbf{B})$ can be computed efficiently.

The remaining part of the algorithm is an iteration during which we maintain a matrix $Z \in L^\epsilon(\mathbf{B})$. If Z has full rank we are done. Otherwise we try all basis elements Z_i and scalars λ from a sufficiently large subset $S \subseteq \mathbb{F}$, either to obtain a matrix $Z' = Z + \lambda Z_i$ which is of higher rank than Z, or, if every such Z' is of rank no more than that of Z, conclude that Z is of the highest rank. We intend to use the following well known fact. Let $A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ be $(r' + r'')$ by $(r' + r'')$ block matrices, where A_{11} is an r' by r' matrix of rank r' and B_{22} is a nonzero r'' by r'' matrix. Let $r = r' + r''$. Then the matrix $B + \lambda A$ has rank larger than r' for some λ from a sufficiently large set of scalars. Formally (see e.g. [IKS10, Lemma 2.2]),

Lemma 30. *Let $A, B \in M(r,\mathbb{F})$ and let $S \subseteq \mathbb{F}$ such that $|S| > r$. If $B\ker(A) \not\subseteq \mathrm{im}(A)$ then $\mathrm{rk}(\lambda A + B) > \mathrm{rk}(A)$ for all but at most r $\lambda \in S$.*

Unfortunately, we are unable to show – and probably it is not true in general — that Lemma 30 becomes applicable to Z (as A) and at least one of the basis elements Z_i (as B), when we consider $L^\epsilon(\mathbf{B})$ as it is obviously given to us (i.e., a space of n by n matrices). However, there is another representation of $L^\epsilon(\mathbf{B})$ as a matrix space in which it provably does. And this is the point where $*$-algebras enter the picture.

To see the details, assume that $\mathbf{B} = E\mathbf{B}'$ where $E \in \mathrm{GL}(n,\mathbb{F})$ and $\mathbf{B}' \in S^\epsilon(n,\mathbb{F})^m$. Since $\mathbf{B}'$ is non-degenerate, we can identify $\mathrm{Adj}(\mathbf{B}') \subseteq M(n,\mathbb{F})^{op} \oplus M(n,\mathbb{F})$ as a subalgebra of $M(n,\mathbb{F})$ by projecting to the second component (see Section 2). Then $L^\epsilon(\mathbf{B}')$ is the set of $*$-symmetric elements in $\mathrm{Adj}(\mathbf{B}')$. Moreover, it is not difficult to see that $L^\epsilon(\mathbf{B}) = L^\epsilon(\mathbf{B}')E^{-1}$. Now for $Z \in L^\epsilon(\mathbf{B})$, consider the following composite linear map, $Z \mapsto ZE \mapsto \overline{ZE} \mapsto \ell_{\overline{ZE}}$, where $\overline{ZE} = ZE + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B}'))$, and $\ell_{\overline{ZE}}$ is the action of $\overline{ZE}$ on the factor $\mathrm{Adj}(\mathbf{B}')/\mathrm{Rad}(\mathrm{Adj}(\mathbf{B}'))$ (see also Section 2). The following lemma ensures that this gives a representation of $L^\epsilon(\mathbf{B})$ to which Lemma 30 becomes applicable, provided that we can compute it. Its proof is in Section 4.4.

Lemma 31. *Let $\mathfrak{A}$ be a semisimple $*$-algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$. Let $a \in \mathfrak{A}$ be a $*$-symmetric zero-divisor. Then there exists a $*$-symmetric element $b \in \mathfrak{A}$, such that $b\mathrm{Ann}_r(a) \not\subseteq a\mathfrak{A}$, where $\mathrm{Ann}_r(\cdot)$ denotes the set of right annihilators.*

Indeed, if b is as in Lemma 31 in a semisimple $\mathfrak{A}$, then viewing a and b as linear maps on $\mathfrak{A}$ (by multiplication from the left), Lemma 30 gives that we have that for some $\lambda \in S \subseteq \mathbb{F}$, $|S| > \dim(\mathfrak{A})$, $\dim((b + \lambda a)\mathfrak{A}) > \dim(a\mathfrak{A})$. (When working with non-semisimple algebras, we also make use the simple fact that an element of an algebra is a unit if and only if it is a unit modulo the radical.)

Thus we wish to work with $\mathrm{Adj}(\mathbf{B}')$ and the dimension of the image of the left multiplication of its symmetric elements, that is, dimension of right ideals of the form $X\mathrm{Adj}(\mathbf{B}')$, $X \in L^\epsilon(\mathbf{B}')$ – modulo the radical of $\mathrm{Adj}(\mathbf{B}')$. But as $\mathbf{B}'$ is not in our hand, $\mathrm{Adj}(\mathbf{B}')$ and $L^\epsilon(\mathbf{B}')$ are not either. In fact $\mathbf{B}'$ is not even uniquely determined by $\mathbf{B}$. These difficulties can be overcome as follows.

- For $\mathrm{Adj}(\mathbf{B}')$, though $\mathbf{B}$ is not ϵ-symmetric, we may still define the adjoint algebra of $\mathbf{B}$ as $\mathrm{Adj}(\mathbf{B}) = \{A \oplus D \in M(n,\mathbb{F})^{op} \oplus M(n,\mathbb{F}) \mid \forall i \in [m], A^t B_i = B_i D\}$. However, while $\mathrm{Adj}(\mathbf{B}')$ is naturally a $*$-algebra by $(A \oplus D)^* = D \oplus A$, $\mathrm{Adj}(\mathbf{B})$ is not. But the following relation is easy to verify: $A \oplus D \in \mathrm{Adj}(E\mathbf{B}') \Leftrightarrow E^t A E^{-t} \oplus D \in \mathrm{Adj}(\mathbf{B}')$. This is because

$A^tEB'_i = EB'_iD \Leftrightarrow E^{-1}A^tEB'_i = B'_iD \Leftrightarrow (E^tAE^{-t})^tB'_i = B'_iD$. So the projection of $\mathrm{Adj}(\mathbf{B})$ to the second component coincides with the projection of $\mathrm{Adj}(\mathbf{B}')$ to the second component.

- To get around the lack of $L^\epsilon(\mathbf{B}')$ is trickier. We first observe that $L^\epsilon(E\mathbf{B}F) = F^tL^\epsilon(\mathbf{B})E^{-1}$. Since $\mathbf{B} = E\mathbf{B}'$, $L^\epsilon(\mathbf{B}) = L^\epsilon(\mathbf{B}')E^{-1}$ so any $Z \in L^\epsilon(\mathbf{B})$ equals XE^{-1} for some $X \in L^\epsilon(\mathbf{B}')$. Then consider $XL^\epsilon(\mathbf{B}')$: we have $XL^\epsilon(\mathbf{B}') = XE^{-1}EL^\epsilon(\mathbf{B}') = ZL^\epsilon(\mathbf{B}'E^t) = ZL^\epsilon(\epsilon\mathbf{B}''^tE^t) = ZL^\epsilon(\epsilon(E\mathbf{B}')^t) = ZL^\epsilon(\epsilon\mathbf{B}^t)$. Here we use the assumption that $\mathbf{B}' \in S^\epsilon(n,\mathbb{F})^m$.

As $L^\epsilon(\mathbf{B}') \subseteq \mathrm{Adj}(\mathbf{B}')$, $L^\epsilon(\mathbf{B}')\mathrm{Adj}(\mathbf{B}') = \mathrm{Adj}(\mathbf{B}')$. Therefore, for any $Z \in L^\epsilon(\mathbf{B})$, $ZL^\epsilon(\epsilon\mathbf{B}^t)\mathrm{Adj}(\mathbf{B}) = XL^\epsilon(\mathbf{B}')\mathrm{Adj}(\mathbf{B}') = X\mathrm{Adj}(\mathbf{B}')$ for some $X \in L^\epsilon(\mathbf{B}')$. Noting that $L^\epsilon(\mathbf{B})$, $L^\epsilon(\epsilon\mathbf{B}^t)$, and $\mathrm{Adj}(\mathbf{B})$ are what we can compute, this allows us to work with the right ideals of $\mathrm{Adj}(\mathbf{B}')$ generated by $X \in L^\epsilon(\mathbf{B}')$ without knowing the hidden $\mathbf{B}'$.

The arguments above lead to the following algorithm, assuming that $|\mathbb{F}| > n^2$ and $\mathrm{Rad}(\mathfrak{A})$ can be computed efficiently over $\mathbb{F}$. Fix $S \subseteq \mathbb{F}$ of size $> n^2$, and perform the following:

1. Compute a basis of $L^\epsilon(\mathbf{B}) = \langle Z_1, \ldots, Z_s\rangle$, and choose some $Z \in L^\epsilon(\mathbf{B})$.
2. If Z is full-rank, return Z. Otherwise, compute $R_Z = ZL^\epsilon(\epsilon\mathbf{B}^t)\mathrm{Adj}(\mathbf{B})$.
3. If there exist $i \in [\ell]$ and $\lambda \in S$ such that $\dim(R_{\lambda Z+Z_i} + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B}))) > \dim(R_Z + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B}))$, let $Z \leftarrow \lambda Z + Z_i$ and go to Step (2). Otherwise return "Not ϵ-symmetrizable".

It is clear that the algorithm uses polynomially many arithmetic operations, and over number fields the bit sizes are controlled well. The correctness follows from Lemma 31, and we only need to exclude a false negative outcome. Assume to this end that $\mathbf{B}$ is ϵ-symmetrizable and $Z \in L^\epsilon(\mathbf{B})$ is not of full rank. With $X = ZE$ and $X_i = Z_iE$, we have $R_Z + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B})) = ZL^\epsilon(\epsilon\mathbf{B}^t)\mathrm{Adj}(\mathbf{B}) + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B})) = XL^\epsilon(\mathbf{B}')\mathrm{Adj}(\mathbf{B}') + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B}'))$, which is essentially (that is, modulo $\mathrm{Rad}(\mathrm{Adj}(\mathbf{B}))$) the image of $\ell_{\overline{X}}$, where $\overline{X}$ stands for the residue class of X modulo $\mathrm{Rad}(\mathrm{Adj}(\mathbf{B}))$. As Z is not of full rank, X is not of full rank either, and hence $X\mathrm{Adj}(\mathbf{B}') < \mathrm{Adj}(\mathbf{B}')$. Then, as $\mathrm{Rad}(\mathrm{Adj}(\mathbf{B}'))$ is the intersection of the maximal right ideals of $\mathrm{Adj}(\mathbf{B}')$, $X\mathrm{Adj}(\mathbf{B}') + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B}')) < \mathrm{Adj}(\mathbf{B}')$. For $i = 1, \ldots, s$, we have that $R_{\lambda Z+Z_i} + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B}))$ is essentially the image of $\lambda\ell_{\overline{X}} + \ell_{\overline{X}_i}$. Now by Lemma 31, if $\ell_{\overline{X}}$ is not of full rank, that is $XL^\epsilon(\mathbf{B}') + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B}')) \neq \mathrm{Adj}(\mathbf{B}')$, or, equivalently, $ZL^\epsilon(\mathbf{B}') + \mathrm{Rad}(\mathrm{Adj}(\mathbf{B})) \neq \mathrm{Adj}(\mathbf{B})$, then there exists an element λ and a linear combination b of the $\ell_{\overline{X}_i}$ such that $\lambda\ell_{\overline{X}} + b$ has larger rank that that of $\ell_{\overline{X}}$. By linearity, b can be chosen from $\ell_{\overline{X}_i}$ $(i = 1, \ldots, s)$. But then $R_{\lambda Z+Z_i}$ will be, modulo $\mathrm{Rad}(\mathrm{Adj}(\mathbf{B}'))$, indeed bigger than R_Z.

4.2 When $|\mathbb{F}|$ is large enough

Suppose $|\mathbb{F}| = \Omega(n^4)$. We shall extend the algorithm in Section 4.1 to work without relying on the presence of the radical of $\mathrm{Adj}(\mathbf{B})$. To that end we need some objects to measure "progress" modulo the radical without actually having the radical at hand. These objects are the right ideals which are, as modules, projective. We summarize here definition and the basic facts known about them, see [Pie82], Chapter 6, in particular Section 6.4 for details.

Let $\mathfrak{A}$ be an algebra of dimension d with identity. Projective modules are direct summands of free modules. More specifically, free right $\mathfrak{A}$ modules are just direct sums of copies of the right $\mathfrak{A}$-module $\mathfrak{A}$ itself, and we say that a submodule M_1 is a direct summand of the module M, if M is the direct sum of M_1 and another submodule M_2. Right ideals that are projective

modules are just the direct summands of $\mathfrak{A}$. A right ideal P is projective if it is generated by an idempotent: $P = e\mathfrak{A}$ for some idempotent $e \in \mathfrak{A}$. (This is equivalent to saying that e is a left identity element of P.) Every projective right $\mathfrak{A}$-module P can be decomposed into the direct sum of indecomposable projective modules. A projective module P is indecomposable, if and only if its *head* $P/P\mathrm{Rad}(\mathfrak{A})$ is a simple $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$-module. Two indecomposable projective modules are isomorphic if and only if their heads are isomorphic. Projective indecomposable modules are also called principal indecomposable, and they appear as projective right ideals of $\mathfrak{A}$. Every projective right $\mathfrak{A}$-module can be decomposed into a direct sum of principal indecomposable modules. The decomposition is, up to the isomorphism types of the principal indecomposable with multiplicities, is unique. This theory, when specialized to right ideals of $\mathfrak{A}$, gives that right ideals that are projective as modules (for brevity, we will call them projective right ideals) can be decomposed into direct sums of indecomposable right ideals, which are also projective. The multiplicities of the various principal indecomposable modules in P are the same as the number of various simple components of the factor $P/\mathrm{Rad}(\mathfrak{A})P$ as a right module over the semisimple algebra $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$. Note that for a projective right ideal P, we have $\mathrm{Rad}(\mathfrak{A})P = P \cap \mathrm{Rad}(\mathfrak{A})$, which can be shown easily using a generating idempotent. It follows that $P/\mathrm{Rad}(\mathfrak{A})P \cong (P + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$. Here we have that $(P + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$ is a right ideal of $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$. If $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$ is the direct sum of simple algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_\ell$, then $(P + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$ is decomposed into the direct sum $P_1, \ldots, P_\ell$ where $P_i = ((P + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})) \cap \mathfrak{A}_i$. Now each P_i can be decomposed into a sum of minimal right ideals of $\mathfrak{A}_i$, where the number of such components are the multiplicities of the various principal indecomposables in the decomposition of P.

Here we show that, given a right ideal J of $\mathfrak{A}$, a projective right ideal P can be computed with the property that $(P + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A}) = (J + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$. By the discussion above, the module structure of P depends only on the factor $(J + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$, and hence can be used to (partially) compare right ideals modulo the radical. The key property is the equivalent characterization of projective right ideals as those generated by idempotents of $\mathfrak{A}$, see [Pie82], Section 6.4.

Proposition 32. *Let $\mathfrak{A}$ be a finite dimensional algebra with identity and let J be a non-nilpotent right ideal of $\mathfrak{A}$. Then in deterministic polynomial time one can compute a right ideal J_0 contained in J generated by an idempotent e such that $e + \mathrm{Rad}(\mathfrak{A})$ is a left identity element of $(J + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$.*

Proof. To compute J_0, it is sufficient to find an idempotent e of J with the property as in the statement. As J is not nilpotent, one can find a non-nilpotent element and even an idempotent e in J, as shown in Fact 11 and the remark following its proof. Compute the right ideal $J'' = \{x - ex : x \in J\}$. Obviously $e\mathfrak{A} \cap J'' = 0$. If J'' is nilpotent, then e is as requested. Otherwise find an idempotent f in J''. We have $ef = 0$ and $(e + fe)^2 = ee + fefe + efe + fee = e + fe$. So if $fe \neq 0$, then we can replace e with $e + fe$ which generates a right ideal larger than $e\mathfrak{A}$. If $fe = 0$, then $(e + f)^2 = e + f$ whence we can proceed with $e + f$ in place of e.

Over a number field, some care is needed to ensure that size of the data representing the idempotent e do not explode. In order to do this, we fix a basis for J and express e in terms of that basis. Let $n = \dim J$. We consider the matrix representation of J on itself by action from the left. Then for every element $x \in J$, let $N_x = \{v \in J : x^n v = 0\}$ be the generalized 0-eigenspace of x. We have $J = x^n J \oplus N_x$, and there is an idempotent e_x in the subalgebra generated by x^n with $e_x J = x^n J$. Assume that we have an idempotent $e \in J$ at hand with $\dim(eJ) = r$. Considering

e as an n by n matrix, we have that $e = e^n$ has rank r: there is an r by r submatrix whose determinant of e^n is nonzero. We consider this determinant for the nth power of the matrix of a generic element x form J. This determinant has degree at most $nr < n^2$ in the coordinates of x. We have at hand e, that is a specific assignment for the coordinates on which this polynomial takes a nonzero value. Given a subset Λ of size n^2 of $\mathbb{F}$, we can replace the first coordinate of e by an element of Λ such that the for the new element x, its power x^n has rank at least r. Then we can proceed with the second coordinate, and so on, see [dGIR96, Lemma 2.2] for a formal statement. When finished, we have an element x of small coordinates such that x^n has still rank at least r. Now the identity element e_x of the subalgebra of J generated by x^n will have the same rank and still moderate coordinates. (This algebra is spanned by $x^n, x^{2n}, \ldots, x^{n^2}$.) We replace e with e_x and continue increasing its rank if $(1-e)J$ is not nilpotent. □

We call the right ideal J_0 as in Proposition 32 the projective module associated to J, and denote it by $P(J)$. For a nilpotent right ideal J we set $P(J) = 0$. As the decomposition of $P(J)$ into principal indecomposables reflects faithfully the decomposition of $(J + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$ into simple modules, the map $J \mapsto P(J)$ is *monotone* in J modulo the radical, in the following sense: if $(J + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$ is isomorphic to a proper submodule of $(J' + \mathrm{Rad}(\mathfrak{A}))/\mathrm{Rad}(\mathfrak{A})$, then $P(J)$ is isomorphic to a proper submodule of $P(J')$.

Fact 33. *Let $\mathfrak{A}$ be a finite dimensional semisimple algebra with identity, let $\mathfrak{A}_1, \ldots, \mathfrak{A}_\ell$ be the simple components of $\mathfrak{A}$, and let $\pi_j : \mathfrak{A} \to \mathfrak{A}_j$, $j \in [\ell]$, be the corresponding projections. Suppose that e and f are idempotents in $\mathfrak{A}$ such that the rank of $\pi_j(e)$ is the same as that of $\pi_j(f)$ for $j = 1, \ldots, \ell$. Then $e\mathfrak{A}$ and $f\mathfrak{A}$ are isomorphic as right $\mathfrak{A}$-modules.*

Proof. Indeed, for each individual j, $\pi_j(e\mathfrak{A})$ and $\pi_j(f\mathfrak{A})$, respectively, are direct sums of minimal right ideals of $\mathfrak{A}_j$, which are, as modules, isomorphic copies of the same simple right $\mathfrak{A}$-module S_j. Here we used the fact that, for the simple algebra $\mathfrak{A}_j$, up to isomorphism there is one simple right module, which is present in $\mathfrak{A}_j$ as a minimal right ideal. The multiplicity of S_j in the decomposition of $e\mathfrak{A}$ (resp. $f\mathfrak{A}$) is then just the quotient of the rank of $\pi_j(e)$ (resp. $\pi_j(f)$) by the dimension of S_j. □

Now we are ready to upgrade the algorithm in Section 4.1 to work without knowing the radical of $\mathrm{Adj}(\mathbf{B})$.

Proposition 34. *Let $\mathfrak{A}$ be an m-dimensional algebra with identity, let a be a zero-divisor in $\mathfrak{A}$ and let $b \in \mathfrak{A}$ such that $a + \mathrm{Rad}(\mathfrak{A})$ and $b + \mathrm{Rad}(\mathfrak{A})$ behave like a and b in Lemma 31. If S is a subset of the base field of size $\Omega(m^2)$, then for at least one $\lambda \in S$ we have $\dim P((\lambda a + b)\mathfrak{A}) > \dim P(a\mathfrak{A})$.*

Proof. We have that, modulo $\mathrm{Rad}(\mathfrak{A})$, $\lambda a + b$ generates a right ideal that has dimension higher than the one generated by a for at least one field element λ from S if $|S| = \Omega(m)$. If S is even larger, say $|S| = \Omega(m^2)$, then S will contain such a field element λ with the additional property that, the projection of $(\lambda a + b)\mathfrak{A}$ to any of the simple components of $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$ has dimension at least as high as that for the projection of $a\mathfrak{A}$. The existence of such a λ is ensured by applying Lemma 30 iteratively to the projections to the simple components. Then, by Fact 33, the right $\mathfrak{A}$-module $a\mathfrak{A} + \mathrm{Rad}(\mathfrak{A})$ can be embedded into $(\lambda a + b)\mathfrak{A} + \mathrm{Rad}(\mathfrak{A})$ as a proper submodule. By monotonicity as explained in the paragraph before Fact 33, $P(a\mathfrak{A})$ is isomorphic to a proper submodule of $P((\lambda a + b)\mathfrak{A})$. □

4.3 When $|\mathbb{F}|$ is small

The algorithm in Section 4.1, upgraded in Section 4.2, runs in polynomial time even over a number field, but has the disadvantage of relying on the field to be large enough. In this subsection, we present an algorithm that works even for small fields. However, the disadvantage of this algorithm is that, over a number field it seems difficult to bound the bit sizes of intermediate data. Still, combining these two algorithms together we are able to cover all fields, so this proves Theorem 8.

As explained in Section 4.1, w.l.o.g. we can assume $\mathbf{B}$ to be non-degenerate. The following Lemma 35 is the key to this algorithm. Its proof is put in Section 4.4.

Lemma 35. *Let $\mathbb{F}$ be a field of characteristic not 2. Let $\mathfrak{A}$ be a finite dimensional $*$-algebra over $\mathbb{F}$ with an identity element. Let a be a $*$-symmetric element of $\mathfrak{A}$ such that the right ideal $a\mathfrak{A}$ has a left identity element. Then the right annihilator $\mathrm{Ann}_r(a) = \{b \in \mathfrak{A} : ab = 0\}$ of a is generated, as a right ideal, by a $*$-symmetric element of $\mathfrak{A}$.*

We remark that the condition that $a\mathfrak{A}$ has a left identity element is just equivalent to that $a\mathfrak{A}$ is projective as a right $\mathfrak{A}$-module; see Section 4.2 for this concept.

We shall only sketch the idea behind the algorithm in the following; a rigorous algorithm can be extracted without much difficulty.

Suppose $\mathbf{B} = E\mathbf{B}'$ where $E \in \mathrm{GL}(n, F)$ and $\mathbf{B}' \leq S^\epsilon(n, \mathbb{F})$. We claim that $L^\epsilon(\mathbf{B}')$ cannot be spanned by nilpotent elements. Indeed, assume the contrary. Let $\mathfrak{A} = \mathrm{Adj}(\mathbf{B}')$, which is a $*$-algebra as $\mathbf{B}'$ is ϵ-symmetric. Then $I \otimes *$ is an involution of $\overline{\mathfrak{A}} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{A}$, where $\overline{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$. We identify $\mathfrak{A}$ with the subalgebra $1 \otimes \mathfrak{A}$ and use $*$ for $I \otimes *$. The $*$-symmetric elements of $\overline{\mathfrak{A}}$ are $\overline{\mathbb{F}}$-linear combinations of $*$-symmetric elements of $\mathfrak{A}$. Using this, we may assume that $\mathbb{F}$ is algebraically closed. Then the $*$-simple components of the factor of $\mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$ contain $*$-symmetric idempotents whose images are rank one or two matrices under some irreducible representation of $\mathfrak{A}$. It follows that any basis for $L^\epsilon(\mathbf{B}')$ contains an element whose image under a matrix representation of $\mathfrak{A}$ has nonzero trace. Such an element cannot be nilpotent.

Thus any basis of $L^\epsilon(\mathbf{B}) = L^\epsilon(\mathbf{B}')E^{-1}$ contains an element of the form $Z = XE^{-1}$ where X is a non-nilpotent element of $L^\epsilon(\mathbf{B}')$. Now consider the subspace $XL^\epsilon(\mathbf{B}')X$. This set equals the set of the $*$-symmetric elements of the subalgebra $X\mathfrak{A}X$. This subalgebra is not nilpotent, as it contains the non-nilpotent element X. Therefore, just like above, an arbitrary basis for $XL^\epsilon(\mathbf{B}')X$ contains a non-nilpotent element. It follows that an arbitrary basis for $L^\epsilon(\mathbf{B}')$ (which may differ from the basis which X is chosen from) contains an element Y such that XYX is not nilpotent. In particular, a basis for $L^\epsilon(\epsilon\mathbf{B}^t) = EL^\epsilon(\mathbf{B}')$ contains an element Z' of the form $Z' = EY$ where XYX is not nilpotent. Now consider the sequences $X_k = X(YX)^k$ and $Y_k = Y(XY)^k$, $k \geq 0$. We have $X_0 = X$, $Y_0 = Y$, $X_{k+1} = XY_kX$, and $Y_{k+1} = YX_kY$. Furthermore $X_{k+1}E^{-1} = (XE^{-1})(EY_k)(XE^{-1})$ and $EY_{k+1} = (EY)(X_kE^{-1})(EY)$, which gives an efficient method for computing X_kE^{-1} and EY_k. The kernels of X_k form a nondecreasing chain of linear spaces. Therefore if k is large enough, then $\ker X_\ell = \ker X_k$ for $\ell > k$. The sequences consisting of the kernels of Y_k, as well as those consisting of the images of X_k and the images of Y_k, stabilize as well. From $Y_{2k+1} = YX_kY_k$, we infer that for sufficiently large k the kernel of X_kY_k is the same as that of Y_k, and the image of X_kY_k is the same as that of X_k. Analogous equalities hold for the kernel and for the image of Y_kX_k. These properties of the pair X_k, Y_k imply that the image of Y_k is a direct complement of the kernel of X_k, and the image of X_k is a direct complement of the kernel of Y_k.

As $X_kY_k = X_kE^{-1}EY_k$, we can efficiently compute the product $X_kY_k \in \mathrm{Adj}(\mathbf{B}')$, which cannot be zero. Note that if X_kY_k is invertible, then X is also invertible, and the XE^{-1} in our hand sends

$\mathbf{B}$ to $\mathbf{B}'$, which solves the problem. So in the following we assume X_kY_k has a non-trivial kernel.

Similarly to the stabilization argument above, we may assume that k is large enough so that the kernel of X_kY_k in the left regular representation $\mathrm{Adj}(\mathbf{B}')$ is a direct complement of the image. This mean that the right annihilator of X_kY_k in $\mathrm{Adj}(\mathbf{B}')$ (which is the same as that of Y_k) and the right ideal generated by X_kY_k (which is also generated by X_k) are complementary to each other and the same holds for the product Y_kX_k.

We claim that there exists $\mathbf{B}'' \leq S^\epsilon(n,\mathbb{F})$ such that $\mathbf{B} = E'\mathbf{B}''$ for some invertible E' and $X_kY_k \in L^\epsilon(\mathbf{B}'')$. To see this, consider an element $Z \in L^\epsilon(\mathbf{B}')$ which is a generator of the right annihilator of X_k as a right ideal in $\mathrm{Adj}(\mathbf{B}')$. Such Z exists by Lemma 35. Put $W = Y_k + Z$. Then $W \in L^\epsilon(\mathbf{B}')$, and W is invertible since Y_k and Z are generators of right ideals of $\mathrm{Adj}(\mathbf{B}')$ complementary to each other. We also have $X_kW = X_k(Y_k + Z) = X_kY_k$. Let $\mathbf{B}'' = W^{-1}\mathbf{B}'$. Then, W^{-1} is an invertible element of $L^\epsilon(\mathbf{B}')$, so we have $\mathbf{B}'' \leq S^\epsilon(n,\mathbb{F})$. Furthermore, $L^\epsilon(\mathbf{B}'') = L^\epsilon(W^{-1}\mathbf{B}') = L^\epsilon(\mathbf{B}')W$. In particular, $X_kY_k = X_kW \in L^\epsilon(\mathbf{B}')W = L^\epsilon(\mathbf{B}'')$.

Let J (resp. K) be the image (resp. the kernel) of X_kY_k. From $X_kY_k \in L^\epsilon(\mathbf{B}'')$ we infer $J = K^{\perp_{\mathbf{B}''}}$. Let $J' = K^{\perp_{\mathbf{B}}}$ and $K' = J^{\perp_{\mathbf{B}}}$. These subspaces can be computed efficiently. Let U_0 be an invertible linear map that maps J to J' and K to K'. Then by replacing $\mathbf{B}$ with $U_0^t\mathbf{B}$ we can arrange that $J = K^{\perp_{\mathbf{B}}}$ as well. Then the problem can be reduced to the subspaces J and K.

4.4 Two lemmas about $*$-algebras

For the next two lemmas, we depend crucially on the structure of $*$-algebras as described in the first paragraph of "Structure of $*$-algebras" in Section 2. We begin with a claim which is used in the proofs of both.

Claim 36. *Let $\mathfrak{A}$ be a semisimple $*$-algebra, and let $a \in \mathfrak{A}$ be a $*$-symmetric zero divisor, such that there is no proper $*$-symmetric idempotent e with $ae = ea$. Then a is nilpotent and $\mathfrak{A}$ is $*$-simple. Furthermore, either*

(i) $\mathfrak{A} \cong M(n,\mathcal{D})$ for a division algebra $\mathcal{D}$ and a, as an n by n matrix over $\mathcal{D}$ has just one Jordan block;

(ii) There exists a proper idempotent $f \in \mathfrak{A}$ with $af = fa$ and $f^* = 1 - f$, $f\mathfrak{A}f \cong M(n,\mathcal{D})$ for some division algebra $\mathcal{D}$ such that faf, as an n by n matrix over $\mathcal{D}$ has just one Jordan block.

We remark that case (ii) captures two sub-cases: either f is in the center of $\mathfrak{A}$, and $\mathfrak{A}$ is of exchange type; or $\mathfrak{A} \cong M_{2n}(\mathcal{D})$ and a, as an $2n$ by $2n$ matrix, has two Jordan blocks of size n.

Proof. Let $\mathcal{C}$ be the centralizer of a in $\mathfrak{A}$, that is, $\mathcal{C} = \{x \in \mathfrak{A} : xa = ax\}$. Then it is straightforward to see that $\mathcal{C}$ is a $*$-subalgebra of $\mathfrak{A}$ containing a. Then, as subalgebras generated by non-nilpotent zero divisors do contain nontrivial idempotents, a is nilpotent. Furthermore, as the center of $\mathfrak{A}$ is contained in $\mathcal{C}$, there are no $*$-invariant central idempotents in $\mathfrak{A}$. In other words, $\mathfrak{A}$ is $*$-simple: it is either simple or of exchange type consisting of two simple components.

Furthermore, every $*$-symmetric element of $\mathcal{C}$ is either nilpotent or invertible, that is, $\mathcal{C}$ is Osborn-local, whence $\mathcal{C}/\mathrm{Rad}(\mathcal{C})$ is an Osborn-division algebra. By Osborn's theorem [Osb70, Theorem 2], $\mathcal{C}/\mathrm{Rad}(\mathcal{C})$ cannot contain three or more pairwise orthogonal idempotents, and if it contains any proper idempotent then it also contains a proper idempotent $\overline{f}$ with $\overline{f}^* = 1 - \overline{f}$. We claim that in the latter case there exists a proper idempotent f in $\mathcal{C}$ such that $f^* = 1 - f$. Such an f can be constructed using the following iteration. Let J be an ideal of $\mathcal{C}$ contained in $\mathrm{Rad}(\mathcal{C})$, and f be an

element of $\mathcal{C}$ such that $f^2 - f \in J$ and $f + f^* - 1 \in J$. Initially, $J = \mathrm{Rad}(\mathcal{C})$ and f is an arbitrary element of the coset $\overline{f}$. The iterative step starts with arranging that $f^2 - f \in J^2$: this can be done by any standard lifting technique, e.g., by replacing f with $3f^2 - 2f^3 \in f + J$ (see [DK94], proof of Lemma 3.2.1). Then, as this new f is from the same residue class modulo J as the old one, the property $f + f^* - 1 \in J$ is preserved. Next we put $r = \frac{1}{2}(f + f^* - 1) \in J$ and $f' = f - r$. Then we have $f' + f'^* - 1 = f + f^* - 2r - 1 = 0 \in J^2$. Furthermore, $f^{*2} - f^* \in J^2$ can be rewritten as $((1 - f) + 2r)^2 - (1 - f) - 2r \in J^2$, which implies $2r - 2fr - 2rf + (f^2 - f) + 4r^2 \in J^2$, whence $-r+rf+fr \in J^2$. It follows that $f'^2-f' = f^2-rf-fr+r^2-(f-r) = (f^2-f)-(rf+fr-r)+r^2 \in J^2$. Therefore, we can proceed with f' in place of f and J^2 in place of J. Note that if $\mathfrak{A}$ is of exchange type, then f can be even chosen as the identity element of one of the simple components of $\mathfrak{A}$. In any case, the subalgebra $f\mathfrak{A}f$ must be simple, so isomorphic to $M(n, \mathcal{D})$ for some n and $\mathcal{D}$. Assume that faf has more than one Jordan blocks. Let $e \in f\mathfrak{A}f$ be the block diagonal matrix which is the identity in one of the Jordan blocks of a and zero elsewhere. Then e is an idempotent commuting with a with $ef = e \neq f$. Then $e^* \in f^*\mathfrak{A}f^*$ and $e + e^*$ is a proper idempotent commuting with a. So faf, as an n by n matrix over $\mathcal{D}$, must consist of single (nilpotent) Jordan block of size n.

If $\mathcal{C}$ does not contain proper idempotents then $\mathfrak{A}$ is itself simple, isomorphic to $M(n, \mathcal{D})$ and again, a has just one (nilpotent) Jordan block of size n. □

Lemma 35, restated. Let $\mathbb{F}$ be a field of characteristic not 2. Let $\mathfrak{A}$ be a finite dimensional $*$-algebra over $\mathbb{F}$ with an identity element. Let a be a $*$-symmetric element of $\mathfrak{A}$ such that the right ideal $a\mathfrak{A}$ has a left identity element. Then the right annihilator $\mathrm{Ann}_r(a) = \{b \in \mathfrak{A} : ab = 0\}$ of a is generated, as a right ideal, by a $*$-symmetric element of $\mathfrak{A}$.

Proof. Note that $e \in \mathfrak{A}$ is a left identity element of the right ideal $a\mathfrak{A}$ if and only if $ea = a$ and there exists $d \in \mathfrak{A}$ such that $e = ad$. Let e be such an element. Then $e^* = d^*a$ is a right identity element of the left ideal $\mathfrak{A}a$. We claim $\mathrm{Ann}_r(a)$ is the right ideal of $\mathfrak{A}$ generated by the idempotent $1 - e^*$. Indeed, assume that $ab = 0$. Then $e^*b = d^*ab = 0$. Conversely, if $e^*b = 0$ then $ab = ae^*b = 0$. Thus $b \in \mathrm{Ann}_r(a)$ if and only if $e^*b = 0$. The latter equality is equivalent to that $b = (1 - e^*)b'$ for some $b' \in \mathfrak{A}$.

Next we show that we may assume that $\mathfrak{A}$ is semisimple. To see this, let e be an idempotent as above. Then $\mathrm{Ann}_r(a) = (1 - e^*)\mathfrak{A}$. Let ϕ be the projection $\mathfrak{A} \to \overline{\mathfrak{A}} := \mathfrak{A}/\mathrm{Rad}(\mathfrak{A})$. We denote the involution of $\overline{\mathfrak{A}}$ induced also by $*$. Obviously, $\phi(e)$ is an idempotent in $\phi(a)\overline{\mathfrak{A}}$ with $\phi(e)\phi(a) = \phi(a)$. It follows that the right annihilator of $\phi(a)$ is generated by $1 - \phi(e^*)$, whence it coincides with $\phi(\mathrm{Ann}_r(a))$. Similarly, the left annihilator of $\phi(a)$ is the left ideal of $\Phi(\mathfrak{A})$ generated by $(1 - \phi(e))$ and it coincides with $\phi(\mathrm{Ann}_l(a))$. It follows that

$$\mathrm{Ann}_r(\phi(a)) \cap \mathrm{Ann}_l(\phi(a)) = \phi(\mathrm{Ann}_r(a) \cap \mathrm{Ann}_l(a)). \tag{1}$$

(The annihilators on the left hand side are understood as inside $\overline{\mathfrak{A}}$.) Assume that the assertion holds in $\overline{\mathfrak{A}}$. Then there is an element $\overline{b}$ of $\overline{\mathfrak{A}}$ such that $\overline{b}^* = \overline{b}$, and $\overline{b}$ generates the right annihilator of $\phi(\mathfrak{A})$ in $\overline{\mathfrak{A}}$. Notice that $\overline{b}$ annihilates $\phi(a)$ from the left as well. Therefore, by Equation 1, $\overline{b}$ has a preimage b in $\mathrm{Ann}_r(a) \cap \mathrm{Ann}_l(a)$. We have $b - b^* \in \mathrm{Ann}_r(a) \cap \mathrm{Ann}_l(a) \cap \mathrm{Rad}(\mathfrak{A})$. Therefore, by replacing b with $b - \frac{1}{2}(b - b^*)$ we can arrange that $b^* = b$. We have $b\mathfrak{A} + \mathrm{Rad}(\mathfrak{A}) = \mathrm{Ann}_r(a) + \mathrm{Rad}(\mathfrak{A})$. The isomorphism theorem, applied to the linear spaces $b\mathfrak{A} + (\mathrm{Ann}_r(\mathfrak{A}) \cap \mathrm{Rad}(\mathfrak{A}))$, $\mathrm{Ann}_r(a)$, and $\mathrm{Rad}(\mathfrak{A})$, gives $b\mathfrak{A} + (\mathrm{Ann}_r(\mathfrak{A}) \cap \mathrm{Rad}(\mathfrak{A}))/(\mathrm{Ann}_r(a) \cap \mathrm{Rad}(\mathfrak{A}) \cong \mathrm{Ann}_r(a))/(\mathrm{Ann}_r(a) \cap \mathrm{Rad}(\mathfrak{A}))$. It follows that $b\mathfrak{A} + (\mathrm{Ann}_r(a) \cap \mathrm{Rad}(\mathfrak{A})) = \mathrm{Ann}_r(a)$. From $\mathrm{Ann}_r(a) = (1 - e^*)\mathfrak{A}$, we infer that the

radical of $\mathrm{Ann}_r(a)$ as a right $\mathfrak{A}$-module is $\mathrm{Ann}_r(a)\mathrm{Rad}(\mathfrak{A}) = (1-e^*)\mathrm{Rad}(\mathfrak{A}) = \mathrm{Ann}_r(a) \cap \mathrm{Rad}(\mathfrak{A})$. Thus b generates the right $\mathfrak{A}$-module $\mathrm{Ann}_r(a)$ modulo its radical, whence $b\mathfrak{A} = \mathrm{Ann}_r(a)$. Therefore we may indeed assume that $\mathfrak{A}$ is semisimple. Furthermore, by going over the $*$-simple components, we can assume that $\mathfrak{A}$ is even $*$-simple, that is, $\mathfrak{A}$ is either a simple algebra or a direct sum $\mathfrak{B} \oplus \mathfrak{B}^{\mathrm{op}}$ where $\mathfrak{B}$ is a simple algebra and $(\beta, \beta')^* = (\beta', \beta)$.

Assume that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^{\mathrm{op}}$. Then a is of the form (α, α) and $\mathrm{Ann}_r(a)$ consists of pairs (β, β') where $\beta \in \mathrm{Ann}_r(\alpha)$ and $\beta' \in \mathrm{Ann}_l(\alpha)$. Let δ be an invertible element of $\mathfrak{B}$ such that $\alpha\delta$ is an idempotent. Then $1 - \alpha\delta$ is a generator for the right annihilator and for the left annihilator of $\alpha\delta$ inside $\mathfrak{B}$ at the same time. Note that the latter is the same as the left annihilator of α. Put $\gamma = \delta(1 - \alpha\delta)$. Then $\alpha\gamma = 0$ and $\gamma\alpha = 0$. Also, the dimensions of the one-sided ideals generated by γ are the same as those generated by $1 - \alpha\delta$. Therefore γ generates as one sided ideals both the left and the right annihilators of α inside $\mathfrak{B}$. It follows that (γ, γ) is a generator for $\mathrm{Ann}_r(a)$ as a right ideal of $\mathfrak{A}$.

The rest of the proof is for the case where $\mathfrak{A}$ is a simple algebra: $\mathfrak{A} \cong M(n, \mathcal{D})$, where $\mathcal{D}$ is a division algebra. Note that in the (semi-)simple case every one-sided ideal is generated by an idempotent.

We first consider the following case: suppose we have $f \in \mathfrak{A}$ which is a proper idempotent in $\mathfrak{A}$ such that $f^* = f$ and $fa = af$. If $b \in \mathrm{Ann}_r(a)$ then $afb = fab = 0$ and $a(1-f)b = (1-f)ab = 0$, whence $\mathrm{Ann}_r(a)$ is decomposed into the direct sum of $f\mathrm{Ann}_r(a) = f\mathfrak{A} \cap \mathrm{Ann}_r(a)$ and $(1-f)\mathrm{Ann}_r(a) = (1-f)\mathfrak{A} \cap \mathrm{Ann}_r(a)$. From the fact that, in a simple algebra, an arbitrary right ideal J is generated by the subspace Jg for any nonzero idempotent element g, we infer that $f\mathrm{Ann}_r(a) = f\mathrm{Ann}_r(a)f\mathfrak{A}$. We claim that $f\mathrm{Ann}_r(a)f$ is the right annihilator of faf in the subalgebra $f\mathfrak{A}f$. Indeed, if $ab = 0$ then $faffbf = fabf = 0$, demonstrating $f\mathrm{Ann}_r(a)f \subseteq f\mathfrak{A}f \cap \mathrm{Ann}_r(faf)$. To see the reverse inclusion let $fbf \in \mathrm{Ann}_r(faf)$. Then $0 = fafbf = afbf$, whence $fbf \in \mathrm{Ann}_r(a)$ and $fbf = f^2bf^2 \in f\mathrm{Ann}_r(a)f$. Assume by induction that the statement of the lemma holds in the simple $*$-invariant subalgebra $f\mathfrak{A}f$. Then there exists an element fb_1f with $fb_1^*f = fb_1f$ generating $f\mathrm{Ann}_r(a)f$ as a right ideal of $f\mathfrak{A}f$. Then by the discussion above, the right ideal of $\mathfrak{A}$ generated by fb_1f is $f\mathrm{Ann}_r(a)$. Similarly, we can use induction to show the existence of b_2 with $(1-f)b_2(1-f) = (1-f)b_2^*(1-f)$ such that $(1-f)b_2(1-f)$ generates the right ideal $(1-f)\mathrm{Ann}_r(a)$. Then the element $b = fb_1f + (1-f)b_2(1-f)$ is $*$-symmetric generator for the right ideal $\mathrm{Ann}_r(a)$.

We then consider the case when there are no proper $*$-symmetric idempotents in the simple algebra $\mathfrak{A} \cong M(n, \mathcal{D})$ commuting with a. Then, by Claim 36 a is nilpotent, and, as an n by n matrix over $\mathcal{D}$, either has one Jordan block of size n, or has two Jordan blocks of $\frac{n}{2}$. Then the annihilator of a is generated by a^{n-1} or by $a^{\frac{n}{2}-1}$, respectively. □

Lemma 31, restated. Let $\mathfrak{A}$ be a semisimple $*$-algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$. Let $a \in \mathfrak{A}$ be a $*$-symmetric zero-divisor. Then there exists a $*$-symmetric element $b \in \mathfrak{A}$, such that $b\mathrm{Ann}_r(a) \not\subseteq a\mathfrak{A}$, where $\mathrm{Ann}_r(\cdot)$ denotes the set of right annihilators.

Proof. Assume that there exists a proper idempotent f in $\mathfrak{A}$ such that $f^* = f$ and $fa = af$. Then either $fa = af = faf$ is a zero-divisor in $f\mathfrak{A}f$, or $(1-f)a(1-f)$ is a zero-divisor in $(1-f)\mathfrak{A}(1-f)$. (For, if $c_1 \in f\mathfrak{A}f$ such that $f = fafc_1$, and $c_2 \in (1-f)\mathfrak{A}(1-f)$ such that $(1-f) = (1-f)a(1-f)c_2$, then from $fc_1 = c_1$ and $fc_2 = 0$ we infer that $a(c_1 + c_2) = ((faf) + (1-f)a(1-f))(c_1 + c_2) = fafc_1 + (1-f)a(1-f)c_2 = f + (1-f) = 1$.) Assume that faf is a zero-divisor in $f\mathfrak{A}f$. Then by induction, there exist $b_1, c_1 \in f\mathfrak{A}f$ such that $b_1^* = b_1$, $fafc_1 = 0$ and $b_1c_1 \notin faff\mathfrak{A}f$. We have $c_1 \in \mathrm{Ann}_r(a)$ because $0 = fafc_1 = af^2c_1 = afc_1 = ac_1$. We claim that $b_1c_1 \notin a\mathfrak{A}$. Indeed,

assume that $b_1c_1 = ad$ for some $d \in \mathfrak{A}$. Then $fb_1 = b_1$ and $c_1f = c_1$ imply $b_1c_1 = fadf$ and $fadf = faffdf \in faf\mathfrak{A}f$, a contradiction with the assumption.

Based on the above, it is sufficient to prove the assertion when $\mathfrak{A}$ has no proper $*$-symmetric idempotents commuting with a. Then, by Claim 36, $\mathfrak{A}$ is $*$-simple and a is nilpotent. Furthermore, either $\mathfrak{A} \cong M(n, \mathcal{D})$ for some division algebra $\mathcal{D}$ and a has a single Jordan block of size n, or there exists an idempotent f of $\mathfrak{A}$ such that $f^* = 1 - f$, $fa = af$, $f\mathfrak{A}f \cong M(n, \mathcal{D})$ for some division algebra $\mathcal{D}$, and $fa = faf$ has just one Jordan block. The latter case covers two cases from the point of view of the structure of $\mathfrak{A}$: it can be either simple or a sum of two simple components, corresponding to the case whether or not f is central in $\mathfrak{A}$.

In the latter case we consider an isomorphism $\phi : f\mathfrak{A}f \to M(n, \mathcal{D})$ such that $\phi(fa)$ is in Jordan normal form. Then let $b_1 \in f\mathfrak{A}f$ such that $\phi(b_1)$ is everywhere zero except in the lower left corner. Then $b = b_1 + b_1^*$ will do.

When $\mathfrak{A} \cong M(n, \mathcal{D})$ and a has a single Jordan block of size n, to prove the lemma, it is enough to show the following: there exists an appropriate basis for $\mathcal{D}^n$ such that the following holds. First, a, as an n by n matrix, is of Jordan normal form. Second, there is a matrix that is everywhere zero, except at the lower left corner corresponding to a $*$-symmetric element of $\mathfrak{A}$. To this end, let f be a right identity element of the left ideal $\mathfrak{A}a^{n-1}$. Then f is a primitive idempotent in $\mathfrak{A}$, and f^* is a left identity element of the right ideal $a^{n-1}\mathfrak{A}$. As $\mathfrak{A}a^n = a^n\mathfrak{A} = 0$, we have $fa = af^* = 0$ and $ff^* = 0$. We claim that we can arrange that $f^*f = 0$ as well. Indeed, setting $f' = f - f^*f$, we have $f'^*f' = f^*f - f^*ff^* - f^*ff + f^*ff^*f = f^*f - 0 - f^*f - 0 = 0$, $f' = (1 - f^*)f \in \mathfrak{A}f$, and $ff' = f$. The latter two properties show that f' is an identity element of $\mathfrak{A}f$. We replace f with f'. Then f and f^* are orthogonal primitive idempotents.

The subspace $f\mathfrak{A}f^*$ is $*$-invariant. Assume that $f\mathfrak{A}f^*$ does not contain nonzero $*$-symmetric elements. Then for every $c \in f^*\mathfrak{A}f$ we have $c^* = -c$ (other wise $c^* + c$ would be nonzero and $*$-symmetric for some nonzero c.) Put $g = f + f^*$. Then the subalgebra $\mathfrak{A}' = g\mathfrak{A}g$ is isomorphic to $M(2, \mathcal{D})$. Let $\phi : g\mathfrak{A}g \mapsto M(2, \mathcal{D})$ be an isomorphism that maps f and f^* to the block diagonal idempotent matrices $\mathrm{diag}(0, 1)$ and to $\mathrm{diag}(1, 0)$, respectively. Then $\phi(a^{n-1})$ is a matrix which is nonzero exactly at the upper right corner, and $\phi(f\mathfrak{A}f^*)$ consist of matrices whose entries are all zero except possibly that at the lower left corner. It follows that there exists element $c \in f^*\mathfrak{A}f$ such that $\phi(c)$ is nonzero only at the lower left corner, where the entry is the inverse (in $\mathcal{D}$) of the upper right entry of $\phi(a^{n-1})$. For this c we have $a^{n-1}c = f^*$ and $ca^{n-1} = f$. It follows that the subspace spanned by f^*, f, a^{n-1} and c form a subalgebra. (It is actually isomorphic to the algebra of the 2 by 2 matrices over the base field.) The assumption $c^* = -c$ implies that this subalgebra is $*$-invariant. However, it is straightforward to verify that the restriction of $*$ does not give an involution on this subalgebra: $(ca^{n-1})^* = f^* \neq -f^* = -a^{n-1}c$.

Thus there exists a nonzero $*$-symmetric element $c \in f\mathfrak{A}f^*$. We have $a^{n-1} \in \mathrm{Ann}_r(a)$ and $ca^{n-1} \in f\mathfrak{A}f \setminus \{0\}$. Then $ca^{n-1} \notin a\mathfrak{A}$. □

A Comparison with the result of Berthomieu et al. [BFP15]

Recall that in [BFP15], the algorithm works under the two conditions: (1) there exists a non-degenerate form in the linear span of the given form, and (2) the underlying field is large enough. The algorithm needs to find a solution possibly from an extension field. To compare our algorithm with theirs, we first present an algorithm that works under conditions (1) and (2) but does not require going over an extension field. It follows the general principle of our algorithm in Section 3

and suggests the role of a hidden $*$-algebra. This allows us to explain what the algorithm of [BFP15] is like, and why our algorithm avoids using extension fields.

Suppose we are given two tuples of symmetric matrices $\mathbf{B} = (B_1, \ldots, B_m)$ and $\mathbf{C} = (C_1, \ldots, C_m)$, $B_i, C_j \in M(n, \mathbb{F}_q)$ where q is an odd prime power. We aim to find $X \in \mathrm{GL}(n, \mathbb{F}_q)$ such that $\forall i \in [m]$, $X^t B_i X = C_i$. The regularity condition (1) as well as the field size condition (2) in [BFP15] imply that we can compute and therefore assume w.l.o.g. B_1 is non-singular. Therefore B_1 and C_1 must be isometric and we can transform C_1 to B_1 using techniques from Section 3.5.6, so in the following we assume $B_1 = C_1$.

Now define an involution $*$ on $M(n, \mathbb{F}_q)$ by $A^* = B_1^{-1} A^t B_1$. Note that $A^t = B_1 A^* B_1^{-1}$. Then the equation $X^t B_i X = C_i$ is equivalent to $B_1 X^* B_1^{-1} B_i X = C_i$. For $i = 1$ this is just $X^* X = I_n$, or, $X^* = X^{-1}$; in other words, X is a $*$-unitary element. For $i = 2, \ldots, m$, let $B_i' = B_1^{-1} B_i$ and $C_i' = B_1^{-1} C_i$, and consider the system of equations $B_i' X = X C_i'$. If a $*$-unitary element X satisfies $B_i' X = X C_i'$ for $i \in \{2, \ldots, m\}$, then it also satisfies $B_i'^* X = X C_i'^*$ for $i \in \{2, \ldots, m\}$. We use the algorithm for module isomorphism problem to compute a solution $A \in \mathrm{GL}(n, \mathbb{F}_q)$ to the $2m - 2$ equations $B_i' X = X C_i'$ and $B_i'^* X = X C_i'^*$, $i = 2, \ldots, m$. Let $D = \{Y \in M(n, \mathbb{F}_q) \mid \forall i \in \{2, \ldots, m\}, B_i' Y = Y B_i', B_i'^* Y = Y B_i'^*\}$. It is easy to verify that D is a $*$-subalgebra of $M(n, \mathbb{F}_q)$. The set of all solutions to these equations is just $\{YA : Y \in D\}$. The question then becomes to find some such Y so that $(YA)^* YA = I_n$, that is, $Y^* Y = (AA^*)^{-1}$. Note that $A^* A$ is a $*$-symmetric element of D. The problem then becomes to solve the decomposition problem for this $*$-symmetric element in the $*$-symmetric algebra D, which can be solved using the method from Section 3.

The above procedure just differs from the main algorithm in [BFP15] in that the latter algorithm does not solve the decomposition problem in D, but in the smaller (commutative) algebra generated by $A^* A$.

Now we explain why using extension fields is necessary in [BFP15]. Consider the following instance: $m = 2$, $B_1 = C_1 = I_n$, and $B_2 = C_2 = \mathrm{diag}(\omega, \ldots, \omega, 1, \ldots, 1)$ where $\omega \in \mathbb{F}$ is a non-square and appears $2k$ times in B_2. Then by techniques from Section 3.5.6, $B_2 = A^t A$ for some $A \in M(n, \mathbb{F})$. As $(B_1, B_2) = (C_1, C_2)$ it is trivial that these two tuples are isometric. Following the above procedure, we see that $B_2' = C_2' = B_2 = C_2$ as $B_1 = I_n$ and $*$ is just the transposition. Suppose the algorithm for the module isomorphism returns to us A as the solution. Note that A is a valid solution to $B_2 X = X B_2$, as $A^t AA = AA^t A \Leftrightarrow A^t A = AA^t \Leftrightarrow B = B^t$. Then we need to solve $Y^t Y = (AA^t)^{-1} = B_2^{-1}$ where Y is from the centralizing algebra of B_2, which is possible by taking $Y = A^{-1}$. On the other hand, if we insist Y to be from the algebra generated by B_2, this would not possible over $\mathbb{F}$, by noting that there is no diagonal square root of B_2 over $\mathbb{F}$. Therefore [BFP15] would need to go over an extension field to locate a solution.

Acknowledgements. Part of this research was accomplished while the first author was visiting the Centre for Quantum Technologies, National University of Singapore. His research was also partially supported by the Hungarian National Research, Development and Innovation Office – NKFIH, Grant K115288. Y. Q.'s research was supported by the Australian Research Council DECRA DE150100720.

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