

# Computing explicit isomorphisms with full matrix algebras over $\mathbb{F}_q(x)$

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## Abstract

We propose a polynomial time  $f$ -algorithm (a deterministic algorithm which uses an oracle for factoring univariate polynomials over  $\mathbb{F}_q$ ) for computing an isomorphism (if there is any) of a finite dimensional  $\mathbb{F}_q(x)$ -algebra  $\mathcal{A}$  given by structure constants with the algebra of  $n$  by  $n$  matrices with entries from  $\mathbb{F}_q(x)$ . The method is based on computing a finite  $\mathbb{F}_q$ -subalgebra of  $\mathcal{A}$  which is the intersection of a maximal  $\mathbb{F}_q[x]$ -order and a maximal  $R$ -order, where  $R$  is the subring of  $\mathbb{F}_q(x)$  consisting of fractions of polynomials with denominator having degree not less than that of the numerator.

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## 1 Introduction

Decomposing finite dimensional associative algebras over a field  $\mathbb{K}$  include the tasks of isolating the radical, computing simple components of the radical-free part and finding minimal one-sided ideals within these simple components. In this paper we consider the case  $\mathbb{K} = \mathbb{F}_q(x)$  where  $\mathbb{F}_q$  is the finite field having  $q$  elements ( $q$  is a prime power). Decomposing algebras over  $\mathbb{F}_q(x)$  can be applied for example to factorization problems in certain skew polynomial rings, see the work [9] of Giesbrecht and Zhang and the recent paper [10] by Gómez-Torrecillas, Lobillo and Navarro. The first two tasks mentioned above can be accomplished by the polynomial time  $f$ -algorithm proposed in the work of the first and the third authors with Szántó [17]. The third problem, finding minimal one-sided ideals in simple algebras appears to be more difficult. In this paper we propose a solution which works in the special case when the algebra happens to be isomorphic to the full matrix algebra  $M_n(\mathbb{F}_q(x))$ .

If  $\mathcal{A}$  is a  $\mathbb{K}$ -algebra isomorphic to  $M_n(\mathbb{K})$ , then finding a minimal left ideal (or, more generally, finding an irreducible  $\mathcal{A}$ -module) is equivalent to constructing an isomorphism  $\phi : \mathcal{A} \rightarrow M_n(\mathbb{K})$ . Indeed, the matrices having possibly nonzero entries only in the first column form a minimal left ideal in  $M_n(\mathbb{K})$ , so the inverse image under  $\phi$  is a minimal left ideal in  $\mathcal{A}$ . Conversely, if  $M$  is an irreducible (that is, an  $n$ -dimensional)  $\mathcal{A}$ -module, then the action of  $\mathcal{A}$  on  $M$  gives an isomorphism  $\mathcal{A} \cong M_n(\mathbb{K})$ . Therefore the task of finding a minimal left ideal is also known as the *explicit isomorphism problem*.

Recall, that for an algebra  $\mathcal{A}$  over a field  $\mathbb{K}$  and a  $\mathbb{K}$ -basis  $a_1, \dots, a_m$  of  $\mathcal{A}$  over  $\mathbb{K}$  the products  $a_i a_j$  can be expressed as linear combinations of the  $a_i$ :

$$a_i a_j = \gamma_{ij1} a_1 + \gamma_{ij2} a_2 + \dots + \gamma_{ijm} a_m.$$

The elements  $\gamma_{ijk} \in \mathbb{K}$  are called structure constants. In this paper an algebra is considered to be given as a collection of structure constants.

Here we consider the explicit isomorphism problem for  $\mathbb{K} = \mathbb{F}_q(x)$ . For the case  $\mathbb{K} = \mathbb{F}_q$  the polynomial time f-algorithm given in [21] by the third author gives a solution. See also [13] for related deterministic methods. Recently the first and the third authors with Schicho [16] found an algorithm for solving the explicit isomorphism problem in the case of number fields. Their algorithm is a polynomial time ff-algorithm (it is allowed to call oracles for factoring polynomials over finite fields and for factoring integers), assuming that the degree of the matrix algebra and the degree of the number field over  $\mathbb{Q}$  are bounded. They combined algebraic techniques with tools from lattice geometry. Some improvements were given in [14]. Their results have various applications, for instance in arithmetic geometry (see [4], [5], [6]).

The structure of the paper will be the following. First we develop the necessary notions concerning polynomial lattices. In Section 2 we summarize the main tools for handling lattices over  $\mathbb{F}_q[x]$ . The orthogonality defect inequality and the basis reduction algorithm of Lenstra [18] are discussed here. We shall also use extensions by Paulus [19].

In the next section we state and prove certain facts about maximal orders over polynomial rings. Then we use them to construct maximal  $\mathbb{F}_q[x]$  and  $\mathbb{F}_q[\frac{1}{x}]$  orders in  $\mathcal{A}$ . The algorithms run in polynomial time if one is allowed to call oracles for factoring univariate polynomials over finite fields (it is a polynomial f-algorithm).

Let  $R$  be the subring of  $\mathbb{F}_q(x)$  consisting of those rational functions where the degree of the denominator is at least as high as the degree of the numerator. The main structural result of the paper is that the intersection of a maximal  $R$ -order and a maximal  $\mathbb{F}_q[x]$ -order is a finite dimensional  $\mathbb{F}_q$ -algebra which contains a primitive idempotent of  $\mathcal{A}$ . This theorem and the resulting algorithms are described in Section 4: we propose an algorithm to find a primitive idempotent of  $\mathcal{A}$ . Finally we arrive at the following theorem:

**Theorem 1.** *Let  $\mathcal{A}$  be isomorphic to  $M_n(\mathbb{F}_q(x))$ , and given by structure constants. Then there exists a polynomial (in  $n$  and in the size of the structure constants) f-algorithm which finds an explicit isomorphism between  $\mathcal{A}$  and  $M_n(\mathbb{F}_q(x))$ .*

Together with the polynomial time randomized algorithms of Cantor and Zassenhaus [3] (or, when  $q$  is a power of a prime bounded by a constant, with the deterministic method of Berlekamp [2]), this gives a randomized polynomial time solution in general (and a deterministic polynomial time algorithm for small characteristic) for the explicit isomorphism problem in the special case  $\mathbb{K} = \mathbb{F}_q(x)$ . We remark that the main ideas of this paper can be extended to the case  $\mathbb{K} = \mathbb{F}(x)$  for various fields  $\mathbb{F}$  of constants. (One just needs efficient methods for decomposing

finite dimensional algebras over  $\mathbb{F}$ , and lattice basis reduction over  $\mathbb{F}[x]$ .) However, extending our algorithms to finding minimal left ideals in algebras which are isomorphic to full matrix algebras over finite *extensions* of  $\mathbb{F}_q(x)$  looks more difficult.

Our main aim was in this work to show the existence of a polynomial time f-algorithm for the explicit isomorphism problem over  $\mathbb{F}_q(x)$ . No attempt has been made to optimize exponents and implied constants in the time bound. Those would require substantial further work. Our approach, in return, allowed a relatively short description of the methods and arguments.

## 2 Lattices over function fields

Most of our definitions and lemmas come from the seminal paper [18] of A. K. Lenstra. He introduced the notion of reduced basis and found an algorithm which finds a shortest vector in polynomial time in lattices over  $\mathbb{F}_q[x]$  (he considered sublattices of  $\mathbb{F}_q[x]^m$ ). Note that the analogous problem is NP-hard in the case of integer lattices [1]. First we state certain definitions about  $\mathbb{F}_q[x]$ -lattices in  $\mathbb{F}_q(x)^m$ .

**Definition 2.** Let  $f, g \in \mathbb{F}_q[x]$ . Then we set  $|\frac{f}{g}| = \deg(f) - \deg(g)$ . We will refer to  $|\cdot|$  as the valuation (or degree) of an element of  $\mathbb{F}_q(x)$ . We set  $|0| = -\infty$ . Let  $\mathbf{v} = (v_1, \dots, v_m)^T \in \mathbb{F}_q(x)^m$ . Then the valuation (or degree) of the vector  $\mathbf{v}$  is  $|\mathbf{v}| = \max(|v_1|, \dots, |v_m|)$ .

**Definition 3.**  $L$  is a full lattice in  $\mathbb{F}_q(x)^m$  if  $L = \{\alpha_1 \mathbf{b}_1 + \dots + \alpha_m \mathbf{b}_m \mid \alpha_i \in \mathbb{F}_q[x]\}$  where  $\mathbf{b}_1, \dots, \mathbf{b}_m$  is a basis (over  $\mathbb{F}_q(x)$ ) in  $\mathbb{F}_q(x)^m$ .

**Definition 4.** Let  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m \in \mathbb{F}_q(x)^m$ . Then the orthogonality defect  $OD(\mathbf{b}_1, \dots, \mathbf{b}_m)$  is defined as  $OD(\mathbf{b}_1, \dots, \mathbf{b}_m) = \sum_{i=1}^m |\mathbf{b}_i| - |\det(B)|$  where  $B$  is the matrix whose columns are the  $\mathbf{b}_i$ , ( $i = 1, \dots, m$ ).

The following lemma is from [18]. However, there it is stated in a slightly weaker form than we need it in this paper. So we state and prove the lemma here as well. The proof is also from [18].

**Lemma 5.** Let  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m \in \mathbb{F}_q(x)^m$  be linearly independent and  $\mathbf{a} = \sum_{i=1}^m \alpha_i \mathbf{b}_i$  where  $\alpha_i \in \mathbb{F}_q[x]$ . Then the following holds for every  $i$ :

$$|\alpha_i| \leq |\mathbf{a}| + OD(\mathbf{b}_1, \dots, \mathbf{b}_m) - |\mathbf{b}_i| \quad (1)$$

*Proof.* Consider the  $\alpha_i$  as unknowns. In this case we have  $m$  linear equations and  $m$  variables so we can use Cramer's rule. Note that  $\{\mathbf{b}_i\}_{i=1}^m$  is a basis so the determinant of the coefficient matrix  $B$  is non-zero. By Cramer's rule  $\alpha_i$  is equal to the quotient of 2 determinants. In other words  $\alpha_i$  multiplied by the determinant of the lattice is equal to the determinant where the  $i$ th column of  $B$  is switched to  $\mathbf{a}$ . Since these two sides are equal, their valuations are equal also (on both sides we have elements from  $\mathbb{F}_q(x)$ ). Note that the valuation of a determinant can be bounded from above by the sum of the valuations of its columns. To formalize this last sentence:

$$\begin{aligned} |\alpha_i| + |\det(B)| &\leq |\mathbf{b}_1| + |\mathbf{b}_2| + \dots + |\mathbf{b}_{i-1}| + |\mathbf{b}_{i+1}| + \dots + |\mathbf{b}_m| + |\mathbf{a}| \\ &= \sum_{i=1}^m |\mathbf{b}_i| - |\mathbf{b}_i| + |\mathbf{a}|. \end{aligned}$$

After rearranging we obtain the result. □

An implication of this lemma is the following. If we have a vector with small valuation, then the coefficients corresponding to a basis are also small, if the orthogonality defect of the basis is small. This also suggests that an ideal basis is one whose orthogonality defect is 0. This motivates the following definition.

**Definition 6.** A basis  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m \in \mathbb{F}_q(x)^m$  is called reduced if  $OD(\mathbf{b}_1, \dots, \mathbf{b}_m) = 0$ .

Lenstra proposed a polynomial time method [18, Algorithm 1.7] to compute reduced bases of sublattices of  $\mathbb{F}_q[x]^m$ . We quote [18, Proposition 1.14] below.

**Proposition 7.** Let  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$  be over  $\mathbb{F}_q(x)$  linearly independent vectors from  $\mathbb{F}_q[x]^m$  and let  $L$  be the  $\mathbb{F}_q[x]$ -lattice they generate. Let  $M = \max_{1 \leq i \leq m}(|\mathbf{b}_i|)$  and let  $M' = \max(M, 1)$ . Then there exists an algorithm which takes  $O(m^3 M'(OD(\mathbf{b}_1, \dots, \mathbf{b}_m) + 1))$  arithmetic operations in  $\mathbb{F}_q$  and returns a reduced basis  $\mathbf{c}_1, \dots, \mathbf{c}_m$  of  $L$  for which we have  $|\mathbf{c}_i| \leq M$  ( $i = 1, \dots, m$ ).

This result can be extended to find a reduced basis of a full lattice in  $\mathbb{F}_q(x)^m$ . Let us assume that we have a basis  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$  in  $\mathbb{F}_q(x)^m$ . Let  $L$  be the  $\mathbb{F}_q[x]$ -lattice generated by these vectors and let  $B$  be the matrix with columns  $\mathbf{b}_1, \dots, \mathbf{b}_m$ . Let  $\gamma$  be the least common multiple of all the denominators of the entries of  $B$ . We consider the lattice  $L'$  generated by  $\gamma\mathbf{b}_1, \dots, \gamma\mathbf{b}_m$ . Note that  $L' \in \mathbb{F}_q[x]^m$ . So using Lenstra's algorithm one can find a reduced basis  $\mathbf{c}_1, \dots, \mathbf{c}_m$  in  $L'$ . Note that  $|\det L'| = |\det L| + m|\gamma|$ . This implies that choosing  $\mathbf{b}'_i = \frac{1}{\gamma}\mathbf{c}_i$  we get a reduced basis of  $L$ . Since the orthogonality defect of  $\mathbf{b}_1, \dots, \mathbf{b}_m$  is the same as the orthogonality defect of  $\gamma\mathbf{b}_1, \dots, \gamma\mathbf{b}_m$ , we obtain the following:

**Proposition 8.** Let  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$  be a basis in  $\mathbb{F}_q(x)^m$  and let  $L$  be the  $\mathbb{F}_q[x]$ -lattice they generate. Let  $\gamma$  be the least common multiple of all the denominators for the entries of  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$ . Let  $M = |\gamma| + \max_{1 \leq i \leq m}(|\mathbf{b}_i|)$  and let  $M' = \max(M, 1)$ . Then there exists an algorithm which takes  $O(m^3 M'(OD(\mathbf{b}_1, \dots, \mathbf{b}_m) + 1))$  arithmetic operations in  $\mathbb{F}_q$  and returns a reduced basis of  $L$ .

*Proof.* Lenstra's method [18, Algorithm 1.7] together with its analysis [18, Proposition 1.14] gives the result.  $\square$

Given an integer  $k$ , the set of elements of the lattice whose valuation is smaller than  $k$  is a finite dimensional  $\mathbb{F}_q$ -vector space (this is a consequence of Lemma 5), and a basis of this vector space can also be computed efficiently.

The algorithm of Proposition 8 finds a reduced basis of a lattice which is given by a basis. However, one can ask the following question: what happens if the lattice is only given by an  $\mathbb{F}_q[x]$ -module generating system? In such situations an algorithm by Paulus [19, Algorithm 3.1.] is applicable. It finds a reduced basis of a lattice in  $\mathbb{F}_q(x)^m$  given by a system of generators. We shall make use of the fact that the valuations of the reduced basis obtained by Paulus' algorithm will not be greater than those of the given generators.

### 3 Maximal orders over $\mathbb{F}_q[x]$

#### 3.1 Preliminaries

In this subsection we assume that  $R$  is a principal ideal domain with quotient field  $\mathbb{K}$  and  $\mathcal{A}$  is a central simple algebra isomorphic to  $M_n(\mathbb{K})$ . Recall that an  $R$ -order in  $\mathcal{A}$  is a full  $R$ -lattice

which is at the same time a subring of  $\mathcal{A}$  containing the identity element. Maximal orders are orders maximal with respect to inclusion. We start with rephrasing [20, Theorem 21.6], specialized to this setting.

**Proposition 9.** *Let  $\mathcal{A} = \text{Hom}_{\mathbb{K}}(V, V)$  where  $V$  is a vector space of dimension  $n$  over  $\mathbb{K}$ . Let  $L$  be any full  $R$ -lattice in  $V$ . Then  $\text{Hom}_R(L, L)$ , identified with the subring*

$$\mathcal{O}(L) = \{a \in \mathcal{A} : aL \leq L\}$$

*of  $\mathcal{A}$ , is a maximal  $R$ -order in  $\mathcal{A}$ , and all maximal orders are of this form.*

In terms of matrices, the second statement of the theorem gives the following.

**Corollary 10.** *Assume that  $\Lambda$  is a maximal  $R$ -order in  $M_n(\mathbb{K})$ . Then there exists an invertible matrix  $P \in M_n(\mathbb{K})$  such that  $\Lambda = PM_n(R)P^{-1}$ .*

*Proof.* The theorem with  $V = \mathbb{K}^n$  gives that every maximal  $R$ -order in  $M_n(\mathbb{K})$  is  $\mathcal{O}(L)$  for a full  $R$ -lattice  $L$  in  $\mathbb{K}^n$ . Let  $P$  be a matrix whose columns are an  $R$ -basis of  $L$ .  $\square$

We remark that this claim can be found for quaternion algebras in [22, Exercise 4.2].

Our eventual aim is to construct a maximal  $R$ -order in  $M_n(\mathbb{K})$ . We will construct an initial order  $\Lambda_0$  in a rather straightforward way and iteratively enlarge it. Strictly speaking, our initial object  $\Lambda_0$  will not be an order. We say that an  $R$ -subalgebra  $\Lambda$  of  $\mathcal{A}$  is an *almost  $R$ -order* in  $\mathcal{A}$  if it is a full  $R$ -lattice in  $\mathcal{A}$ . Thus orders are almost orders containing the identity element of  $\mathcal{A}$ . It turns out that if  $\Lambda_0$  is an almost  $R$ -order, then the  $R$ -lattice generated by  $\Lambda_0$  and the identity element of  $\mathcal{A}$  is an  $R$ -order.

Discriminants enable us to control the depth of chains of (almost) orders and will also be useful in representing orders efficiently. The *reduced trace*,  $\text{tr } a$ , of an element  $a$  of an  $\mathcal{A}$  is simply the trace of  $a$  as an  $n$  by  $n$  matrix. (This is well defined by the Noether-Skolem theorem.) To compute reduced traces efficiently, it is not necessary to know an isomorphism  $\mathcal{A} \cong M_n(\mathbb{K})$ . If  $n$  is not divisible by the characteristic of  $\mathbb{K}$ , then  $\text{tr } a$  is  $\frac{1}{n}$  times the trace of the image of  $a$  under the regular representation of  $a$ . In general, the reduced trace can be computed by taking an appropriate coefficient of the  $n$ th root of the characteristic polynomial of the regular representation. This is because the regular representation of  $\mathcal{A}$  decomposes as a direct sum of  $n$  copies of the standard  $n$ -dimensional (irreducible) representation.

The *bilinear trace form* on  $\mathcal{A}$  is the symmetric bilinear function  $(a, b) \mapsto \text{tr } ab$ . As the matrix corresponding to an element of an almost  $R$ -order  $\Lambda$  is similar to a matrix with entries of  $R$ , the reduced trace of any element of  $\Lambda$  is from  $R$ . The discriminant  $d(\Lambda)$  can be defined as the principal ideal of  $R$  generated by the determinant of the Gram matrix  $(\text{tr } b_i b_j)_{i,j=1}^{n^2}$  where  $b_1, \dots, b_{n^2}$  are an  $R$ -basis for  $\Lambda$ . It is nonzero and independent of the choice of the basis. We can loosely think of  $d(\Lambda)$  as an element of  $R$ , defined up to a unit of  $R$ . As the bilinear trace form is non-degenerate, we have the following (see [20, Exercise 10.3]).

**Proposition 11.** *Let  $\Lambda$  and  $\Gamma$  be almost  $R$ -orders in  $\mathcal{A}$  such that  $\Lambda \subseteq \Gamma$ . Then  $d(\Gamma) | d(\Lambda)$  and  $\Lambda = \Gamma$  if and only if  $d(\Gamma) = d(\Lambda)$ .*

The following statement gives an  $R$ -lattice as an upper bound for  $R$ -orders containing a given almost order. An extension to more general rings  $R$  is used in the proof of [20, Theorem 10.3]. For orders over principal ideal domains it is stated explicitly in [15, Proposition 2.2]. As we need a slight generalization to almost orders, we give a proof for completeness.

**Proposition 12.** *Let  $\Lambda$  and  $\Gamma$  be almost  $R$ -orders in  $\mathcal{A}$  such that  $\Lambda \subseteq \Gamma$ . Then  $\Gamma \subseteq \frac{1}{d}\Lambda$  where  $d = d(\Lambda)$ .*

*Proof.* Let  $b_1, \dots, b_{n^2}$  be an  $R$ -basis for  $\Lambda$ . Then an element  $a \in \Gamma$  can be written as  $a = \sum_{i=1}^{n^2} \alpha_i b_i$  with  $\alpha_i \in \mathbb{K}$  ( $i = 1, \dots, n^2$ ). For  $j = 1, \dots, n^2$  put  $\beta_j = \text{tr } ab_j$ . Then the elements  $\beta_j$  are in  $R$  because the elements  $ab_j$  are in the almost order  $\Gamma$  which is contained in an  $R$ -order and hence have reduced trace from  $R$ . By linearity, we have  $\sum_i \alpha_i \text{tr } b_i b_j = \beta_j$ . Cramer's rule gives that each  $\alpha_i$  is a quotient of an element of  $R$  and  $d$ , which means that  $a \in \frac{1}{d}\Lambda$ .  $\square$

An algorithmic consequence is that it is possible to represent  $R$ -orders containing a given almost order  $\Lambda$  as submodules of the factor module  $\frac{1}{d}\Lambda/\Lambda$ . This will be particularly useful when  $R = \mathbb{F}_q[x]$ , in which case this factor is an  $n^2 \deg d$ -dimensional vector space over  $\mathbb{F}_q$ .

Our algorithm for computing maximal orders is an adaptation of the method proposed by the first and third authors for the case  $R = \mathbb{Z}$  in [15]. The method is discussed in the context of global fields in the Ph. D. thesis of the first author [12]. The algorithm finds a maximal order in a separable algebra over a global field. The algorithm proposed in this paper works also for separable algebras in a similar fashion. However, we only consider the case of full matrix algebras as some minor details (e.g. those regarding the trace form) are simpler in this case (and this is the only case we need later on).

For completeness, we include proofs of statements that are not rigorously proved for general principal ideal rings in [15].

Let  $M$  be a full  $R$ -lattice in  $\mathcal{A}$ . Then the left order of  $M$  is defined by

$$O_l(M) = \{a \in \mathcal{A} \mid aM \subseteq M\}.$$

The set  $O_l(M)$  is known to be an  $R$ -order of  $\mathcal{A}$ , see [20, Chapter 8]. It actually follows from the fact that  $O_l(M)$  is isomorphic to the intersection of two  $R$ -algebras: the image of  $\mathcal{A}$  under the left regular representation and  $\text{Hom}_R(M, M)$  (embedded into  $\text{Hom}_{\mathbb{K}}(\mathcal{A}, \mathcal{A})$ ).

The next two lemmas will be important tools for the algorithm which finds maximal orders. The first one reduces the question of enlarging an order over  $R$  to a similar task for  $R_\pi$ -orders where  $\pi$  is a prime element of  $R$ . Here  $R_\pi \leq \mathbb{K}$  denotes the localization of  $R$  at the prime ideal  $R\pi$ , that is,  $R_\pi = \{\frac{\alpha}{\beta} : \alpha, \beta \in R \text{ with } \pi \nmid \beta\}$ . If  $\Gamma$  is an  $R$ -order in  $\mathcal{A}$ , then  $\Gamma_\pi = R_\pi\Gamma$  is an  $R_\pi$ -order.

**Lemma 13.** *Let  $\pi$  be a prime element of  $R$  and  $\Gamma$  be an  $R$ -order in  $\mathcal{A}$ . Suppose that  $J$  is an ideal of  $\Gamma_\pi$  such that  $J \geq \pi\Gamma_\pi$  and  $O_l(J) > \Gamma_\pi$ . Put  $I = \Gamma \cap J$ . Then we have  $I \geq \pi\Gamma$  and  $O_l(I) > \Gamma$ .*

This lemma is stated for  $R = \mathbb{Z}$  in [15, Lemma 2.7]. The proof goes through for any principal domain  $R$ . We include it for completeness.

*Proof.* Clearly  $I \geq \pi\Gamma$  and  $I$  is an ideal of  $\Gamma$ . We also have  $J = R_\pi I$ . Let  $a \in O_l(J) \setminus \Gamma_\pi$ . Let  $a_1, a_2, \dots, a_t$  be a generating set of  $I$  as an  $R$ -module. Then these elements generate  $J$  as an  $R_\pi$ -module whence for  $i = 1, \dots, t$  we have

$$aa_i = \frac{\alpha_{i1}}{\beta_{i1}}a_1 + \dots + \frac{\alpha_{it}}{\beta_{it}}a_t, \tag{2}$$

where  $\alpha_{ij}, \beta_{ij} \in R$  and  $\pi$  does not divide  $\beta_{ij}$ . Now put  $\beta = \prod_{i,j} \beta_{ij}$ . Then  $\beta aa_i$  is in  $I$  ( $i = 1, \dots, t$ ), whence  $\beta aI \leq I$  and consequently  $\beta a \in O_l(I)$ . Finally we observe that  $\beta a$  is not in  $\Gamma$  since  $\beta$  is not divisible by  $\pi$  and therefore  $\beta a \in \Gamma$  would imply  $a \in \Gamma_\pi$ . The proof is complete.  $\square$

The next simple statement is stated in [15, Proposition 2.8] for  $R = \mathbb{Z}$ . It enables us to use  $\Lambda$  in place of  $\Lambda_\pi$  in computations regarding sufficiently large one or two-sided ideals of  $\Lambda_\pi$ .

**Proposition 14.** *Let  $\Lambda$  be an  $R$ -order in  $\mathcal{A}$  and  $\pi$  be a prime of  $R$ . Then the map  $\Phi : x \mapsto x + \pi\Lambda_\pi (x \in \Lambda)$  induces an isomorphism of rings  $\Lambda/\pi\Lambda \cong \Lambda_\pi/\pi\Lambda_\pi$ .*

*Proof.* Clearly  $\Phi : \Lambda \rightarrow \Lambda_\pi/\pi\Lambda_\pi$  is an epimorphism of rings. It is straightforward to check that its kernel is  $\pi\Lambda$ .  $\square$

Now we quote some further theorems and definitions from [15]. The next statement is [15, Proposition 3.1].

**Proposition 15.** *Let  $\Lambda_\pi$  be an  $R_\pi$ -order in  $\mathcal{A}$ . Then the residue class ring  $\overline{\Lambda}_\pi = \Lambda_\pi/\pi\Lambda_\pi$  is an algebra with identity element over the residue class field  $\overline{R}_\pi = R_\pi/\pi R_\pi \cong R/\pi R$  and  $\dim_{\mathbb{K}} \mathcal{A} = \dim_{\overline{R}_\pi} \overline{\Lambda}_\pi$ . If  $\Phi : \Lambda_\pi \rightarrow \overline{\Lambda}_\pi$  is the canonical epimorphism, then  $\pi\Lambda_\pi \subseteq \text{Rad}(\Lambda_\pi) = \Phi^{-1}\text{Rad}(\overline{\Lambda}_\pi)$  and  $\Phi$  induces a ring isomorphism  $\Lambda_\pi/\text{Rad}(\Lambda_\pi) \cong \overline{\Lambda}_\pi/\text{Rad}(\overline{\Lambda}_\pi)$ .*

Now we will introduce the important concept of extremal orders:

**Definition 16.** *Let  $\Lambda_\pi$  and  $\Gamma_\pi$  be  $R_\pi$ -orders in  $\mathcal{A}$ . We say that  $\Gamma_\pi$  radically contains  $\Lambda_\pi$  if and only if  $\Gamma_\pi \supseteq \Lambda_\pi$  and  $\text{Rad}(\Gamma_\pi) \supseteq \text{Rad}(\Lambda_\pi)$ . This is a partial ordering on the set of  $R_\pi$ -orders. Orders maximal with respect to this partial ordering are called extremal.*

The next statement is [15, Proposition 4.1].

**Proposition 17.** *An  $R_\pi$ -order  $\Lambda_\pi$  of  $\mathcal{A}$  is extremal if and only if  $\Lambda_\pi = O_l(\text{Rad}(\Lambda_\pi))$ .*

Finally, we quote [15, Proposition 4.5].

**Proposition 18.** *Let  $\Lambda_\pi \subset \Gamma_\pi$  be  $R_\pi$ -orders in  $\mathcal{A}$ . Suppose that  $\Lambda_\pi$  is extremal and  $\Gamma_\pi$  is minimal among the  $R_\pi$ -orders properly containing  $\Lambda_\pi$ . Then there exists a two-sided ideal  $I$  of  $\Lambda_\pi$  minimal among those containing  $\text{Rad}(\Lambda_\pi)$  such that  $O_l(I) \supseteq \Gamma_\pi$ .*

## 3.2 The algorithm

We start with a high-level description of the algorithm over a general principal ideal domain  $R$ . Let  $R$  be a principal ideal domain,  $\mathbb{K}$  its field of fractions. Suppose that an algebra  $\mathcal{A}$ , isomorphic to  $M_n(\mathbb{K})$  is given by structure constants  $\gamma_{ij}^k$  ( $i, j, k = 1, \dots, n^2$ ) from  $\mathbb{K}$  with respect to a basis  $a_1, \dots, a_{n^2}$ . We assume that these structure constants are represented as fractions of pairs of elements from  $R$ . Let  $\delta$  be a common multiple (e.g., the product or the l. c. m.) of the denominators. Then  $a'_i = \delta a_i$  ( $i = 1, \dots, n^2$ ) will be a basis with structure constants  $\delta\gamma_{ij}^k \in R$ . Therefore the  $R$ -submodule  $\Lambda_0$  of  $\mathcal{A}$  with basis  $a'_1, \dots, a'_{n^2}$  is an almost  $R$ -order.

We shall compute the discriminant  $d = d(\Lambda_0)$ . Let  $S = \{\pi_1, \dots, \pi_r\}$  be the set of the prime factors of  $d$ . Observe, that the discriminant of any  $R$ -order conjugate to  $M_n(R)$  is 1. This also holds for  $R_\pi$ -orders for any prime element  $\pi$ . Therefore, by Corollary 10 and by Proposition 11,  $\Lambda_{0\pi}$  is a maximal  $R_\pi$ -order for any prime  $\pi$  not in  $S$ .

Starting with the order  $\Lambda$  obtained by taking the  $R$ -module generated by  $\Lambda_0$  and the identity element, for each prime in  $S$  we test constructively whether  $\Lambda_\pi$  is a maximal  $R_\pi$  order using the two tests described below. By constructiveness we mean that in the "no" case we construct an  $R$ -order  $\Gamma \supsetneq \Lambda$ . If any of the tests finds such a  $\Gamma$ , then we proceed with  $\Gamma$  in place of  $\Lambda$ .

Otherwise, if  $\Lambda_\pi$  passes the tests for every  $\pi \in S$  then we conclude that  $\Lambda$  is already maximal. By Proposition 11 the number of such rounds is at most the number of the prime divisors of  $d$ , counted with multiplicities.

The first test is used to constructively decide whether  $\Lambda_\pi$  is an extremal  $R_\pi$ -order by checking if  $O_l(\text{Rad}(\Lambda_\pi)) = \Lambda_\pi$  (Proposition 17). To this end, we compute the ideal  $I = \text{Rad}(\Lambda_\pi) \cap \Lambda$ . By Lemma 13,  $\Lambda$  passes the test if and only if  $O_l(I) = \Lambda$ . Otherwise  $\Gamma = O_l(I)$  is an order strictly containing  $\Lambda$ . To compute  $I$ , we work with the  $n^2$ -dimensional  $R/\pi R$ -algebra  $\mathcal{B} = \Lambda/\pi\Lambda$ . From Propositions 14 and 15 we infer that  $I$  is the inverse image of  $\text{Rad}(\mathcal{B})$  with respect to the canonical map  $\Lambda \rightarrow \mathcal{B}$ .

If  $\Lambda_\pi$  passes the first test, then we proceed with the test of Proposition 18: if there exists an ideal  $J$  of  $\Lambda_\pi$  minimal among the two-sided ideals properly containing  $\text{Rad}(\Lambda_\pi)$  such that  $O_l(J) > \Lambda_\pi$ , then we construct an  $R$ -order  $\Gamma$  that properly contains  $\Lambda$ . Like for the first test, we can work in the  $R/\pi R$ -algebra  $\mathcal{B} = \Lambda/\pi\Lambda$ . Let  $J_1, \dots, J_m$  denote the minimal two-sided ideals of  $\mathcal{B}$  which contain  $\text{Rad}(\mathcal{B})$ . We have  $m \leq n^2$ . Let  $I_i$  denote the inverse image of  $J_i$  with respect to the map  $\Lambda \rightarrow \mathcal{B}$ . As in the first case we obtain, that we have to compute the rings  $O_l(I_i)$  for  $i = 1, \dots, m$ . We can stop when  $\Lambda < O_l(I_i)$  is detected, because then we have an order properly containing  $\Lambda$ .

### 3.3 The case $R = \mathbb{F}_q[x]$

We continue with details of the key ingredients of an efficient algorithm for  $R = \mathbb{F}_q[x]$  following the lines above. These will give an  $f$ -algorithm whose running time is polynomial in the size of the input. The input is an array of  $n^6$  structure constants represented as fractions of polynomials. We assume that the numerators are of degree at most  $d_N$  and the denominators are of degree at most  $d_D$ . Thus the size of the input is around  $n^6(d_D + d_N) \log q$ .

The l. c. m. of the denominators and hence a basis for the initial order  $\Lambda_0$  can be computed in polynomial time. The degree of this common denominator is at most  $n^6 d_D$ , whence  $\Lambda_0$  will have a basis  $a'_1, \dots, a'_{n^2}$ , where each  $a'_j$  is  $a_j$ , multiplied by a polynomial of degree at most  $n^6 d_D$ . The structure constants for the basis  $a'_1, \dots, a'_{n^2}$  are polynomials of degree at most  $n^6 d_D + d_N$ . The discriminant  $d = d(\Lambda_0)$  can be efficiently computed in a direct way following the definition. The entries of the matrices for the images of  $a'_j$  at the regular representation, written in terms of the basis  $a'_1, \dots, a'_{n^2}$  are just structure constants for the basis  $a'_1, \dots, a'_{n^2}$ . Therefore these entries are polynomials of degree bounded by  $n^6 d_D + d_N$  and hence the entries of the Gram matrix of the bilinear trace form are polynomials of degree  $2n^6 d_D + 2d_N$ . To compute  $d(\Lambda_0)$ , let  $n = p^r k$  where  $p$  is the characteristic of  $\mathbb{F}_q$  and  $k$  is relatively prime to  $p$ . Then the characteristic polynomial of  $a'_i a'_j$ , is the  $n$ th power of the characteristic polynomial of  $a'_i a'_j$  as an  $n$  by  $n$  matrix. Therefore it is of the form

$$\begin{aligned} (X^n - (\text{tr } a'_i a'_j) X^{n-1} + \dots)^n &= (X^{nk} - k(\text{tr } a'_i a'_j) X^{nk-1} + \dots)^{p^r} \\ &= X^{n^2} - (k \text{tr } a'_i a'_j)^{p^r} X^{n^2 - p^r} + \dots \end{aligned}$$

It follows that  $d(\Lambda_0)$  is a polynomial  $D_0$  of degree at most  $2n^8 d_D + 2n^2 d_N$ .

By Proposition 12, we have  $\Lambda \leq \frac{1}{D_0} \Lambda_0$  for any  $\mathbb{F}_q[x]$ -order  $\Lambda \geq \Lambda_0$ . Therefore we can represent  $\Lambda$  as the  $\mathbb{F}_q[x]$ -submodule  $\Lambda/\Lambda_0$  of the factor module  $\frac{1}{D_0} \Lambda_0/\Lambda_0$ . This factor module is an  $n^2 \deg D_0$ -dimensional vector space over the field  $\mathbb{F}_q$ . In fact, the elements  $\frac{x^k}{D_0} a'_i + \Lambda_0$  ( $i = 1, \dots, n^2$ ,  $k = 0, \dots, \deg D_0 - 1$  form an  $\mathbb{F}_q$ -basis) and we represent  $\Lambda/\Lambda_0$  by an  $\mathbb{F}_q$ -basis written in terms of this basis. Notice that the ideals  $I$  whose left order  $O_l(I)$  we compute



throughout the algorithm are all (left)  $\Lambda_0$ -submodules of  $\frac{1}{D_0}\Lambda_0$  containing  $D_0\Lambda_0$ . Observe next that the multiplication of  $\mathcal{A}$  induces an  $\mathbb{F}_q$ -bilinear map  $\mu$  from  $\frac{1}{D_0}\Lambda_0/\Lambda_0 \times I/D_0I$  to  $\frac{1}{D_0}I/I$ . For  $a \in \frac{1}{D_0}\Lambda$  and  $b \in I$ , one can set

$$\mu(a + \Lambda_0, b + D_0I) = ab + I.$$

This is well defined as  $(\frac{1}{D_0}\Lambda_0)(D_0I) = \Lambda_0I \subseteq I$ . Taking an  $\mathbb{F}_q$ -basis  $b_1, \dots, b_t$  of  $I/D_0I$ , the factor  $O_l(I)/\Lambda_0$  can be computed as the intersection of the kernels of the linear maps  $\mu(\cdot, b_i)$  ( $i = 1, \dots, t$ ). As the dimensions are bounded by polynomials in  $n$  and in the degree of  $D_0$ , for every  $I$  possibly occurring in the algorithm,  $O_l(I)$  is computable in polynomial time. Given an intermediate order  $\Lambda$ , we can compute the candidate ideals  $I$  by computing the radical of  $\mathcal{B} = \Lambda/g\Lambda$  for the irreducible factors  $g$  of  $D_0$  and the minimal two-sided ideals of  $\mathcal{B}$  containing the radical and finally by taking inverse images of these at the map  $\Lambda \rightarrow \mathcal{B}$ . As  $\mathcal{B}$  is an  $n^2 \deg g$ -dimensional vector space over  $\mathbb{F}_q$ , its radical and the minimal two-sided ideals containing it can be computed in time polynomial in the input size using for example the deterministic method of the third author [21]. The minimal two-sided ideals containing the radical, that is, the simple components of  $\mathcal{B}/\text{Rad}(\mathcal{B})$  can be found by the deterministic  $f$ -algorithm of Friedl and the third author [8].

For  $\alpha_{ik} \in \mathbb{F}_q$  ( $i = 1, \dots, n^2$ ,  $k = 0, \dots, \deg D_0 - 1$ ), the combination  $\sum_{i=1}^{n^2} \sum_{k=0}^{\deg D_0 - 1} \alpha_{ik} \frac{x^k}{D_0} a'_i$  of  $a'_1, \dots, a'_{n^2}$  has coefficients whose numerators and denominators are polynomials of degree at most  $\deg D_0 \leq 2n^8 d_D + 2n^2 d_N$ . Together with  $a'_1, \dots, a'_{n^2}$ , such representatives for an  $\mathbb{F}_q$ -basis of  $\Lambda/\Lambda_0$  give a system of generators over  $\mathbb{F}_q[x]$  for  $\Lambda$ . When  $\Lambda$  turns out to be maximal, then we can use the lattice reduction algorithm by Paulus [19] to obtain a basis for  $\Lambda$  consisting of combinations of  $a'_1, \dots, a'_{n^2}$  with coefficients having numerators and denominators also of degree at most  $\deg D_0 \leq 2n^8 d_D + 2n^2 d_N$ . (Here we make use of the nature of the reduction algorithm: it never increases the maximum degree of the coordinates of the intermediate generators.) This gives us the following theorem:

**Theorem 19.** *Let  $\mathcal{A}$  be isomorphic to  $M_n(\mathbb{F}_q(x))$  given by structure constants having numerators and denominators of degree at most  $d_C \geq 1$ . A maximal  $\mathbb{F}_q[x]$ -order  $\Lambda$  can be constructed by an  $f$ -algorithm running in time  $(n + d_C + \log q)^{O(1)}$ . The output of the algorithm is an  $\mathbb{F}_q[x]$ -basis for  $\Lambda$  whose elements are linear combinations in the original basis of  $\mathcal{A}$  with coefficients which are ratios of polynomials of degree at most  $(2n^8 + n^6 + 2n^2)d_C$ .*

Notice that  $\sum_{j=0}^d \alpha_j x^j = x^d \sum_{j=0}^d \alpha_{d-j} \frac{1}{x^j}$ . Therefore a fraction of two polynomials in  $x$  of degree at most  $d$  can also be written as a fraction of two polynomials in  $\frac{1}{x}$  also of degree at most  $d$ . Therefore Theorem 19 gives the following.

**Corollary 20.** *Let  $\mathcal{A}$  and  $d_C$  be as in Theorem 19. Then a maximal  $\mathbb{F}_q[\frac{1}{x}]$ -order  $\Delta$  can be constructed by an  $f$ -algorithm running in time  $(n + d_C + \log q)^{O(1)}$ . The output of the algorithm is an  $\mathbb{F}_q[\frac{1}{x}]$ -basis for  $\Delta$  whose elements are linear combinations in the original basis of  $\mathcal{A}$  with coefficients which are ratios of polynomials (in  $x$ ) of degree at most  $(2n^8 + n^6 + 2n^2)d_C$ .*

We remark that we will actually need an  $\mathbb{F}_q[\frac{1}{x}]_{(\frac{1}{x})}$ -basis for a maximal  $\mathbb{F}_q[\frac{1}{x}]_{(\frac{1}{x})}$ -order. Obviously, for this an  $\mathbb{F}_q[\frac{1}{x}]$ -basis for an  $\mathbb{F}_q[\frac{1}{x}]$ -order  $\Delta$  whose localization at the prime  $\frac{1}{x}$  is maximal, will do. Therefore it will be actually sufficient to apply the main steps of the order increasing algorithm only for the prime  $\frac{1}{x}$  of  $\mathbb{F}_q[\frac{1}{x}]$ .

## 4 Finding a rank 1 idempotent in $\mathcal{A}$

Let  $R \subseteq \mathbb{F}_q(x)$  be the set of rational functions having degree at most 0 (note that the 0 polynomial has degree  $-\infty$  hence it also belongs to  $R$ ). Thus, if  $f, g \in \mathbb{F}_q[x]$ ,  $g \neq 0$ , then  $\frac{f}{g} \in R$  iff  $\deg f \leq \deg g$ . It is easy to see that  $R$  is a subring of  $\mathbb{F}_q(x)$ . Actually  $R$  is the valuation ring for the valuation  $-\deg$  of  $\mathbb{F}_q(x)$ . An alternative view is that  $R = \mathbb{F}_q[\frac{1}{x}]_{(\frac{1}{x})}$ , the localization of the ring  $\mathbb{F}_q[\frac{1}{x}]$  at the prime ideal  $(\frac{1}{x})$ . (In fact, one readily verifies that the elements of  $R$  are precisely the functions of the form  $f(\frac{1}{x})/g(\frac{1}{x})$ , where  $f, g$  are univariate polynomials over  $\mathbb{F}_q$  and the constant term of  $g$  is not 0.) Thus  $R$  is a discrete valuation ring, and as such, a principal ideal ring.

The main structural result of the paper is the following theorem. It identifies a finite subalgebra  $C$  of modest size in  $\mathcal{A}$ , which contains a primitive idempotent of  $\mathcal{A}$ .

**Theorem 21.** *Let  $\mathcal{A} \cong M_n(\mathbb{F}_q(x))$  and let  $\Lambda$  be a maximal  $\mathbb{F}_q[x]$ -order in  $\mathcal{A}$ . Also, let  $R$  be the subring of  $\mathbb{F}_q(x)$  discussed above, that is, the set of rational functions of degree at most zero. Let  $\Delta$  be a maximal  $R$ -order in  $\mathcal{A}$ . Let  $b_1, \dots, b_{n^2}$  be an  $\mathbb{F}_q[x]$ -basis of  $\Lambda$ , and for  $j = 1, \dots, n^2$  let  $d_j$  be the smallest integer such that  $\frac{1}{x^{d_j}}b_j \in \Delta$ . Let  $d_{\min} = \min\{d_j : 1 \leq j \leq n^2\}$   $d_{\max} = \max\{d_j : 1 \leq j \leq n^2\}$ . Then*

(i) *For every element  $a \in \Lambda \cap \Delta$  we have  $a = \sum \alpha_i b_i$ , where the  $\alpha_i$  are polynomials in  $\mathbb{F}_q[x]$  of degree at most  $n^2 d_{\max} - d_{\min}$ .*

(ii)  *$\Lambda \cap \Delta$  contains a primitive idempotent of  $\mathcal{A}$ .*

*Proof.* Let  $\phi : \mathcal{A} \rightarrow M_n(\mathbb{F}_q(x))$  be an algebra isomorphism such that  $\phi(\Delta) = M_n(R)$ . (Such a  $\phi$  exists by Corollary 10.) We show that the  $\mathbb{F}_q[x]$ -lattice  $\phi(\Lambda)$  in  $M_n(\mathbb{F}_q(x))$  (the latter considered as  $\mathbb{F}_q(x)^{n^2}$ ) has determinant 1. To see this, let  $B$  be the matrix whose columns form an  $\mathbb{F}_q[x]$ -basis for the  $\mathbb{F}_q[x]$  lattice  $\phi(\Lambda)v \subset \mathbb{F}_q(x)^n$  where  $v$  is a nonzero vector from  $\mathbb{F}_q(x)^n$ . Then  $\phi(\Lambda) = BM_n(\mathbb{F}_q[x])B^{-1}$ . The claim on the determinant follows from that the standard lattice  $\mathbb{F}_q[x]^{n^2}$  has determinant one and from that the conjugation  $X \mapsto BXB^{-1}$ , considered as an  $\mathbb{F}_q(x)$ -linear transformation on  $\mathbb{F}_q(x)^{n^2}$ , has determinant one. For the latter, notice that multiplication by  $B^{-1}$  from the right is similar to a block diagonal matrix consisting of  $n$  copies of  $B^{-1}$ , and hence has determinant  $(\det B^{-1})^n$ , while multiplication by  $B$  from the left has determinant  $(\det B)^n$ .

Let  $C = \Lambda \cap \Delta$ . As  $\Delta = \phi^{-1}(M_n(R))$ ,  $C$  can be characterized as the set of the elements  $a$  of  $\Lambda$  such that  $\phi(a)$  has no entries of positive degree. As both  $\Delta$  and  $\Lambda$  are  $\mathbb{F}_q$ -algebras, so is  $C$ .

Notice that for  $0 \neq a \in \mathcal{A}$  the degree of  $\phi(a) \in M_n(\mathbb{F}_q(x))$  (the maximum of the degrees of the entries of the matrix  $\phi(a)$ ) is just the minimal (possibly negative) integer  $r$  such that  $\frac{1}{x^r}\phi(a) \in M_n(R)$ , or, equivalently,  $x^{-r}a \in \Delta$ . It follows that the degrees of the entries of  $\phi(b_j)$  are bounded by  $d_{\max}$  and hence the orthogonality defect of the basis  $\phi(b_1), \dots, \phi(b_{n^2})$  for  $\phi(\Lambda)$  is at most  $n^2 d_{\max}$ , because  $|\det \phi(\Lambda)| = 0$ . Therefore, for  $a = \sum_{j=1}^{n^2} \alpha_j b_j \in C$  Lemma 5 gives that  $\alpha_j$  has degree at most  $n^2 d_{\max} - d_{\min}$ , showing statement (i).

To establish statement (ii), consider an invertible matrix  $B \in M_n(\mathbb{F}_q(x))$  for which  $\phi(\Lambda) = B^{-1}M_n(\mathbb{F}_q[x])B$ . Let us consider the lattice  $L_1 = B^{-1}\mathbb{F}_q[x]^n$  in  $\mathbb{F}_q(x)^n$ . The determinant of  $L_1$  is obviously  $\det B^{-1}$ . Let us denote by  $\delta$  be the degree of  $\det B$ . Let  $B^{-1}u_1, \dots, B^{-1}u_n$ , with  $u_i \in \mathbb{F}_q[x]^n$ , be an  $\mathbb{F}_q[x]$ -basis of orthogonality defect zero for  $L_1$ . One can obtain such a basis by lattice basis reduction. Similarly, let  $L_2 = B^T\mathbb{F}_q[x]$ . Then  $L_2$  is an  $\mathbb{F}_q[x]$ -lattice

having determinant  $\det B$ . Let  $B^T u'_1, \dots, B^T u'_n$ , with  $u'_i \in \mathbb{F}_q[x]^n$ , be a basis of defect zero for  $L_2$ . Now we define a graph. We connect  $u_i$  with  $u'_j$  with an edge if  $u'_j{}^T u_i \neq 0$ . This defines a bipartite graph having these  $2n$  vectors as vertices satisfying Hall's criterion for having a perfect matching. (A set of  $s$  vectors from  $u_1, \dots, u_n$  having less than  $s$  neighbors would span a subspace of dimension  $s$  having an orthocomplement having dimension larger than  $n - s$ .) By changing the order of  $u'_j$ s we arrange that  $u'_i{}^T u_i \neq 0$  ( $i = 1, \dots, n$ ). We have

$$\sum_{j=1}^n (|B^{-1}u_j| + |B^T u'_j|) = \sum_{j=1}^n |B^{-1}u_j| + \sum_{j=1}^n |B^T u'_j| = -\delta + \delta = 0,$$

whence there exists at least one index  $i$ , such that the maximum degree of the coordinates of  $B^{-1}u_i$  and the maximum degree of the coordinates of  $B^T u'_i$  add up to at most zero. Let  $i$  be such an index and let  $S$  resp.  $S'$  be the matrix whose first column is  $u_i$  resp.  $u'_i$ , and whose remaining entries are zero. Now  $Z = B^{-1}SS'^T B$  is a matrix whose entries are of degree at most zero. Also,  $Z \in \phi(\Lambda)$ . Therefore  $\phi^{-1}(Z)$  is in  $C$ . Furthermore,  $Z$  has rank one as it is similar to  $SS'^T = u_i u'_i{}^T$ . Also, as  $(u_i u'_i{}^T)^2 = \mu u_i u'_i{}^T$  where  $\mu = u'_i{}^T u_i \neq 0$ . It follows that the minimal polynomial of  $Z$  over  $\mathbb{F}_q(x)$  as well as that of  $\phi^{-1}(Z)$  is  $X^2 - \mu X$  with a nonzero  $\mu \in \mathbb{F}_q(x)$ . As  $\phi^{-1}(Z) \in \Lambda \cap \Delta$ , we have  $\mu \in \mathbb{F}_q[x] \cap R = \mathbb{F}_q$ . Now  $e = \frac{1}{\mu} \phi^{-1}(Z)$  is an idempotent in  $C$  such that  $\phi(e)$  has rank one.  $\square$

*Remark 22.* We give an example of a  $C$  which is not isomorphic to a full matrix algebra over  $\mathbb{F}_q(x)$ . Let  $\Lambda = B^{-1}M_2(\mathbb{F}_q[x])B$  where  $B$  is the following matrix:

$$\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix}.$$

Let  $\Gamma = M_2(R)$ , i.e. those matrices whose degree is at most 0. Then  $C = \Gamma \cap \Lambda$  is generated as an  $\mathbb{F}_q$  vector space by the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{t} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{t^2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that  $C$  has dimension 5 over  $\mathbb{F}_q$ , hence it cannot be isomorphic to  $M_2(\mathbb{F}_q)$ . As a matter of fact, it is not even semisimple. The radical of  $C$  consists of those matrices whose diagonal entries are 0. Finally, note that  $C/\text{Rad } C \cong \mathbb{F}_q \oplus \mathbb{F}_q$ .

For finding a primitive idempotent of  $\mathcal{A}$  inside  $C$  we can use the method described in the proof of the following lemma.

**Lemma 23.** *Let  $C$  be the finite  $\mathbb{F}_q$ -algebra from Theorem 21, and let  $e_1, \dots, e_r$  be a complete system of orthogonal primitive idempotents in  $C$ . Then there exists an  $i$  such that  $e_i$  is a rank 1 idempotent in  $\mathcal{A}$ .*

*Having a basis of  $C$  at hand (a subset of  $\mathcal{A}$ ), one can find such an idempotent by a polynomial time  $f$ -algorithm.*

*Proof.* We note first, that the identity element of  $\mathcal{A}$  is in  $C$ , hence  $C$  has idempotents. Let  $x \in C$  be an element which is a rank 1 idempotent in  $\mathcal{A}$ . By Theorem 21 such an  $x$  exists. Next observe that there exists an index  $i$ , for which  $e_i x$  is not in the radical of  $C$ . For, otherwise  $\sum_{i=1}^r e_i x = x$  would be in the radical of  $C$ , which is impossible, as  $x$  is not nilpotent. Let us denote this

primitive idempotent  $e_i$  by  $e$ . Since  $ex$  is not in the radical of  $C$ , the right ideal  $exC$  it generates in  $C$  contains a nonzero idempotent  $f$ . Indeed, we can consider this right ideal as an  $\mathbb{F}_q$ -algebra which is not nilpotent. Hence if we factor out its radical, then we have a nonzero idempotent there ([7, Corollary 2.2.5]), which can be lifted to an idempotent in  $exC$  ([7, Corollary 3.1.2]). Write  $f = exy$  with a suitable  $y \in C$ . We have  $ef = e(exy) = e^2xy = exy = f$ . We verify now that both  $fe$  and  $e - fe$  are idempotent elements:

$$(fe)^2 = fefe = f(ef)e = ffe = fe$$

and

$$(e - fe)^2 = e^2 + (fe)^2 - efe - fee = e + fe - fe - fe = e - fe.$$

Furthermore, they are orthogonal:

$$fe(e - fe) = (fee) - (fe)^2 = fe - (fe)^2 = 0$$

and

$$(e - fe)fe = (ef)e - (fe)^2 = fe - (fe)^2 = 0.$$

Since  $e$  is a primitive idempotent, one has either  $fe = 0$  or  $fe = e$ . We show that the first case cannot happen. If  $fe = 0$  then  $fef = 0$ . However,  $fef = f^2 = f$  which is not zero. This implies that  $fe = e$ , and  $e = exye$ . Since  $x$  had rank 1 in  $\mathcal{A}$ ,  $e$  also has rank 1 in  $\mathcal{A}$ .

As for the computational part of the statement, first one computes a Wedderburn-Malcev complement in  $C$ : a subalgebra  $B$  of  $C$  which is isomorphic to  $C/\text{Rad}(C)$ . This can be done in deterministic polynomial time using the algorithm of [11, Theorem 3.1]. Then we can use for example the polynomial time  $f$ -algorithms of [8] and [21] to compute a complete system of primitive idempotents in  $B$ . To calculate ranks, we can use the fact that for  $a \in \mathcal{A}$  the left ideal  $a\mathcal{A}$  has dimension  $rn$  over  $\mathbb{F}_q(x)$  where  $r$  is the rank of  $a$  (considered as an  $n$  by  $n$  matrix).  $\square$

We prove a bound on  $d_{\min}$  and  $d_{\max}$  in the case when  $\Lambda$  and  $\Delta$  are the maximal orders constructed in Theorem 19 and Corollary 20, respectively.  $\Lambda$  is an  $\mathbb{F}_q[x]$ -order and  $\Delta$  is viewed as an  $R$ -order here.

**Lemma 24.** *For the pair of maximal orders as above, we have  $d_{\max} \leq (2n^8 + 2n^6 + 2n^2)d_C$  and  $d_{\min} \geq -2(2n^8 + n^6 + 2n^2)d_C$*

*Proof.* For short, we write  $L = (2n^8 + n^6 + 2n^2)d_C$ . Let  $a_1, \dots, a_{n^2}$  be the input basis of  $\mathcal{A}$  we use in the algorithms of Theorem 19 and Corollary 20. We know that the numerators and denominators of the structure constants for  $\mathcal{A}$  are polynomials of degree at most  $d_C$ . Let  $g^*(1/x)$  be the smallest common denominator of the structure constants when written as rational functions in  $\frac{1}{x}$ . The degree of  $g^*$  is at most  $n^6d_C$ . We know that the  $g^*(1/x)a_i$  are in the starting almost  $\mathbb{F}_q[\frac{1}{x}]$ -order  $\Delta_0$ , hence they are also in  $\Delta$ . Also, one can then write

$$g^* \left( \frac{1}{x} \right) = \frac{1}{x^\ell} h \left( \frac{1}{x} \right)$$

where  $h(y) \in \mathbb{F}_q[y]$  and  $h(0) \neq 0$ . We have here  $\ell \leq n^6d_C$ . We claim that  $\frac{1}{x^{n^6d_C}}a_i \in \Delta$  hold for every  $i$ . Indeed

$$\frac{1}{x^{n^6d_C}}a_i = \frac{1}{x^{n^6d_C - \ell}} \cdot g^* \left( \frac{1}{x} \right) a_i.$$

Here the first factor is in  $R$ , the second is in  $\Delta$ , thus giving the claim.

We know from Theorem 19 that every basis element  $b_j$  of  $\Lambda$  is a linear combination of the  $a_i$  with coefficients  $\alpha_i \in \mathbb{F}_q(x)$ , and the numerator as well as the denominator of  $\alpha_i$  has degree at most  $L$ . We claim now that  $\frac{1}{x^{n^6 d_C + L}} b_j \in \Delta$ . Indeed, we have

$$\frac{1}{x^{n^6 d_C + L}} \alpha_i a_i = \left( \frac{1}{x^{n^6 d_C}} a_i \right) \cdot \left( \frac{1}{x^L} \alpha_i \right).$$

The first factor is in  $\Delta$ , the second is in  $R$  and the upper bound follows.

As for  $d_{min}$ , we observe that the coefficients for the elements of  $\Lambda$  in the basis  $\{a_i\}$  are rational functions of degree at least  $-L$  (Theorem 19). Similarly, by Corollary 20 the coefficients for the elements of  $\Delta$  in the basis  $\{a_i\}$  are rational functions of degree at most  $L$ . It follows that for  $d < -2L$  the element  $\frac{1}{x^d} b_j$  can not be in  $\Delta$ , as the coefficient  $\frac{1}{x^d} \alpha_i$  has degree at least  $L + 1$ .  $\square$

Now we turn to the algorithmic task of finding (an  $\mathbb{F}_q$ -basis of)  $C$ .

**Lemma 25.** *Let  $b_1, \dots, b_{n^2}$  be the  $\mathbb{F}_q[x]$ -basis of  $\Lambda$  constructed by the algorithm of Theorem 19, and let  $u_1, \dots, u_{n^2}$  be the  $R$ -basis of  $\Delta$  constructed by the method of Corollary 20. From these data we can construct an  $F_q$ -basis of  $C$  in deterministic polynomial time.*

*Proof.* We consider the elements of  $\mathcal{A}$  as vectors in the basis  $u_1, \dots, u_{n^2}$ . This way the elements of  $\mathcal{A}$  can be viewed as vectors from  $\mathbb{F}_q(x)^{n^2}$  in the usual way: an element  $a \in \mathcal{A}$  with  $a = \sum_{j=1}^{n^2} \alpha_j u_j$  is represented by the vector

$$(\alpha_1, \dots, \alpha_{n^2})^T \in \mathbb{F}_q(x)^{n^2}.$$

Observe, that a vector as above represents an element of  $\Delta$  iff  $|\alpha_i| \leq 0$  holds for every  $i$ . Consider now the vectors  $b'_i \in \mathbb{F}_q(x)^{n^2}$  representing the basis elements  $b_i$  of  $\Lambda$ . They generate a full  $\mathbb{F}_q[x]$ -lattice (corresponding to  $\Lambda$ ) in  $\mathbb{F}_q(x)^{n^2}$ . We next compute a reduced basis  $c_1, \dots, c_{n^2}$  of this lattice. An element

$$a = \sum_{i=1}^{n^2} \beta_i c_i \quad \text{with } \beta_i \in \mathbb{F}_q[x] \text{ for } i = 1, \dots, n^2$$

represents an element of  $C = \Lambda \cap \Delta$  iff  $|a| \leq 0$ . We claim that this latter condition is equivalent to the set of inequalities

$$|\beta_i c_i| = |\beta_i| + |c_i| \leq 0, \quad i = 1, \dots, n^2.$$

Indeed, as the  $\{c_i\}$  is a reduced  $\mathbb{F}_q[x]$ -basis, from Lemma 5 we obtain that

$$|\beta_i| \leq |a| + OD(c_1, \dots, c_{n^2}) - |c_i| = |a| - |c_i| \tag{3}$$

for every  $i$ , hence if  $|a| \leq 0$  then  $|\beta_i c_i| \leq 0$  for every  $i$ . Conversely,  $|\beta_i c_i| \leq 0$  for every  $i$  obviously implies that  $|a| \leq 0$ . We conclude that the elements  $x^j c_i$  such that  $1 \leq i \leq n^2$  and  $j$  is a natural number with  $j + |c_i| \leq 0$  form an  $\mathbb{F}_q$ -basis of  $C$ . Theorem 21 and Lemma 24 provide a polynomial upper bound for the dimension of  $C$  over  $\mathbb{F}_q$ , and hence on the number of such elements  $x^j c_i$ .<sup>1</sup>

The algorithmic subtasks involved here: change of basis from the input basis to the basis  $\{u_i\}$ , and the lattice basis reduction both can be done in deterministic polynomial time, hence from  $\Lambda$  and  $\Delta$  we obtain  $C$  in polynomial time.  $\square$

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<sup>1</sup>A polynomial bound for the dimension of  $C$  follows also simply from the polynomiality of the algorithm described here.

The main steps of our algorithm for finding a rank 1 idempotent element  $e \in \mathcal{A}$  are as follows.

- 
1. Construct a maximal  $\mathbb{F}_q[x]$ -order  $\Lambda$  and a maximal  $R$ -order  $\Delta$ , by the f-polynomial time algorithms of Theorem 19, and Corollary 20, respectively.
  2. Compute an  $\mathbb{F}_q$ -basis of the finite algebra  $C = \Lambda \cap \Delta$  using the polynomial time algorithm of Lemma 25.
  3. With the polynomial time f-algorithm of Lemma 23 find a complete system  $e_1, \dots, e_r$  of orthogonal primitive idempotents in  $C$ , and then select an  $e_i$  among them which has rank 1 in  $\mathcal{A}$ . Finally output this element  $e = e_i$ .
- 

*Proof of Theorem 1.* The correctness and the timing for the first Step follows immediately from Theorem 19, and Corollary 20. These, and Lemma 24 imply that  $C$  admits polynomial size description. Then Lemma 25 settles Step 2. Correctness and polynomiality for the last Step is provided by Lemma 23.  $\square$

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