How to Teach Control System Optimization
(A Practical Decompisition Approach for the Optimization of *TDOF* Control Systems)

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Abstract: The optimality in a generic two-degree of freedom control system can be decomposed into three major steps, because the control error has three major parts: design-, realizability- and modeling-loss. The second term can be made zero for inverse stable processes only. This decomposition opens new ways for practical optimization of two-degree-of-freedom (*TDOF*) systems and helps the construction of new algorithms for robust identification and control. It is more reasonable to teach the optimization of control systems using these components.

Keywords: two-degree-of-freedom control systems, performance, robustness, optimality

1. INTRODUCTION

Control system optimization is usually based on the error signal or the error transfer function of the closed-loop. The last one is called sensitivity function (*SF*), so any such optimization procedure is strongly connected to the sensitivity or the robustness of control systems. Optimization in classical control theory is a one step procedure. Historically first it was the optimization of an integral criterion, recently the $\mathcal{H}_\infty$ and/or $\mathcal{H}_2$ norm formulated for the control error signal. These procedures were definitely one step methods, when the difficulty arised only from the strongly nonlinear constrained mathematical programming problem, so the education is also concentrated to these problems. These methods do not analyze the internal properties of the control error and the different contributing parts of the sensitivity.

This is why we suggest a decomposition of the original problem, where the separate tasks can be easily understood and are well scaled in the selection parameters and factors. The introduced new decomposition helps to analyze the reachable minimum of the different components, so it is possible to see the theoretical limits of the optimization of control systems and much more understandable for students.

2. CONTROL ERROR DECOMPOSITION

Assume that the pulse transfer function of the discrete-time plant to be controlled is factorable as

$$S = S_+ \bar{S}_- = S_+ S_- z^{-d} = \frac{\mathcal{B}}{\mathcal{A}} = \frac{\mathcal{B}_+ \mathcal{B}_-}{\mathcal{A}} z^{-d} \tag{1}$$

where $S_+ = \mathcal{B}_+ / \mathcal{A}$ means the *inverse stable* (*IS*) and $S_- = \mathcal{B}_-$ the *inverse unstable* (*IU*) factors, respectively. z^{-d} corresponds to the discrete time-delay, where d is the integer multiple of the sampling time. (In a practical case the factor S_- can incorporate the underdamped zeros and neglected poles providing realizability, too).

In a practical case only the model M of the process is known. Assume that the discrete-time model M is similarly factorable as the process in (1)

$$M = M_+ \bar{M}_- = M_+ M_- z^{-d_m} = \frac{\hat{\mathcal{B}}}{\hat{\mathcal{A}}} z^{-d_m} = \frac{\hat{\mathcal{B}}_+ \hat{\mathcal{B}}_-}{\hat{\mathcal{A}}} z^{-d_m} \tag{2}$$

where $M_+ = \hat{\mathcal{B}}_+ / \hat{\mathcal{A}}$ means the *IS*, $M_- = \hat{\mathcal{B}}_-$ the *IU* factors, respectively. z^{-d_m} corresponds to the model discrete time delay, usually $z^{-d_m} = z^{-d}$ is assumed (Wang (1988)).

Introduce the additive

$$\Delta = S - M \tag{3}$$

and relative model errors

$$\ell = \frac{\Delta}{M} = \frac{S - M}{M} \tag{4}$$

The *complementary sensitivity function* (*CSF*) of a *one-degree of freedom* (*ODOF*) control system denoted by T is

$$T = \frac{RS}{1 + RS} = \hat{T} \frac{1 + \ell}{1 + \hat{T}\ell} \quad ; \quad \hat{T} = \frac{RM}{1 + RM} \tag{5}$$

where R is the pulse transfer function of the regulator in the feedback control loop and $\hat{T}$ is the *CSF* of the model based *ODOF* system. Let P_w denote the prescribed *CSF*, which can be considered as the design goal. The *SF* denoted by E, which can be expressed as $E = 1 - T$, can be decomposed into additive components according to different principles (Keviczky et al. (2015), (2018a), (2018b)):

$$E = \underbrace{\left(1 - P_{\mathrm{w}}\right)}_{E_{\mathrm{des}}} + \overbrace{\underbrace{\left(P_{\mathrm{w}} - \hat{T}\right)}_{E_{\mathrm{real}}} - \underbrace{\left(T - \hat{T}\right)}_{E_{\mathrm{id}}}}^{E_{\mathrm{perf}}} = E_{\mathrm{des}} + E_{\mathrm{real}} + E_{\mathrm{id}} \\ = E_{\mathrm{cont}} + E_{\mathrm{id}} \tag{6}$$

Here $E_{\mathrm{des}} = \left(1 - P_{\mathrm{w}}\right)$ is the design, $E_{\mathrm{real}} = \left(P_{\mathrm{w}} - \hat{T}\right)$ is the realizability, $E_{\mathrm{id}} = -\left(T - \hat{T}\right) = \hat{T} - T$ is the modeling (or identification) degradation, respectively. Furthermore $E_{\mathrm{cont}} = \left(1 - \hat{T}\right)$ and $E_{\mathrm{perf}} = \left(P_{\mathrm{w}} - T\right)$ are the overall control and performance degradations, respectively. The *SF* depends on the model-based *SF* ($\hat{E} = 1 - \hat{T}$) as

$$E = \frac{1}{1 + RS} = \hat{E}\frac{1}{1 + \hat{T}\ell} = \hat{E} + E_{\mathrm{id}}\,;\, \hat{E} = \frac{1}{1 + RM} \tag{7}$$

The term E_{id} can be further simplified

$$E_{\mathrm{id}} = E - \hat{E} = \hat{T} - T = -\frac{\hat{T}\hat{E}\ell}{1 + \hat{T}\ell} = -\hat{T}E\ell\Big|_{\ell \to 0} \approx -\hat{T}\hat{E}\ell \tag{8}$$

It is easy to see that $\left|\hat{T}\hat{E}\right|$ has its maximum at the cross over frequency ω_{c}, which means that the model minimizing E_{id} is the most accurate around this medium frequency range. (Note that the accuracy of the estimated model at a given frequency is inverse proportional to the weight in the modeling error at that frequency. The realizability and identification degradations can be called as systematic (E_{syst}) and random (E_{rand}) components, too.

For a two-degree of freedom (*TDOF*) control system (Horowitz, 1963) it is reasonable to request the design goals by two stable and usually strictly proper transfer functions P_{r} and P_{w}, that are partly capable to place desired poles in the tracking and the regulatory transfer functions, furthermore they are usually referred as reference signal and output disturbance predictors. They can even be called as reference models, so reasonably $P_{\mathrm{r}}\left(\omega = 0\right) = 1$ and $P_{\mathrm{w}}\left(\omega = 0\right) = 1$ are selected.

Assuming that the overall *CSF* of a *TDOF* control system is $T_{\mathrm{r}} = FT$, where F is the pulse transfer function of the reference signal filter, then similar decomposition can be introduced for the tracking error function $E_{\mathrm{r}} = 1 - T_{\mathrm{r}}$ as for E in (6):

$$E_{\mathrm{r}} = \left(1 - P_{\mathrm{r}}\right) + \left(P_{\mathrm{r}} - \hat{T}_{\mathrm{r}}\right) - \left(T_{\mathrm{r}} - \hat{T}_{\mathrm{r}}\right) = E_{\mathrm{des}}^{\mathrm{r}} + E_{\mathrm{real}}^{\mathrm{r}} + E_{\mathrm{id}}^{\mathrm{r}} \tag{9}$$

The overall transfer function of the *TDOF* system is

$$T_{\mathrm{r}} = \hat{T}_{\mathrm{r}}\frac{1 + \ell}{1 + \hat{T}\ell} \tag{10}$$

The term $E_{\mathrm{id}}^{\mathrm{r}}$ can be further simplified

$$E_{\mathrm{id}}^{\mathrm{r}} = \hat{T}_{\mathrm{r}} - T_{\mathrm{r}} = -\frac{\hat{T}_{\mathrm{r}}\hat{E}\ell}{1 + \hat{T}\ell} = -\hat{T}_{\mathrm{r}}E\ell\Big|_{\ell \to 0} \approx -\hat{T}_{\mathrm{r}}\hat{E}\ell \tag{11}$$

In an ideal control system it is required to follow the transients required by P_{r} and P_{w} (more exactly $\left(1 - P_{\mathrm{w}}\right)$), i.e., the ideal overall transfer characteristics of the *TDOF* control system would be

$$y^{\mathrm{o}} = P_{\mathrm{r}}y_{\mathrm{r}} - \left(1 - P_{\mathrm{w}}\right)w = y_{\mathrm{r}}^{\mathrm{o}} + y_{\mathrm{w}}^{\mathrm{o}} \tag{12}$$

while a practical, realizable control can provide only

$$y = T_{\mathrm{r}}y_{\mathrm{r}} - Ew = T_{\mathrm{r}}y_{\mathrm{r}} - \left(1 - T\right)w \\ \hat{y} = \hat{T}_{\mathrm{r}}y_{\mathrm{r}} - \hat{E}w = \hat{T}_{\mathrm{r}}y_{\mathrm{r}} - \left(1 - \hat{T}\right)w \tag{13}$$

for the true (y) and model-based ($\hat{y}$) closed-loop control output signals.

Express the deviation between the ideal (y^{o}) and the realizable best (y) closed-loop output signals as

$$\Delta y = y^{\mathrm{o}} - y = \left(P_{\mathrm{r}} - T_{\mathrm{r}}\right)y_{\mathrm{r}} - \left(P_{\mathrm{w}} - T\right)w = \\ = E_{\mathrm{perf}}^{\mathrm{r}}y_{\mathrm{r}} - E_{\mathrm{perf}}^{\mathrm{w}}w \tag{14}$$

where $E_{\mathrm{perf}}^{\mathrm{r}}$ is the performance degradation for tracking and $E_{\mathrm{perf}}^{\mathrm{w}} = E_{\mathrm{perf}}$ is the performance degradation for the disturbance rejection (or control) behaviors, respectively. Similar equation can be obtained for the deviation between the ideal (y^{o}) and the model based ($\hat{y}$) closed-loop outputs

$$\Delta\hat{y} = y^{\mathrm{o}} - \hat{y} = \left(P_{\mathrm{r}} - \hat{T}_{\mathrm{r}}\right)y_{\mathrm{r}} - \left(P_{\mathrm{w}} - \hat{T}\right)w = \\ = E_{\mathrm{real}}^{\mathrm{r}}y_{\mathrm{r}} - E_{\mathrm{real}}^{\mathrm{w}}w \tag{15}$$

where $E_{\mathrm{perf}}^{\mathrm{r}}$ is the realizability degradation for tracking and $E_{\mathrm{perf}}^{\mathrm{w}} = E_{\mathrm{perf}}$ is the realizability degradation for the disturbance rejection (control) behaviors, respectively. So

$$\Delta y = \Delta\hat{y} - \left(E_{\mathrm{id}}^{\mathrm{r}}y_{\mathrm{r}} - E_{\mathrm{id}}^{\mathrm{w}}w\right) \tag{16}$$

It is important to note that the term E_{real} (and $E_{\mathrm{real}}^{\mathrm{r}}$) can be made zero for *IS* processes only, however, for *IU* plants the reachable minimal value of E_{real} (and $E_{\mathrm{real}}^{\mathrm{r}}$) always depends on the invariant factors and never becomes zero. In the sequel *YP* based control system will be discussed.

2.1 YOULA-parameterization

If the applied regulator design is based on the *YOULA-parameterization* (*YP*) (Maciejowski, 1989), (Keviczky et al. (2015)), (Keviczky et al. (2018a)), then the realizable best and the model based regulators are

$$R = \frac{Q}{1-QS} \quad ; \quad \hat{R} = \frac{Q}{1-QM} \tag{17}$$

where Q is the YOULA parameter. Thus the *CSF*'s of the true and model-based *ODOF* control systems are

$$T = \frac{\hat{R}S}{1+\hat{R}S} = \frac{QM\left(1+\ell\right)}{1+QM\ell} \quad ; \quad \hat{T} = \frac{RM}{1+RM} = QM \tag{18}$$

Only in case of *YP* one can also compute the realizable best *CSF*

$$T_* = \frac{RS}{1+RS} = QS = QM\left(1+\ell\right) = \hat{T}\left(1+\ell\right) \tag{19}$$

The *SF* of the model based and true closed-loops are now

$$\hat{E} = \frac{1}{1+\hat{R}M} = 1-QM \tag{20}$$

and

$$E = \frac{1}{1+\hat{R}S} = \frac{1-QM}{1+QM\ell} = \frac{\hat{E}}{1+\hat{T}\ell} \tag{21}$$

The realizable best *SF*, corresponding to T_* is

$$E_* = \frac{1}{1+RS} = 1-QS = 1-QM\left(1+\ell\right) = \hat{E} - \hat{T}\ell \tag{22}$$

The decomposition of the *SF* is

$$\begin{aligned} E &= \left(1-P_{\mathrm{w}}\right) + \left(P_{\mathrm{w}} - \hat{T}\right) - \left(T - \hat{T}\right) = E_{\mathrm{des}} + E_{\mathrm{real}} + \\ &+ E_{\mathrm{id}} = \left(1-P_{\mathrm{w}}\right) + \left(P_{\mathrm{w}} - QM\right) - \frac{QM\left(1-QM\right)}{1+QM\ell}\ell \end{aligned} \tag{23}$$

where the identification degradation is

$$E_{\mathrm{id}} = -\left.\frac{QM\left(1-QM\right)}{1+QM\ell}\ell\right|_{\ell\to 0} \approx -QM\left(1-QM\right)\ell \tag{24}$$

It is interesting to note that for the realizable best case the decomposition of $E_* = 1 - T_*$ results in

$$\begin{aligned} E_* &= 1-QS = E_{\mathrm{des}} + E_{\mathrm{real}} + E_{\mathrm{id}}^* = \\ &= E_{\mathrm{des}} + E_{\mathrm{perf}}^* = \left(1-P_{\mathrm{w}}\right) + \left(P_{\mathrm{w}} - QS\right) \end{aligned} \tag{25}$$

where

$$E_{\mathrm{id}}^* = -QM\ell = \frac{1}{E}E_{\mathrm{id}} \tag{26}$$

This last expression is different from the form (8), because at the optimal point, when $M = S$, the *Y-parameterized* closed-loop virtually opens, therefore the weighting by $\hat{E}$ is missing here.

The decomposition of the tracking error function for the *YP* is

$$\begin{aligned} E_{\mathrm{r}} &= 1 - T_{\mathrm{r}} = \left(1-P_{\mathrm{r}}\right) + \left(P_{\mathrm{r}} - Q_{\mathrm{r}}M\right) - \left(T_{\mathrm{r}} - \hat{T}_{\mathrm{r}}\right) = \\ &= E_{\mathrm{des}}^{\mathrm{r}} + E_{\mathrm{real}}^{\mathrm{r}} + E_{\mathrm{id}}^{\mathrm{r}} \end{aligned} \tag{27}$$

where

$$E_{\mathrm{id}}^{\mathrm{r}} = -\left.\frac{Q_{\mathrm{r}}M\left(1-QM\right)}{1+QM\ell}\ell\right|_{\ell\to 0} \approx -Q_{\mathrm{r}}M\left(1-QM\right)\ell \tag{28}$$

3. A *GTDOF* CONTROLLER FOR STABLE LINEAR PLANTS

In many practical cases the plant to be controlled is stable, and a *TDOF* control system is required because of the high performance double tracking and regulatory requirements (Horowitz, 1963). An ideal solution for this task is the *generic two-degree of freedom* (*GTDOF*) scheme introduced in Keviczky (1995). This framework and topology is based on the *YP* (Maciejowski, 1989), (Keviczky et al. (2015)), (Keviczky et al. (2018a)) providing stabilizing regulators for open-loop stable plants and capable to handle the plant time-delay, too.

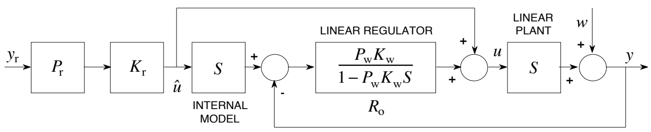

Fig. 1. The *generic TDOF* (*GTDOF*) control system

A *GTDOF* control system is shown in Fig. 1, where w is the output disturbance signal. The realizable best regulator of the *GTDOF* scheme can be given by an explicit form

$$R_* = \frac{Q_*}{1-Q_*S} = \frac{P_{\mathrm{w}}K_{\mathrm{w}}}{1-P_{\mathrm{w}}K_{\mathrm{w}}S} = \frac{P_{\mathrm{w}}G_{\mathrm{w}}S_+^{-1}}{1-P_{\mathrm{w}}G_{\mathrm{w}}S_- z^{-d}} \tag{29}$$

where

$$Q_* = Q_{\mathrm{w}}^* = P_{\mathrm{w}}K_{\mathrm{w}} = P_{\mathrm{w}}G_{\mathrm{w}}S_+^{-1} \tag{30}$$

is the associated optimal *Y-parameter* furthermore

$$Q_{\mathrm{r}}^* = P_{\mathrm{r}}K_{\mathrm{r}} = P_{\mathrm{r}}G_{\mathrm{r}}S_+^{-1};\ K_{\mathrm{w}} = G_{\mathrm{w}}S_+^{-1}\ ;\ K_{\mathrm{r}} = G_{\mathrm{r}}S_+^{-1} \tag{31}$$

The regulator (29) can be considered the generalization of the TRUXAL-GUILLEMIN method for stable processes. It is interesting to see how the transfer characteristics of the closed-loop look like:

$$
\begin{aligned}
y &= P_\mathrm{r} K_\mathrm{r} S y_\mathrm{r} + \left(1 - P_\mathrm{w} K_\mathrm{w} S\right) w = T_\mathrm{r} y_\mathrm{r} + E w = y_* = \\
&= P_\mathrm{r} G_\mathrm{r} S_- z^{-d} y_\mathrm{r} + \left(1 - P_\mathrm{w} G_\mathrm{w} S_- z^{-d}\right) w = y_\mathrm{t} + y_\mathrm{d}
\end{aligned} \tag{32}
$$

where y_t is the tracking (servo) and y_d is the regulating (control or disturbance rejection) independent behavior of the closed-loop response, respectively.

So the delay z^{-d} and S_- can not be eliminated, consequently the ideal design goals P_r and P_w are biased by $G_\mathrm{r} S_-$ and $G_\mathrm{w} S_-$. We can not reach the ideal tracking $y_\mathrm{r}^\mathrm{o} = P_\mathrm{r} y_\mathrm{r}$ and regulatory $y_\mathrm{w}^\mathrm{o} = \left(1 - P_\mathrm{w}\right) w$ behaviors (see (12)), because of the un-compensable time-delay and the so-called invariant factors (mainly zeros) in the *IU* factor S_-. The realizable best transients, corresponding to (13) and (32), is given by $P_\mathrm{r} G_\mathrm{r} S_- z^{-d}$ and $\left(1 - P_\mathrm{w} G_\mathrm{w} S_- z^{-d}\right)$ respectively, where G_r and G_w can optimally attenuate the influence of S_-.

After some straightforward block manipulations the *GTDOF* control system can be transformed to a simpler form shown in Fig. 2. This form is special because the controller consists of two parts. The first part depends only on the design parameters and the invariant process factor (at a selected optimality criterion), while the second one depends only on the realizable inverse model of the plant.

The model based version of the *YP* regulator $\hat{R} = R\left(M\right)$ in the *GTDOF* scheme means that S is substituted by M in equations (29)-(31).

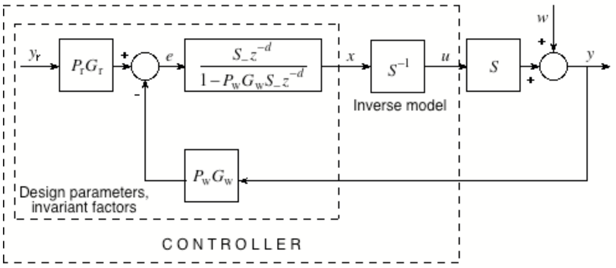

Fig. 2. Simplified form of the *GTDOF* control system

The decomposition of the *SF* in the true *GTDOF* control system by (23) is

$$
\begin{aligned}
E &= E_\mathrm{des} + E_\mathrm{real} + E_\mathrm{id} = \left(1 - P_\mathrm{w}\right) + P_\mathrm{w}\left(1 - G_\mathrm{w} M_- z^{-d}\right) - \\
&- \frac{P_\mathrm{w} G_\mathrm{w} M_- z^{-d}\left(1 - P_\mathrm{w} G_\mathrm{w} M_- z^{-d}\right)}{1 + QM\ell}\,\ell
\end{aligned} \tag{33}
$$

4. OPTIMIZATION SOLUTIONS AND SCHEMES

The optimization of the *GTDOF* control system is usually based on a proper selected norm of $\left|E\right|$. Corresponding cost functions for the tracking and control properties can always be constructed by using the triangle inequality

$$
\begin{aligned}
J_\mathrm{tracking} &\le J_\mathrm{des}^\mathrm{r} + J_\mathrm{real}^\mathrm{r} + J_\mathrm{id}^\mathrm{r} = \left\|E_\mathrm{des}^\mathrm{r}\right\| + \left\|E_\mathrm{real}^\mathrm{r}\right\| + \left\|E_\mathrm{id}^\mathrm{r}\right\| \\
J_\mathrm{control} &\le J_\mathrm{des}^\mathrm{w} + J_\mathrm{real}^\mathrm{w} + J_\mathrm{id}^\mathrm{w} = \left\|E_\mathrm{des}\right\| + \left\|E_\mathrm{real}\right\| + \left\|E_\mathrm{id}\right\|
\end{aligned} \tag{34}
$$

4.1. Minimization of the design loss

The minimization of the first terms can be formulated by

$$
\begin{aligned}
P_\mathrm{r}^\mathrm{opt} &= \arg\left\{\min_{P_\mathrm{r}}\left(J_\mathrm{des}^\mathrm{r}\right)\Big|_{u\in\mathcal{U}}\right\} = \arg\left\{\min_{P_\mathrm{r}} \left\|1 - P_\mathrm{r}\right\|\Big|_{u\in\mathcal{U}}\right\} \\
P_\mathrm{w}^\mathrm{opt} &= \arg\left\{\min_{P_\mathrm{w}}\left(J_\mathrm{des}^\mathrm{w}\right)\Big|_{u\in\mathcal{U}}\right\} = \arg\left\{\min_{P_\mathrm{w}} \left\|1 - P_\mathrm{w}\right\|\Big|_{u\in\mathcal{U}}\right\}
\end{aligned} \tag{35}
$$

The goal of this optimization step is to minimize the design loss. Here the fastest reference models $P_\mathrm{r} = P_\mathrm{r}^\mathrm{opt}$ and $P_\mathrm{w} = P_\mathrm{w}^\mathrm{opt}$ must be found by minimizing the introduced criteria $J_\mathrm{des}^\mathrm{r} = \left\|1 - P_\mathrm{r}\right\|$ and $J_\mathrm{des}^\mathrm{w} = \left\|1 - P_\mathrm{w}\right\|$.

One should expect that this optimization step results in the best reachable reference models corresponding to the existing constraints $u \in \mathcal{U}$ for the control action, where $\mathcal{U}$ is the (mostly amplitude: $\mathcal{U} : \left|u\right| \le 1$ and/or rate) constrained input signal domain. This is usually the boundary of the linear operational domain.

For low (e.g. first) order reference models it is easy to compute the maximum pick (overshoot) of the closed-loop step response with simple algebraic formulas for the reachable bandwidth. With a first order reference model $P_\mathrm{w} = \left(1 + a_1\right) / \left(1 + a_1 z^{-1}\right)$ the inequality, necessary to maintain in case of an amplitude limit U_L, is

$$
\frac{1 + a_\mathrm{w}}{\hat{b}_1} \le U_\mathrm{L} \tag{36}
$$

and the applicable reference model parameter is

$$
a_\mathrm{w} \le \hat{b}_1 U_\mathrm{L} - 1 \tag{37}
$$

It is not so widely known that the robust stability condition $\left\|\hat{T}\ell\right\|_\infty < 1$ (and $\left\|QM\ell\right\|_\infty < 1$ for the *YP* controllers) can also give a constraint for the reachable closed-loop bandwidth formulated by P_w. This condition is very simple for the *GTDOF* system

$$
\left|QM\right| < \frac{1}{\left|\ell\right|} \quad \text{or} \quad \left|\ell\right| < \frac{1}{\left|QM\right|} \quad \forall\omega \tag{38}
$$

Thus the robust stability strongly depends on the model M and how the model-based *Y-parameter* $Q = P_\mathrm{w} G_\mathrm{w} M_+^{-1}$ is selected. In this case

$$\left|QM\ell\right| = \left|P_\mathrm{w} G_\mathrm{w} M_+^{-1} M\ell\right| = \left|P_\mathrm{w} G_\mathrm{w} M\_ z^{-d}\ell\right| = \left|P_\mathrm{w}\ell\right| \tag{39}$$

where $\left|G_\mathrm{w} M\_\right| = 1$, (because of the optimization), furthermore $\left|z^{-d}\right| = 1$ (which is well known) were used, thus finally

$$\sup_{\omega}\left|\ell\right| \le 1/\left|P_\mathrm{w}\right| \quad \text{or} \quad \left\|\ell\right\|_\infty \le 1/\left\|P_\mathrm{w}\right\|_\infty \tag{40}$$

Because the right hand side of this inequality depends only on P_w, which is the reference model for the regulatory property of the *GTDOF* system, this means that this is a special controller structure, where the performance of the closed-loop is directly influenced by the robustness limit (via the selected P_w). Observe that this method can be considered a new kind of loop-shaping via P_r and P_w, which are direct and well understandable design goals.

4.2. Minimization of the relizability loss

The goal of this optimization step is to minimize the realizability loss $J_\mathrm{real}^\mathrm{w}$ using optimal embedded filters $G_\mathrm{r} = G_\mathrm{r}^\mathrm{opt}$ and $G_\mathrm{w} = G_\mathrm{w}^\mathrm{opt}$ attenuating the influence of the invariant model factor $M\_$

$$\begin{aligned} G_\mathrm{r}^\mathrm{opt} &= \arg\left\{\min_{G_\mathrm{r}}\left(J_\mathrm{real}^\mathrm{r}\right)\right\} = \arg\left\{\min_{G_\mathrm{r}}\left\|P_\mathrm{r}\left(1 - G_\mathrm{r} M\_ z^{-d}\right)\right\|\right\} \\ G_\mathrm{w}^\mathrm{opt} &= \arg\left\{\min_{G_\mathrm{w}}\left(J_\mathrm{real}^\mathrm{w}\right)\right\} = \arg\left\{\min_{G_\mathrm{w}}\left\|P_\mathrm{w}\left(1 - G_\mathrm{w} M\_ z^{-d}\right)\right\|\right\} \end{aligned} \tag{41}$$

This task corresponds to the model matching approach of control system design. The realizability degradation is considerably different for *IS* and *IU* processes. For the *IS* case $M\_ z^{-d} = 1$, so there is no optimization problem to be solved and the trivial $G_\mathrm{r} = G_\mathrm{w} = 1$ selections can be used. The realizability degradation is zero now.

For *IU* case the minimization of $J_\mathrm{real}^\mathrm{r}$ and $J_\mathrm{real}^\mathrm{w}$ can be performed in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm spaces (Keviczky et al. (1999)). If using $\mathcal{H}_2$ norm a *DIOPHANTINE* equation (*DE*) should be solved to optimize these filters only and not the whole regulator itself. If the optimality requires a $\mathcal{H}_\infty$ norm, then the *NEVANLINNA-PICK* (*NP*) approximation is applied. Applicable procedures can be found in Wang et al. (1988) and Keviczky et al. (1999).

4.3. Minimization of the modeling loss

The goal of this optimization step is to minimize the identification (or modeling) loss J_id^r via the optimal external excitation $y_\mathrm{r} = y_\mathrm{r}^\mathrm{opt}$ and the optimal model $M = M^\mathrm{opt}$. This is a minimax problem

$$M^\mathrm{opt} = \arg\left\{\min_{M}\left[\max_{y_\mathrm{r}}\left(J_\mathrm{real}^\mathrm{r}\right)\right]\right\} = \arg\left\{\min_{M}\left[\max_{y_\mathrm{r}}\left(\left\|E_\mathrm{id}^\mathrm{r}\right\|\right)\right]\right\} \tag{42}$$

where

$$\begin{aligned} E_\mathrm{id}^\mathrm{r} &= -\left.\frac{P_\mathrm{r} G_\mathrm{r} M\_ z^{-d}\left(1 - P_\mathrm{r} G_\mathrm{r} M\_ z^{-d}\right)}{1 + QM\ell}\ell\right|_{\ell\to 0} \approx \\ &\approx -P_\mathrm{r} G_\mathrm{r} M\_ z^{-d}\left(1 - P_\mathrm{r} G_\mathrm{r} M\_ z^{-d}\right)\ell \end{aligned} \tag{43}$$

So this optimization can be done in two steps. The first step is the so-called optimal input design, where the optimized "maximum variance" type excitation produces the worst maximal modeling error to be minimized in the next identification step.

The same procedure can not be exactly applied for minimizing the loss J_id^w, because the output disturbance w does not depend on us. However, it is simply possible to use P_w instead of P_r in (42) and (43).

In Section 3 it was shown that in the vicinity of the exact model case $M = S$ the term E_id in E becomes E_id^* (see (26)), which can also be used instead of (42)

$$E_\mathrm{id}^* = -QM\ell = -P_\mathrm{r} G_\mathrm{r} M\_ z^{-d}\ell \tag{44}$$

There exist several closed-loop identification (*ID*) schemes to obtain a good model M. There is a natural possibility to perform this task, avoiding the well known "circulating noise" issue (Åström et al. (1984)), namely to apply the *ID* between $\hat{u}$ (see Fig. 1) and y. In this approach (called *KB-parameterization* (Keviczky et al. (2001))) $\hat{u}$ depends on the a-priori model estimate M_i, so only iterative schemes can be constructed. After some straightforward computation the model output (or *ID*) error ε_ID is

$$\begin{aligned} \varepsilon_\mathrm{ID} &= \varepsilon_\mathrm{KB} = y - y_\mathrm{m} = y - M\hat{u} = \\ &= \left.\frac{\left(P_\mathrm{r} G_\mathrm{r} M\_ z^{-d}\right)\left(1 - P_\mathrm{w} G_\mathrm{w} M\_ z^{-d}\right)}{1 + P_\mathrm{w} G_\mathrm{w} M\_ z^{-d}\ell}\right|_{\ell\to 0} \ell y_\mathrm{r} \approx \\ &\approx \left(P_\mathrm{r} G_\mathrm{r} M\_ z^{-d}\right)\left(1 - P_\mathrm{w} G_\mathrm{w} M\_ z^{-d}\right)\ell y_\mathrm{r} = H_\mathrm{KB}\ell y_\mathrm{r} \end{aligned} \tag{45}$$

where

$$\hat{u} = P_\mathrm{r} K_\mathrm{r} y_\mathrm{r} = P_\mathrm{r} G_\mathrm{r} M_+^{-1} y_\mathrm{r} \tag{46}$$

is the model input and $y_\mathrm{m} = M\hat{u}$ the model output. Note that the weighting filters in (43) and (45) are the same. This means that a proper selection of the identification criterion for the model error (45) can also solve the simultaneous minimization of a proper norm of $\left\| E_\mathrm{id}^\mathrm{r} \right\|$.

It is easy to check that $\left| H_\mathrm{KB} \right|$ has its maximum at the cross over frequency ω_c, which means that the model minimizing $\varepsilon_\mathrm{ID} = \varepsilon_\mathrm{KB}$ is the most accurate around this medium frequency range (see Eq. (8)). At the end of the iteration one can switch to Eq. (44) and the accuracy improves according to a weighting factor given by P_r.

5. AN ITERATIVE OPTIMIZATION SCHEME

The introduced decompositions are natural, useful and correspond to the control engineering practice. Based on the previous section the following generic optimal design procedure can be constructed for the model based optimization:

$$\begin{array}{cccc} E = \left(1 - P_\mathrm{w}\right) + & P_\mathrm{w}\left(1 - G_\mathrm{w} M\_ z^{-d}\right) - & P_\mathrm{w} G_\mathrm{w} M\_ z^{-d} \ell & \\ \Downarrow & \Downarrow & \Downarrow & (47) \\ P_\mathrm{w}^\mathrm{opt}\left(M\right) & G_\mathrm{w}^\mathrm{opt}\left(M\right) & M^\mathrm{opt}; y_\mathrm{r}^\mathrm{opt} & \end{array}$$

The solution of this decomposed optimization problem can only be an iterative procedure, because each term depends on the model of the process. A reasonable order of these steps is the following (starting with an available a-priori initial model M_o):

1. Having known an a-priori model M_i and reference models P_r^i and P_w^i solve (41) to get $G_\mathrm{r}^i = G_\mathrm{r}^\mathrm{opt}$ and $G_\mathrm{w}^i = G_\mathrm{w}^\mathrm{opt}$, then compute the model based regulator $\hat{R}_i = R_i\left(M_i\right)$ using (29).

2. Solve (35) to obtain $R_\mathrm{r}^{i+1} = R_\mathrm{r}^\mathrm{opt}$ and $R_\mathrm{w}^{i+1} = R_\mathrm{w}^\mathrm{opt}$. This optimization can be done by simulation in a model based or (if the technological requirements allow) by real experiments in the true closed-loop.

3. Using M_i, P_r^i, P_w^i, G_r^i and G_w^i perform the optimal input design to determine the best external excitation $y_\mathrm{r}^\mathrm{opt}$.

4. Apply this optimal reference signal to the closed-loop and collect the measured output variable y. Compute the auxiliary signal $\hat{u}$. Perform the *ID* step based on $\varepsilon_\mathrm{ID} = \varepsilon_\mathrm{KB}$ to identify the best model $M_{i+1} = M^\mathrm{opt}$.

5. The iterative process is continued from step 1, while a stop condition is not fulfilled.

6. CONCLUSIONS

It is shown that the sensitivity function of a *GTDOF* control system can be decomposed into three major parts, corresponding to the design, realizability and identification degradation.

The minimization of the design loss can be performed in connection with finding the fastest reference models P_r and P_w under available constraints for the control action. The robust stability condition can also be provided by applying proper constraints to the reference model P_w. This step can be considered a new kind of loop-shaping via P_r and P_w, which are direct and well understandable design goals, at the same time.

The realizability degradation is zero for *IS* processes. For *IU* plants the realizability degradation can be minimized in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm spaces.

The minimization of the modeling part is connected to the optimal *ID* of the process model. Properly selected norms can be found which help to minimize both the weighted identification loss and the realizability degradation loss simultaneously.

The new control error decomposition approach gives an excellent new way to teach control system optimization.

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