

A characterization of the Logarithmic Least Squares Method

László Csató*

Institute for Computer Science and Control, Hungarian Academy of Sciences (MTA SZTAKI)
 Laboratory on Engineering and Management Intelligence, Research Group of Operations
 Research and Decision Systems

Corvinus University of Budapest (BCE)
 Department of Operations Research and Actuarial Sciences

Budapest, Hungary

18th September 2018

Abstract

We provide an axiomatic characterization of the Logarithmic Least Squares Method (sometimes called row geometric mean), used for deriving a preference vector from a pairwise comparison matrix. This procedure is shown to be the only one satisfying two properties, correctness in the consistent case, which requires the reproduction of the inducing vector for any consistent matrix, and invariance to a specific transformation on a triad, that is, the weight vector is not influenced by an arbitrary multiplication of matrix elements along a 3-cycle by a positive scalar.

Keywords: Decision analysis; pairwise comparisons; geometric mean; axiomatic approach; characterization

JEL classification number: C44

MSC class: 90B50, 91B08

1 Introduction

Pairwise comparisons are a fundamental tool in many decision-analysis methods such as the Analytic Hierarchy Process (AHP) (Saaty, 1980). However, in real-world applications the judgements of decision-makers may be inconsistent: for example, alternative A is two times better than alternative B , alternative B is three times better than alternative C , but alternative A is not six times better than alternative C . Inconsistency can also be an inherent feature of the data, for example, on the field of sports (Csató, 2013; Bozóki et al., 2016; Chao et al., 2018).

Therefore, a lot of methods have been suggested in the literature for deriving preference values from pairwise comparison matrices. In such cases, it seems to be fruitful to

* e-mail: laszlo.csato@uni-corvinus.hu

follow an axiomatic approach: the introduction and justification of reasonable properties may help to narrow the set of appropriate weighting methods and reveal some crucial features of them. The most important contribution of similar analyses can be an *axiomatic characterization*, when a set of properties *uniquely* determine a preference vector.

Characterization of different methods is a standard tool in social choice theory, for instance, in the case of the Shapley value in game theory (see e.g. [Shapley \(1953\)](#); [van den Brink and Pintér \(2015\)](#)), or for the Hirsch index in scientometrics (see e.g. [Woeginger \(2008\)](#); [Bouyssou and Marchant \(2014\)](#)). This approach has been applied recently for inconsistency indices of pairwise comparison matrices ([Csató, 2018a,b](#)).

[Fichtner \(1984\)](#), presumably the first work on the axiomatizations of weighting methods, characterized the Logarithmic Least Squares Method ([Rabinowitz, 1976](#); [Crawford and Williams, 1980, 1985](#); [De Graan, 1980](#)) by using four requirements, correctness in the consistent case, comparison order invariance, smoothness, and power invariance. [Fichtner \(1986\)](#) showed that substituting power invariance with rank preservation leads to the Eigenvector Method suggested by [Saaty \(1980\)](#).

From this set of axioms, correctness in the consistent case and comparison order invariance are almost impossible to debate. However, according to [Bryson \(1995\)](#), there exists a goal-programming method satisfying power invariance and a slightly modified version of smoothness, which possesses the additional property that the presence of a single outlier cannot prevent the identification of the correct priority vector. While [Fichtner \(1984\)](#) introduces smoothness in terms of differentiable functions and continuous derivatives, the interpretation of [Bryson \(1995\)](#) – a small change in the input does not lead to a large change in the output – seems to be more natural for us. [Cook and Kress \(1988\)](#) approached the problem by focusing on distance measures in order to get another goal programming method on an axiomatic basis.

Smoothness and power invariance can be entirely left out from the characterization of the Logarithmic Least Squares Method. [Barzilai et al. \(1987\)](#) exchange them for a consistency-like axiom by considering two procedures: (1) some pairwise comparison matrices are aggregated to one matrix and the solution is computed for this matrix; (2) the priorities are derived separately for each matrix and combined by the geometric mean; which are required to result in the same preference vector. We think it is not a simple condition immediately to adopt. [Barzilai \(1997\)](#) managed to replace this axiom and comparison order invariance with essentially demanding that each individual weight is a function of the entries in the corresponding row of the pairwise comparison matrix only. Joining to [Dijkstra \(2013\)](#), we are also somewhat uncomfortable with this premise.

To summarize, the problem of weight derivation does not seem to be finally settled by the axiomatic approach. Consequently, it may not be futile to provide another characterization of the Logarithmic Least Squares Method, which hopefully highlights some new aspects of the procedure. This is the main aim of the current paper.

Presenting an axiomatic characterization does not mean that we accept all properties involved as wholly justified and unquestionable or we reject the axioms proposed by previous works. To consider an example from a related topic, although most axiomatic analysis of inconsistency ([Brunelli and Fedrizzi, 2011](#); [Brunelli, 2016, 2017](#); [Brunelli and Fedrizzi, 2015, 2018](#); [Cavallo and D’Apuzzo, 2012](#); [Koczkodaj and Szwarz, 2014](#); [Koczkodaj and Szybowski, 2015](#); [Koczkodaj and Urban, 2018](#)) look for well-motivated axioms that should be satisfied by any reasonable measure, [Csató \(2018b\)](#) does not deal with the appropriate motivation of his axioms, the issue to be investigated is only how they can narrow the set of inconsistency indices.

This paper strictly follows the latter direction, therefore, we only say that *if* one agrees with our axioms, then the geometric mean remains the only choice. The importance of an axiomatic characterization does not depend on a convincing explanation for the desirability of the properties suggested. For example, Arrow's impossibility theorem (Arrow, 1950) can also be interpreted as an axiomatization of the dictatorial rule by universality, Pareto efficiency, and independence of irrelevant alternatives, however, it does not mean that the dictatorial rule is good.

The study is structured as follows. Section 2 presents some definitions on the field of pairwise comparison matrices. Two properties of weighting methods are defined in Section 3, which will provide the characterization of the Logarithmic Least Squares Method in Section 4. Section 5 summarizes our findings.

2 Preliminaries

Assume that n alternatives should be measured with respect to a given criterion on the basis of pairwise comparisons such that $a_{i,j}$ is an assessment of the relative importance of alternative i with respect to alternative j .

Let \mathbb{R}_+^n and $\mathbb{R}_+^{n \times n}$ denote the set of positive (with all elements greater than zero) vectors of size n and matrices of size $n \times n$, respectively.

Definition 2.1. *Pairwise comparison matrix:* Matrix $\mathbf{A} = [a_{i,j}] \in \mathbb{R}_+^{n \times n}$ is a *pairwise comparison matrix* if $a_{j,i} = 1/a_{i,j}$ for all $1 \leq i, j \leq n$.

Any pairwise comparison matrix is well-defined by its elements above the diagonal since we discuss only multiplicative pairwise comparison matrices with the reciprocal property throughout the paper. Let $\mathcal{A}^{n \times n}$ be the set of pairwise comparison matrices of size $n \times n$.

A pairwise comparison matrix $\mathbf{A} \in \mathcal{A}^{n \times n}$ is called *consistent* if $a_{i,k} = a_{i,j}a_{j,k}$ for all $1 \leq i, j, k \leq n$. Otherwise, it is said to be *inconsistent*. Any pairwise comparison matrix is allowed to be inconsistent unless its consistency is explicitly stated.

Definition 2.2. *Weight vector:* Vector $\mathbf{w} = [w_i] \in \mathbb{R}_+^n$ is a *weight vector* if $\sum_{i=1}^n w_i = 1$.

Let \mathcal{R}^n be the set of weight vectors of size n .

Definition 2.3. *Weighting method:* Function $f : \mathcal{A}^{n \times n} \rightarrow \mathcal{R}^n$ is a *weighting method*.

A weighting method associates a weight vector to any pairwise comparison matrix \mathbf{A} such that $f_i(\mathbf{A})$ is the weight of alternative i .

Several weighting methods have been suggested in the literature, see Choo and Wedley (2004) for an overview. This paper discusses two of them, which are among the most popular.

Definition 2.4. *Eigenvector Method (EM) (Saaty, 1980):* The *Eigenvector Method* is the function $\mathbf{A} \rightarrow \mathbf{w}^{EM}(\mathbf{A})$ such that

$$\mathbf{A}\mathbf{w}^{EM}(\mathbf{A}) = \lambda_{\max}\mathbf{w}^{EM}(\mathbf{A}) \quad \text{and} \quad \sum_{i=1}^n w_i^{EM} = 1,$$

where λ_{\max} denotes the maximal eigenvalue, also known as principal or Perron eigenvalue, of matrix \mathbf{A} .

Definition 2.5. *Logarithmic Least Squares Method (LLSM)* (Crawford and Williams, 1980, 1985; De Graan, 1980): The *Logarithmic Least Squares Method* is the function $\mathbf{A} \rightarrow \mathbf{w}^{LLSM}(\mathbf{A})$ such that the weight vector $\mathbf{w}^{LLSM}(\mathbf{A})$ is the optimal solution of the problem:

$$\min_{\mathbf{w} \in \mathcal{R}^n} \sum_{i=1}^n \sum_{j=1}^n \left[\log a_{i,j} - \log \left(\frac{w_i}{w_j} \right) \right]^2. \quad (1)$$

LLSM is sometimes called (row) *geometric mean* because the solution of (1) can be computed as

$$w_i^{LLSM}(\mathbf{A}) = \frac{\prod_{j=1}^n a_{i,j}^{1/n}}{\sum_{k=1}^n \prod_{j=1}^n a_{k,j}^{1/n}}. \quad (2)$$

3 Axioms

In this section, two properties of weighting methods will be discussed.

Axiom 3.1. *Correctness (CO)*: Let $\mathbf{A} \in \mathcal{A}^{n \times n}$ be a consistent pairwise comparison matrix. Weighting method $f : \mathcal{A}^{n \times n} \rightarrow \mathcal{R}^n$ is *correct* if $f_i(\mathbf{A})/f_j(\mathbf{A}) = a_{i,j}$ for all $1 \leq i, j \leq n$.

CO requires the reproduction of the inducing vector for any consistent pairwise comparison matrix. It was introduced by Fichtner (1984) under the name *correct result in the consistent case* and was used by Fichtner (1986), Barzilai et al. (1987) and Barzilai (1997), among others.

Proposition 3.1. *The Eigenvector Method and the Logarithmic Least Squares Method satisfy correctness.*

Definition 3.1. *α -transformation on a triad*: Let $\mathbf{A} \in \mathcal{A}^{n \times n}$ be a pairwise comparison matrix and $1 \leq i, j, k \leq n$ be three different alternatives. An *α -transformation on the triad (i, j, k)* – which is determined by the three alternatives i , j , and k – provides the pairwise comparison matrix $\hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ such that $\alpha > 0$, $\hat{a}_{i,j} = \alpha a_{i,j}$ ($\hat{a}_{j,i} = a_{j,i}/\alpha$), $\hat{a}_{j,k} = \alpha a_{j,k}$ ($\hat{a}_{k,j} = a_{k,j}/\alpha$), $\hat{a}_{k,i} = \alpha a_{k,i}$ ($\hat{a}_{i,k} = a_{i,k}/\alpha$) and $\hat{a}_{\ell,m} = a_{\ell,m}$ for all other elements.

The transformation changes three elements of a pairwise comparison matrix along a 3-cycle. It can reproduce *local consistency*: the choice $\alpha = \sqrt[3]{a_{i,k}/(a_{i,j}a_{j,k})}$ leads to $\hat{a}_{i,j}\hat{a}_{j,k} = \alpha^2 a_{i,j}a_{j,k} = a_{i,k}/\alpha = \hat{a}_{i,k}$.

Naturally, this process modifies all values of the triad, while maybe two of the comparisons are accurate and one contains all the inaccuracy. However, if no further information is available, then our assumption seems to be reasonable.

Axiom 3.2. *Invariance to α -transformation on a triad (IT)*: Let $\mathbf{A}, \hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ be any two pairwise comparison matrices such that $\hat{\mathbf{A}}$ can be obtained from \mathbf{A} through an α -transformation on a triad. Weighting method $f : \mathcal{A}^{n \times n} \rightarrow \mathcal{R}^n$ is *invariant to α -transformation on a triad* if $f(\mathbf{A}) = f(\hat{\mathbf{A}})$.

IT means that the weights of the alternatives are not influenced by α -transformations on triads. It has been inspired by the axiom *independence of circuits* in Bouyssou (1992).

A motivation for invariance to α -transformation on a triad can be the following. Consider a sports competition where player i has defeated player j , player j has defeated player k , while player k has defeated player i , and suppose that the three wins are equivalent. Then the final ranking is not allowed to change if the margins of victories are modified by the same amount. In particular, the three results can be reversed (j beats i , k beats j , and i beats k), or all comparisons can become a draw. *IT* is practically a generalization of this idea.

Proposition 3.2. *The Eigenvector Method violates invariance to α -transformation on a triad.*

Proof. Consider the following pairwise comparison matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1/8 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1/2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1/4 & 1/2 & 1 & 1 \end{bmatrix}.$$

$\hat{\mathbf{A}}$ can be obtained from \mathbf{A} through an α -transformation on the triad $(1, 2, 4)$ by $\alpha = 2$ as $\hat{a}_{1,2} = 2a_{1,2}$, $\hat{a}_{1,4} = a_{1,4}/2$, and $\hat{a}_{2,4} = 2a_{2,4}$. The corresponding weight vectors are

$$\begin{aligned} \mathbf{w}^{EM}(\mathbf{A}) &\approx \begin{bmatrix} 0.4269 & 0.2182 & 0.2182 & 0.1367 \end{bmatrix}^\top \neq \\ &\neq \begin{bmatrix} 0.3941 & 0.2256 & 0.2389 & 0.1413 \end{bmatrix}^\top \approx \mathbf{w}^{EM}(\hat{\mathbf{A}}), \end{aligned}$$

which shows the violation of the axiom *IT*. □

Proposition 3.3. *The Logarithmic Least Squares Method satisfies invariance to α -transformation on a triad.*

Proof. Take two pairwise comparison matrices $\mathbf{A}, \hat{\mathbf{A}} \in \mathcal{A}^{n \times n}$ such that $\hat{\mathbf{A}}$ is obtained from \mathbf{A} through an α -transformation on a triad, namely, they are identical except for $\hat{a}_{i,j} = \alpha a_{i,j}$ ($\hat{a}_{j,i} = a_{j,i}/\alpha$), $\hat{a}_{j,k} = \alpha a_{j,k}$ ($\hat{a}_{k,j} = a_{k,j}/\alpha$), and $\hat{a}_{k,i} = \alpha a_{k,i}$ ($\hat{a}_{i,k} = a_{i,k}/\alpha$). The product of row elements does not change, so $\mathbf{w}^{LLSM}(\mathbf{A}) = \mathbf{w}^{LLSM}(\hat{\mathbf{A}})$ according to (2). □

Corollary 3.1. *The counterexample concerning the Eigenvector Method and *IT* in Proposition 3.2 is minimal with respect to the number of alternatives as in the case of $n = 3$, *EM* and *LLSM* yield the same result (Crawford and Williams, 1985).*

4 Characterization of the Logarithmic Least Squares Method

Theorem 4.1. *The Logarithmic Least Squares Method is the unique weighting method satisfying correctness and invariance to α -transformation on a triad.*

Proof. *LLSM* satisfies both axioms according to Propositions 3.1 and 3.3.

For uniqueness, consider an arbitrary pairwise comparison matrix $\mathbf{A} \in \mathcal{A}^{n \times n}$ and a weighting method $f : \mathcal{A}^{n \times n} \rightarrow \mathbb{R}^n$, which meets correctness and invariance to α -transformation on a triad. Denote by $P_i = \sqrt[n]{\prod_{k=1}^n a_{i,k}}$ the geometric mean of row elements

for alternative i . In order to prove that f is equivalent to the Logarithmic Least Squares Method, it is enough to show that $f_i(\mathbf{A})/f_j(\mathbf{A}) = P_i/P_j$.

The proof will present that every pairwise comparison matrix can be transformed into a consistent matrix by a sequence of α -transformations on a triad carried out over an appropriately chosen set of triads: the triads to be modified always contain the first alternative, while the two others are considered according to the sequence $(n-1, n), (n-2, n), (n-2, n-1), (n-3, n), (n-3, n-1), \dots, (2, n), (2, n-1), \dots, (2, 3)$. Since the Logarithmic Least Squares Method satisfies invariance to α -transformation on a triad, each matrix of the sequence shares the same geometric mean weight vector. The procedure will stop after (maximally) $(n-1)(n-2)/2$ steps, which is the number of triads not containing the first alternative.

Let us introduce the pairwise comparison matrix $\mathbf{A}^{(n-1, n)} \in \mathcal{A}^{n \times n}$ such that $a_{1, n-1}^{(n-1, n)} := \alpha_{n-1, n} a_{n-1, n}$; $a_{1, n}^{(n-1, n)} := a_{1, n} / \alpha_{n-1, n}$; $a_{n-1, n}^{(n-1, n)} := \alpha_{n-1, n} a_{1, n}$ ¹ and $a_{i, j} := a_{i, j}^{(n-1, n)}$ for all other elements, where $\alpha_{n-1, n} = P_{n-1} / (P_n a_{n-1, n})$. Since $\mathbf{A}^{(n-1, n)}$ is obtained from \mathbf{A} through an α -transformation on a triad, $f(\mathbf{A}) = f(\mathbf{A}^{(n-1, n)})$ according to the assumption that f satisfies the axiom *IT*. If $n = 3$, $\mathbf{A}^{(n-1, n)}$ is consistent because $a_{1, 2}^{(2, 3)} = P_1/P_2$, $a_{1, 3}^{(2, 3)} = P_1/P_3$, and $a_{2, 3}^{(2, 3)} = P_2/P_3$, therefore correctness implies $f(\mathbf{A}) = \mathbf{w}^{LLSM}(\mathbf{A})$.

Otherwise, analogous α -transformations on a triad can be implemented until we get $\mathbf{A}^{(i, j)} \in \mathcal{A}^{n \times n}$, where $1 < i < j$ and $a_{k, \ell}^{(i, j)} = P_k/P_\ell$ for all $i \leq k < \ell$. The next step of the algorithm depends on the difference of i and j .

- If $j > i + 1$, then introduce the pairwise comparison matrix $\mathbf{A}^{(i, j-1)} \in \mathcal{A}^{n \times n}$ such that $a_{1, i}^{(i, j-1)} := \alpha_{i, j-1} a_{1, i}^{(i, j)}$; $a_{1, j}^{(i, j-1)} := a_{1, j}^{(i, j)} / \alpha_{i, j-1}$; $a_{i, j-1}^{(i, j-1)} := \alpha_{i, j-1} a_{i, j-1}^{(i, j)}$ and $a_{k, \ell}^{(i, j-1)} := a_{k, \ell}^{(i, j)}$ for all other elements, where $\alpha_{i, j-1} = P_i / (P_{j-1} a_{i, j-1}^{(i, j)})$. It can be checked that $a_{k, \ell}^{(i, j-1)} = a_{k, \ell}^{(i, j)} = P_k/P_\ell$ for all $i \leq k < \ell$, while $a_{i, j-1}^{(i, j-1)} = P_i/P_{j-1}$.
- If $j = i + 1$ and $i > 2$, then define the pairwise comparison matrix $\mathbf{A}^{(i-1, n)} \in \mathcal{A}^{n \times n}$ such that $a_{1, i-1}^{(i-1, n)} := \alpha_{i-1, n} a_{1, i-1}^{(i, n)}$; $a_{1, n}^{(i-1, n)} := a_{1, n}^{(i, n)} / \alpha_{i-1, n}$; $a_{i-1, n}^{(i-1, n)} := \alpha_{i-1, n} a_{i-1, n}^{(i, n)}$ and $a_{k, \ell}^{(i, j-1)} := a_{k, \ell}^{(i, j)}$ for all other elements, where $\alpha_{i-1, n} = P_{i-1} / (P_n a_{i-1, n}^{(i, n)})$. It can be checked that $a_{k, \ell}^{(i-1, n)} = a_{k, \ell}^{(i, n)} = P_k/P_\ell$ for all $i \leq k < \ell$, while $a_{i-1, n}^{(i-1, n)} = P_{i-1}/P_n$.

Finally, $\mathbf{A}^{(2, 3)} \in \mathcal{A}^{n \times n}$ is obtained such that $a_{k, \ell}^{(2, 3)} = P_k/P_\ell$ for all $2 \leq k < \ell$. Furthermore,

$$a_{1, j}^{(2, 3)} = a_{1, j} \frac{\prod_{m=j+1}^n \alpha_{j, m}}{\prod_{m=2}^{j-1} \alpha_{m, j}} = a_{1, j} \left(\prod_{m=j+1}^n \frac{P_j}{P_m} \frac{1}{a_{j, m}} \right) \left(\prod_{m=2}^{j-1} \frac{P_j}{P_m} a_{m, j} \right).²$$

However, $a_{m, j} = 1/a_{j, m}$ due to the reciprocity condition and $\prod_{m=1}^n a_{j, m} = P_j^n$, therefore

$$a_{1, j}^{(2, 3)} = \frac{P_j^{n-2}}{\prod_{m=j+1}^n P_m \prod_{m=2}^{j-1} P_m} \frac{1}{P_j^n} = \frac{1}{P_j} \frac{1}{\prod_{m=2}^n P_m}.$$

It is clear that $P_1 = 1 / (\prod_{m=2}^n P_m)$ as the product of all elements of \mathbf{A} gives one, which leads to

$$a_{1, j}^{(2, 3)} = \frac{P_1}{P_j}$$

¹ For the sake of simplicity, only the elements above the diagonal are indicated.

² Note that $\prod_{m=2}^{j-1} \alpha_{m, j} = 1$ if $j = 2$ and $\prod_{m=j+1}^n \alpha_{m, j} = 1$ if $j = n$.

for all $j \geq 2$. In other words, $\mathbf{A}^{(2,3)} \in \mathcal{A}^{n \times n}$ is a consistent pairwise comparison matrix such that $a_{i,j}^{(2,3)} = P_i/P_j = w_i^{LLSM}(\mathbf{A}^{(2,3)})/w_j^{LLSM}(\mathbf{A}^{(2,3)})$ for all $1 \leq i, j \leq n$. Consequently, $f(\mathbf{A}^{(2,3)}) = \mathbf{w}^{LLSM}(\mathbf{A}^{(2,3)})$ due to correctness. Weighting method f is invariant to α -transformation on a triad, hence

$$f(\mathbf{A}^{(2,3)}) = f(\mathbf{A}^{(2,4)}) = \dots = f(\mathbf{A}^{(n-1,n)}) = f(\mathbf{A}).$$

The Logarithmic Least Squares Method also satisfies *IT* according to Proposition 3.3, verifying the claim that $f(\mathbf{A}) = \mathbf{w}^{LLSM}(\mathbf{A})$. \square

Example 4.1. As an illustration of the proof of Theorem 4.1, consider the following pairwise comparison matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 16 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1/16 & 1 & 1 & 1 \end{bmatrix}, \text{ which leads to } \mathbf{w}^{LLSM}(\mathbf{A}) = \frac{1}{9} \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Since $a_{3,4} \neq w_3^{LLSM}(\mathbf{A})/w_4^{LLSM}(\mathbf{A})$, an α -transformation on the triad (1, 3, 4) should be carried out by $\alpha_{3,4} = [w_3^{LLSM}(\mathbf{A})/w_4^{LLSM}(\mathbf{A})]/a_{3,4} = 2$, which results in the matrix $\mathbf{A}^{(3,4)}$. After that, another α -transformation on the triad (1, 2, 4) is necessary by $\alpha_{2,4} = [w_3^{LLSM}(\mathbf{A})/w_4^{LLSM}(\mathbf{A})]/a_{2,4}^{(3,4)} = 2$ in order to get the matrix $\mathbf{A}^{(2,4)}$:

$$\mathbf{A}^{(3,4)} = \begin{bmatrix} 1 & 1 & 2 & 8 \\ 1 & 1 & 1 & 1 \\ 1/2 & 1 & 1 & 2 \\ 1/8 & 1 & 1/2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{(2,4)} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1/2 & 1 & 1 & 2 \\ 1/2 & 1 & 1 & 2 \\ 1/4 & 1/2 & 1/2 & 1 \end{bmatrix} = \mathbf{A}^{(2,3)}.$$

Finally, an α -transformation on the triad (1, 2, 3) should be carried out by $\alpha_{2,3} = [w_2^{LLSM}(\mathbf{A})/w_3^{LLSM}(\mathbf{A})] = 1$, so the pairwise comparison matrix remains unchanged, $\mathbf{A}^{(2,3)} = \mathbf{A}^{(2,4)}$. It is a consistent matrix, therefore any weighting method satisfying correctness and invariance to α -transformation on a triad should give $\mathbf{w}^{LLSM}(\mathbf{A})$ as the weight vector associated with the pairwise comparison matrix \mathbf{A} .

Proposition 4.1. *CO and IT are logically independent axioms.*

Proof. It is shown that there exist weighting methods, which satisfy one axiom, but do not meet the other:

- [1] *CO*: the Eigenvector Method (see Propositions 3.1 and 3.2);
- [2] *IT*: the flat method such that $f_i(\mathbf{A}) = 1/n$ for all $1 \leq i \leq n$.

\square

5 Conclusions

We have proved *LLSM* to be the *unique* weighting method among the procedures used to derive priorities from reciprocal pairwise comparison matrices, which is correct in the consistent case and invariant to a specific transformation on a triad. The somewhat

surprising fact is that our algorithm aims only to recover local consistency by focusing on a given triad without the consideration of other elements of the pairwise comparison matrix. Hence satisfaction of a local property fully determines a global weight vector.

Naturally, one can debate whether the axiom invariance to α -transformation on a triad should be accepted, but, at least, it reveals an important aspect of the geometric mean, contributing to the long list of its favourable theoretical properties (Barzilai et al., 1987; Barzilai, 1997; Dijkstra, 2013; Čaklović and Kurdija, 2017; Lundy et al., 2017; Csató, 2018c). Furthermore, the violation of this property can be an argument against the Eigenvector Method, a procedure having several other disadvantages, for example, the Pareto inefficiency of the weight vector (Blanquero et al., 2006; Bozóki, 2014; Bozóki and Fülöp, 2018), or the possibility of strong rank reversal in group decision-making (Pérez and Mokotoff, 2016; Csató, 2017).

Some directions for future research are also worth mentioning. First, further axiomatic analysis and characterizations of weighting methods can help in a better understanding of them. Second, α -transformation on a triad seems to be related to inconsistency reduction processes in pairwise comparison matrices (Koczkodaj and Szybowski, 2016; Szybowski, 2018). Third, the Logarithmic Least Squares Methods has been extended to the incomplete case when certain elements of the pairwise comparison matrix are unknown (Bozóki et al., 2010). Axiomatization on this more general domain seems to be promising and within reach, as revealed by Bozóki and Tsyganok (2017), although *LLSM* sometimes behaves strangely on this general domain (Csató and Rónyai, 2016).

Acknowledgements

We would like to thank *Denis Bouyssou* for inspiration and *Sándor Bozóki* for plentiful advice.

Three anonymous reviewers provided beneficial remarks and suggestions.

We are grateful to the audience of our talk at Corvinus Game Theory Seminar on 17 March 2017 for useful comments concerning the proof of Theorem 4.1.

The research was supported by OTKA grant K 111797 and by the MTA Premium Post Doctorate Research Program.

References

- Arrow, K. J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy*, 58(4):328–346.
- Barzilai, J. (1997). Deriving weights from pairwise comparison matrices. *Journal of the Operational Research Society*, 48(12):1226–1232.
- Barzilai, J., Cook, W. D., and Golany, B. (1987). Consistent weights for judgements matrices of the relative importance of alternatives. *Operations Research Letters*, 6(3):131–134.
- Blanquero, R., Carrizosa, E., and Conde, E. (2006). Inferring efficient weights from pairwise comparison matrices. *Mathematical Methods of Operations Research*, 64(2):271–284.

- Bouyssou, D. (1992). Ranking methods based on valued preference relations: A characterization of the net flow method. *European Journal of Operational Research*, 60(1):61–67.
- Bouyssou, D. and Marchant, T. (2014). An axiomatic approach to bibliometric rankings and indices. *Journal of Informetrics*, 8(3):449–477.
- Bozóki, S. (2014). Inefficient weights from pairwise comparison matrices with arbitrarily small inconsistency. *Optimization*, 63(12):1893–1901.
- Bozóki, S., Csató, L., and Temesi, J. (2016). An application of incomplete pairwise comparison matrices for ranking top tennis players. *European Journal of Operational Research*, 248(1):211–218.
- Bozóki, S. and Fülöp, J. (2018). Efficient weight vectors from pairwise comparison matrices. *European Journal of Operational Research*, 264(2):419–427.
- Bozóki, S., Fülöp, J., and Rónyai, L. (2010). On optimal completion of incomplete pairwise comparison matrices. *Mathematical and Computer Modelling*, 52(1-2):318–333.
- Bozóki, S. and Tsyganok, V. (2017). The logarithmic least squares optimality of the geometric mean of weight vectors calculated from all spanning trees for (in)complete pairwise comparison matrices. Manuscript. [arXiv: 1701.04265](https://arxiv.org/abs/1701.04265),.
- Brunelli, M. (2016). Recent advances on inconsistency indices for pairwise comparisons – a commentary. *Fundamenta Informaticae*, 144(3-4):321–332.
- Brunelli, M. (2017). Studying a set of properties of inconsistency indices for pairwise comparisons. *Annals of Operations Research*, 248(1):143–161.
- Brunelli, M. and Fedrizzi, M. (2011). Characterizing properties for inconsistency indices in the AHP. In *Proceedings of the International Symposium on the Analytic Hierarchy Process (ISAHP)*, pages 15–18.
- Brunelli, M. and Fedrizzi, M. (2015). Axiomatic properties of inconsistency indices for pairwise comparisons. *Journal of the Operational Research Society*, 66(1):1–15.
- Brunelli, M. and Fedrizzi, M. (2018). A general formulation for some inconsistency indices of pairwise comparisons. *Annals of Operations Research*, in press. DOI: [10.1007/s10479-018-2936-6](https://doi.org/10.1007/s10479-018-2936-6).
- Bryson, N. (1995). A goal programming method for generating priority vectors. *Journal of the Operational Research Society*, 46(5):641–648.
- Čaklović, L. and Kurdija, A. S. (2017). A universal voting system based on the Potential Method. *European Journal of Operational Research*, 259(2):677–688.
- Cavallo, B. and D’Apuzzo, L. (2012). Investigating properties of the \odot -consistency index. In Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., and Yager, R. R., editors, *Advances in Computational Intelligence: 14th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2012, Catania, Italy, July 9-13, 2012, Proceedings, Part IV*, volume 300 of *Communications in Computer and Information Science*, pages 315–327. Springer, Berlin-Heidelberg.

- Chao, X., Kou, G., Li, T., and Peng, Y. (2018). Jie Ke versus AlphaGo: A ranking approach using decision making method for large-scale data with incomplete information. *European Journal of Operational Research*, 265(1):239–247.
- Choo, E. U. and Wedley, W. C. (2004). A common framework for deriving preference values from pairwise comparison matrices. *Computers & Operations Research*, 31(6):893–908.
- Cook, W. D. and Kress, M. (1988). Deriving weights from pairwise comparison ratio matrices: An axiomatic approach. *European Journal of Operational Research*, 37(3):355–362.
- Crawford, G. and Williams, C. (1980). Analysis of subjective judgment matrices. Interim report R-2572-AF, Rand Corporation, Santa Monica.
- Crawford, G. and Williams, C. (1985). A note on the analysis of subjective judgment matrices. *Journal of Mathematical Psychology*, 29(4):387–405.
- Csató, L. (2013). Ranking by pairwise comparisons for Swiss-system tournaments. *Central European Journal of Operations Research*, 21(4):783–803.
- Csató, L. (2017). Eigenvector Method and rank reversal in group decision making revisited. *Fundamenta Informaticae*, 156(2):169–178.
- Csató, L. (2018a). Axiomatizations of inconsistency indices for triads. Manuscript. [arXiv: 1801.03355](https://arxiv.org/abs/1801.03355).
- Csató, L. (2018b). Characterization of an inconsistency ranking for pairwise comparison matrices. *Annals of Operations Research*, 261(1-2):155–165.
- Csató, L. (2018c). Characterization of the row geometric mean ranking with a group consensus axiom. *Group Decision and Negotiation*, in press. DOI: [10.1177/10.1007/s10726-018-9589-3](https://doi.org/10.1177/10.1007/s10726-018-9589-3).
- Csató, L. and Rónyai, L. (2016). Incomplete pairwise comparison matrices and weighting methods. *Fundamenta Informaticae*, 144(3-4):309–320.
- De Graan, J. G. (1980). Extensions of the multiple criteria analysis method of T. L. Saaty. Report, National Institute for Water Supply, Voorburg.
- Dijkstra, T. K. (2013). On the extraction of weights from pairwise comparison matrices. *Central European Journal of Operations Research*, 21(1):103–123.
- Fichtner, J. (1984). Some thoughts about the mathematics of the Analytic Hierarchy Process. Technical report, Institut für Angewandte Systemforschung und Operations Research, Universität der Bundeswehr München.
- Fichtner, J. (1986). On deriving priority vectors from matrices of pairwise comparisons. *Socio-Economic Planning Sciences*, 20(6):341–345.
- Koczkodaj, W. W. and Szwarz, R. (2014). On axiomatization of inconsistency indicators for pairwise comparisons. *Fundamenta Informaticae*, 132(4):485–500.

- Koczkodaj, W. W. and Szybowski, J. (2015). Axiomatization of inconsistency indicators for pairwise comparisons matrices revisited. Manuscript. [arXiv: 1509.03781v1](https://arxiv.org/abs/1509.03781v1).
- Koczkodaj, W. W. and Szybowski, J. (2016). The limit of inconsistency reduction in pairwise comparisons. *International Journal of Applied Mathematics and Computer Science*, 26(3):721–729.
- Koczkodaj, W. W. and Urban, R. (2018). Axiomatization of inconsistency indicators for pairwise comparisons. *International Journal of Approximate Reasoning*, 94:18–29.
- Lundy, M., Siraj, S., and Greco, S. (2017). The mathematical equivalence of the "spanning tree" and row geometric mean preference vectors and its implications for preference analysis. *European Journal of Operational Research*, 257(1):197–208.
- Pérez, J. and Mokotoff, E. (2016). Eigenvector priority function causes strong rank reversal in group decision making. *Fundamenta Informaticae*, 144(3-4):255–261.
- Rabinowitz, G. (1976). Some comments on measuring world influence. *Conflict Management and Peace Science*, 2(1):49–55.
- Saaty, T. L. (1980). *The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation*. McGraw-Hill, New York.
- Shapley, L. S. (1953). A value for n -person games. In Kuhn, H. W. and Tucker, A. W., editors, *Contributions to the Theory of Games Volume II*, volume 28 of *Annals of Mathematical Studies*, pages 307–317. Princeton University Press, Princeton, New Jersey.
- Szybowski, J. (2018). The improvement of data in pairwise comparison matrices. *Procedia Computer Science*, 126:1006–1013.
- van den Brink, R. and Pintér, M. (2015). On axiomatizations of the Shapley value for assignment games. *Journal of Mathematical Economics*, 60:110–114.
- Woeginger, G. J. (2008). An axiomatic characterization of the Hirsch-index. *Mathematical Social Sciences*, 56(2):224–232.