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Internal stability and loop-transformations:

an overview on LFTs, Möbius transforms and chain scattering

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Abstract: In robust control problems it is often convenient to consider maps between controller sets that are defined by Möbius transformations. Moreover, these loop-transformations are also intimately related to different factorizations, that simplify the structure of the problem.
Starting from a fundamental observation that relates internal stability of the control loops to mere stability of a specific LFT, the paper provides an exhaustive answer to the question under which conditions the internal stability of a control loop is preserved by performing a loop-transformation. As a main result it is shown that Möbius transformations defined by unimodular matrices preserves the internal stability of the loop and an explicit formula is also given that relates the two loops. This result is formulated in a general context that includes LTV or LPV systems as well.
The paper also provides an overview on the different transformation techniques, as Möbius transformations, LFTs, different generalized chain scattering (CSD) transforms, and their interrelations. This knowledge gives a solid basis for those approaches that use an input-output framework in combination with the IQC analysis and synthesis techniques in the solution of linear parameter varying (LPV) design problems.

1. INTRODUCTION AND MOTIVATION

Robust stability and robust performance analysis and synthesis of control systems are usually formulated in a standard form by using the generalized $P - K - \Delta$ structure, i.e., linear fractional transformation (LFT) interconnection structure, which is the basis of control design, in which a design problem for robust quadratic performance is formulated, e.g., $\|\mathfrak{F}_u(M, \Delta)\| \leq 1$ for Δ being any member of a predefined uncertainty set. We denote by $M = \mathfrak{F}_l(P_g, K)$ the nominal controlled system, where P_g is the generalized plant.

In this paradigm the small-gain theorem plays a central role. Actually, in the robust design paradigm, the most decisive role are played by those loop transformation techniques that shape both the plant and the uncertainty in order to make the small-gain condition less-conservative. These techniques range from elementary loop transformations, μ techniques to more sophisticated multiplier searches in the integral quadratic constraints (IQC) framework.

Developing efficient robust analysis and synthesis algorithms for a class of uncertain, linear parameter varying (LPV) systems leads to a renewed interest in certain input-output techniques that conveniently manipulates the original control loop. In combination with IQCs this approach provides a generalization of classical LTI techniques, see, e.g., a coordinate-wise descent similar to the well-known DK-iteration of the μ synthesis, Pfifer and Seiler [2015], Wang et al. [2016].

In this context the most relevant configuration is represented by the generalized small-gain setting, depicted on Figure 1.

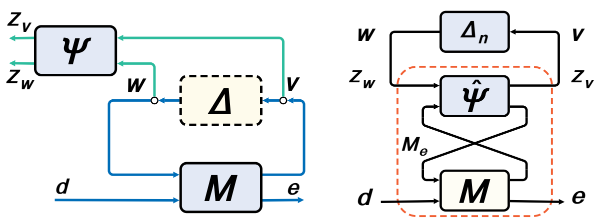

Fig. 1. Generalized small gain setting

The technique consists of the following steps: the IQC formulated for the signal pair $\eta = (v, w)$, $w = \Delta v$ is transformed to an IQC depending on the signal pair $\zeta = (z_v, z_w)$, where $\zeta = \Psi\eta$. In other terms, the uncertainty set Δ is mapped by a Möbius transform defined by Ψ to a new one, say Δ'. Without restricting the generality, we can assume that Δ' is the unit ball Δ_n. In that case Ψ is defined by a so called J-outer factorization of the initial IQC.

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Then, the implicit constraint, see Figure 1(a), is transformed into a usual generalized plant setting, see Figure 1(b). Technically this means that by a suitable transformation, $\Psi \mapsto \hat{\Psi}$, the interconnection is equivalent to the one defined by the Redheffer product $M_e = \hat{\Psi} \star M$. Thus, if the internal stability of these loops is equivalent, we can relate robust performance property of the original loop to the robust stability of the new loop against the two block normalized uncertainty $\mathrm{diag}(\Delta_n, \Delta_n)$.

The motivation in relating Figure 1(a) to Figure 1(b) is to obtain a standard, possibly easier problem that can be handled with the available tools. Moreover, Figure 1(b) is a clear indication that suitable factorizations of M (or P) facilitate this desire. Thus searching for a suitable IQC can be replaced by a systematic search for a suitable factorization.

Recall that the structure of the solutions for the nominal suboptimal $\mathcal{H}_\infty$ problem also fits into this scheme: by a suitable choice for $\hat{\Psi}$ the resultant system $\hat{\Psi} * P$ is inner, i.e., the Möbius map of the unit ball defined by $\hat{\Psi}$ provides the controllers that guarantee $\mathfrak{F}_l(P, K) < 1$.

This paper concentrates on the following question: under what conditions the internal stability of the loop is preserved by performing a loop transformation defined by a Möbius transform. For convenience, in this paper the controller K is transformed; the other case can be obtained by using straightforward manipulations.

This question has already got a partial answer in Ball et al. [1991] based on the scattering approach through the Potapov-Ginsburg transformation $\hat{P}$ of the generalized plant P. However, that method should assume a left or right invertibility of P and does not provide an explicit formula for the transformed configuration. Moreover, due to the augmentation, the technique has limited value in an IQC context.

By using the generalized chain scattering approach, see Tsai and Tsai [1993], this paper is a substantial step forward, providing an exhaustive and explicit answer of the problem. The need for the augmentation is completely eliminated. Thus, the final result and also the proposed technique fits well to the needs of the IQC analysis and synthesis framework. Moreover, the results are not tight to rational LTI systems – they are formulated in a fairly general context that includes, e.g., LPV systems, Balas et al [1997], or infinite-dimensional systems, Unal and Iftar [2008], Unal et al. [2010].

Concerning the factorization step itself often, at least on a conceptual level, a Schur complement based technique is the starting point of the actual algorithms. The paper also brings to the attention of the control community a generalization of this technique which is based on Möbius transforms.

To formulate the problem, let us recall that stability of the basic feedback loop is a continuity property of the connection which is formulated as the requirement that the causal map that relates the signals as follows:

$$\begin{pmatrix} d \\ n \end{pmatrix} = \begin{pmatrix} I & K \\ P & I \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}$$

is invertible on the extended space (the loop is well-posed), and the inverse map $\mathcal{H}(P, K)$, i.e.,

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -K \\ -P & I \end{pmatrix} \begin{pmatrix} (I - KP)^{-1} & 0 \\ 0 & (I - PK)^{-1} \end{pmatrix} \quad (1)$$

the "gang of four" is stable, see Figure 3(a).

If the loop is stable, we will say that (P, K) is stable.

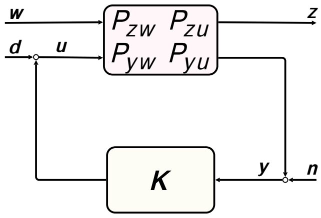
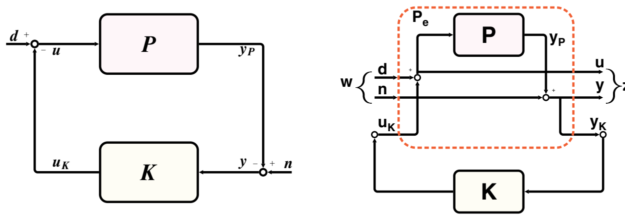


Fig. 2. LFT loop: performance and internal stability

A lower and an upper linear fractional transform (LFT) is defined as $\mathfrak{F}_l(P_g, K) = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}$ and $\mathfrak{F}_u(P_g, \Delta) = P_{yu} + P_{yw}\Delta(I - P_{zw}\Delta)^{-1}P_{zu}$. In the generalized plant paradigm two issues are handled: the loop should be stable and the resulting system, i.e., the LFT $M = \mathfrak{F}_l(P_g, K)$ should satisfy some norm constraints. In general, stability of the LFT loop means that the causal map that relates the signals (z, u, y) to (w, d, n) is invertible on the extended space, and the inverse map $\mathcal{L}(P_g, K) : (w, d, n) \mapsto (z, u, y)$, i.e.,

$$\mathcal{L}(P_g, K) = \begin{bmatrix} \mathfrak{F}_l(P_g, K) & \begin{pmatrix} P_{zu} & 0 \end{pmatrix} \mathcal{H}(P_{yu}, K) \\ \mathcal{H}(P_{yu}, K) \begin{pmatrix} 0 \\ -P_{yw} \end{pmatrix} & \mathcal{H}(P_{yu}, K) \end{bmatrix}, \quad (2)$$

the "gang of nine" is stable, see Figure 2. To distinguish between mere stability of $\mathfrak{F}_l(P_g, K)$ and the internal stability of the loop, i.e., stability of $\mathcal{L}(P_g, K)$, in what follows the former will be termed as "weak-stability" while the latter as stability.



(a) Basic loop (b) Equivalent LFT

Fig. 3. Internal stability vs. weak-stability

Theorem 1.1. (P, K) stable is equivalent to $\mathfrak{F}_l(P_{e,s}, K)$ weak-stable, where

$$P_{e,s} = \left[\begin{array}{cc|c} I & 0 & -I \\ -P & I & P \\ \hline -P & I & P \end{array}\right]. \quad (3)$$

$\mathfrak{F}_l(P_g, K)$ stable is equivalent to $\mathfrak{F}_l(P_{e,p}, K)$ weak-stable, where

$$P_{e,p} = \left[\begin{array}{ccc|c} P_{zw} & P_{zu} & 0 & -P_{zu} \\ 0 & I & 0 & -I \\ -P_{yw} & -P_{yu} & I & P_{yu} \\ \hline -P_{yw} & -P_{yu} & I & P_{yu} \end{array}\right]. \quad (4)$$

Proof. One can either inspect the signals (see, e.g., Figure 3(b)), or can perform the direct computation to obtain the equalities $\mathfrak{F}_l(P_{e,s}, K) = \mathcal{H}(P, K)$ and $\mathfrak{F}_l(P_{e,p}, K) = \mathcal{L}(P_g, K)$. The elementary computation is left out for brevity.

While the equivalence shown in Theorem 1.1 can be considered as known, this particular assertion apparently has been not realized and stated in the control literature yet. The impact of this result is that it relates a seemingly stronger property (internal stability) to a weaker one (weak-stability): the price is that the original problem is embedded in a generalized plant formulation which is slightly bigger. The gain is that on this new LFT we can apply all those nice factorization tricks that might simplify our particular problem (e.g., stabilizability, $\mathcal{H}_\infty$ design, IQC, etc). The only restriction is to keep the well-posedness conditions intact, but this is already a requirement imposed to these manipulations.

Concerning the application of this result for the loop transformation problem we should relate $\mathfrak{F}_l(P_e, K)$ to the transformed loop $\mathfrak{F}_l(\hat{P}_e, \hat{K})$, where P_e is determined by P through (4) or (3). What we actually expect is an equality of the type

$$\mathfrak{F}_l(P_e, K) = \mathfrak{F}_l(M, \mathfrak{F}_l(\hat{P}_e, \hat{K})), \tag{5}$$

where $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{pmatrix}$. If M is stable, we obtain a sufficient condition concerning stability. If M_{12} and M_{21} are stabily invertible, then we obtain an equivalence. This strategy leads to the main result of this paper formulated in Theorem 5.1.

The rest of the paper is dedicated to this project: we will provide the tools and the corresponding background information that not only help the designer in this particular case but which are also applicable in other situations, when factorization and loop-transform is involved. Thus the paper might be interesting for a larger audience.

In what follows Section 2 introduces the basic notations and transformations. It is also pointed out how different factorizations are directly related to these transforms. Section 3 provides the framework, in which the relation of Möbius transformations and LFTs can be understand. After this introduction Section 4 provides the reader with a deeper initialization in the generalized scattering approach than Tsai and Tsai [1993] or even Tsai and Gu [2014]. The proof of the main result of the paper, contained in Section 5, heavily relies on the facts listed in that section. We conclude this paper with some comments and further research questions.

2. SCHUR COMPLEMENT REVISITED

To fix the ideas let us recall the basic notions related to the feedback-connection depicted on Figure 2. While the reader might consider only the linear time invariant setting, where all the matrices are LTI transfer functions, the results of the paper remain valid for the considerably larger set of linear causal systems, e.g., LTV, LPV, etc.

For this more general setting we suppose that signals are elements of the extended spaces related to the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ (e.g., $\mathcal{H}_i = \mathcal{L}^{n_i}[0, \infty)$) endowed by a resolution structure defined by a nest algebra (resolution space) which determines the causality structure on these spaces. For details on the extended space, nest algebras and causality, see Feintuch [1998].

Let us consider the linear map $T : \begin{pmatrix} z \\ y \end{pmatrix} \mapsto \begin{pmatrix} w \\ u \end{pmatrix}$, and its inverse (if exists) described by the operator matrices

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{and} \quad T^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \tag{6}$$

respectively. We will use this notation throughout the rest of the paper.

Möbius transformations, which are usually defined as

$$Z' = \mathfrak{M}_T(Z) = (C + DZ)(A + BZ)^{-1},$$

relate two graph subspaces, $\mathcal{G}_Z$ and $\mathcal{G}_{Z'}$, through the invertible linear operator T, i.e., $\mathcal{G}_{Z'} = T\mathcal{G}_Z$ on the domain

$$\mathrm{dom}\mathfrak{M}_T = \{(A + BZ)^{-1} \text{ exists }\}.$$

Thus they inherit the group structure of the linear operators, i.e.,

$$\mathfrak{M}_P \circ \mathfrak{M}_Q(Z) = \mathfrak{M}_{PQ}(Z). \tag{7}$$

provided that the corresponding expressions exist. Analogously, one can introduce a Möbius transformation that relates inverse graph subspaces according to $\mathcal{G}^{-1}_{Z''} = T\mathcal{G}^{-1}_Z$ as

$$Z'' = \mathfrak{M}_{\tilde{T}}(Z) = (AZ + B)(CZ + D)^{-1}.$$

Remark 2.1. We emphasize, that the equations in

$$\begin{pmatrix} z \\ y \end{pmatrix} = T \begin{pmatrix} w \\ u \end{pmatrix}$$

reflect only relations between the signals, which by no means should be interpreted as if the left hand side were determined by the right hand side, as the notation suggests. To see this, let us suppose that we would like to interchange the role of w and z ("change of arrows", if one make a picture where arrows reflect the signal flow). If A is invertible we can rearrange the terms to obtain the desired map

$$\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} A^{\ 1} & -A^{\ 1}B \\ CA^{\ 1} & D - CA^{\ 1}B \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix}. \tag{8}$$

Apparently this possibility is tightly coupled to the invertibility of A. Let us shift now u to get $\tilde{u} = u - Zw$, where the map Z renders $A+BZ$ invertible. With this trick we obtain a configuration that is suitable to our "change of arrows" task:

$$\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} \hat{A}^{-1} & -\hat{A}^{-1}B \\ \hat{C}\hat{A}^{-1} & D - \hat{C}\hat{A}^{-1}B \end{pmatrix} \begin{pmatrix} z \\ \tilde{u} \end{pmatrix}.$$

with $\hat{A} = A + BZ$ and $\hat{C} = C + DZ$.

While the consequences of the first case concerning factorization are well-known (Schur complement) and also those related to inverse computation (all kind of matrix inversion lemmas), the second possibility has evaded yet the attention of the control community.

Schur complement based manipulations are often a starting point of control relevant factorizations. We conclude this subsection with a result that generalizes this technique and reveals the connection between factorization and the Möbius transform, for details see Harris [1992, 1995].

Proposition 2.1. Let us suppose that $(CK+D)^{-1}$ exists. Then the map T can be written as

$$\begin{pmatrix} I & W \\ 0 & I \end{pmatrix} \begin{pmatrix} A-WC & 0 \\ 0 & CK+D \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}, \tag{9}$$

with $X=(CK+D)^{-1}C$ and $W=(AK+B)(CK+D)^{-1}$. The inverse T^{-1} exists if and only if $(A-WC)^{-1}$ exists; then

$$E=(A-WC)^{-1}+KG,\ F=-EW+K(CK+D)^{-1},$$
$$G=-X(A-WC)^{-1},\ H=(CK+D)^{-1}-GW.$$

Accordingly the map $\mathfrak{M}_{\bar{T}}(Z)=(AZ+B)(CZ+D)^{-1}$ can be written as $T_4\circ T_3\circ T_2\circ T_1$ where

$$T_1(Z)=Z-K,\quad T_3(Z)=(A-WC)Z(CK+D)^{-1},$$
$$T_2(Z)=Z(I+XZ)^{-1},\quad T_4(Z)=Z+W.$$

This factorization reduces to the Schur variant if and only if $0\in\mathrm{dom}\mathfrak{M}_{\bar{T}}$. The specific form of the factorization depends on the position of the square blocks. While in this standard setting we consider them at the main diagonal, in different applications is more convenient to use variants where the invertible block is off diagonal, see, e.g., the scattering transformations of Kimura [1997]. Those formulas can be obtained by applying a suitable permutation (rearrangement of the signals).

3. SYSTEM CONNECTIONS AND LINEAR RELATIONS

If X and Y are two sets, a relation $T\subset X\times Y$ is defined as a set of pairs $(x,y)\in T$, where $x\in X, y\in Y$. If X and Y are linear spaces ($X\oplus Y=X\times Y$) a linear relation T is a linear subspace of $X\oplus Y$. Recall that a linear operator $P: X\mapsto Y$ is equivalent to a special relation which is called the graph (inverse graph) of the operator according to $\mathcal{G}_P=\mathrm{Im}\begin{pmatrix} I \\ P \end{pmatrix}$ and $\mathcal{G}_P^{-1}=\mathrm{Im}\begin{pmatrix} P \\ I \end{pmatrix}$, respectively. In this paper we consider only linear operators and relations. For details on linear relations see, e.g., Arens [1961].

In contrast to an operator, which formulates an explicit constraint among certain signals, relations can be viewed as implicit constraints. In practice often is inconvenient to work with such constraints and it is desirable to eliminate them from our configurations (systems connections). In what follows we will list some standard methods how we can achieve this goal.

It is desirable to relate Möbius transforms to LFTs. An elementary observation reveals that while 0 is always in the domain of an LFT, this is not true for the domain of a Möbius transform, in general. In the example of Remark 2.1 we already see that if $0\in\mathrm{dom}\mathfrak{M}_T$ the equivalent LFT representation can be written as

$$Z'=CA^{-1}+(D-AC^{-1}B)Z(I-(-A^{-1}B)Z)^{-1}A^{-1}.$$

If it is not the case, it was also shown how we can overcome this limitation by applying a suitable shift $Z\mapsto Z-Z_0$. In this way, by applying the chain property (7), one can obtain a Möbius transform that can be represented as an LFT.

These observations show that an LFT can be related to a Möbius transform only if some off diagonal term of its symbol is invertible. Thus, in order to describe an LFT in the general case we need to consider the case when the graph subspaces are related by relations instead of operators.

It turns out that LFTs can be obtained in the same way as the Möbius transformations, by performing some interchange in the signal spaces and by considering linear relations, instead of the linear operators, for details see Shmulyan [1980] and Szabó et al. [2014]. By taking a suitable block permutation Π_l the operator P_g induces a relation $\mathcal{R}_P$ through its graph subspace:

$$\mathcal{R}_P=\Pi_l\mathcal{G}_P\sim\begin{pmatrix} P_{yw} & P_{yu} \\ 0 & I_u \\ \hline I_w & 0 \\ P_{zw} & P_{zu} \end{pmatrix}, \tag{10}$$

called scattering transformation. Evaluating this relation on the graph subspaces $\mathcal{G}_K$ we obtain a graph subspace $\mathcal{G}_F=\mathcal{R}_P\mathcal{G}_K$ provided that $(I-P_{yu}K)$ is invertible. This map is exactly the lower LFT $F=\mathfrak{F}_l(P_g,K)$. Analogously, by considering another permutation Π_r, one can obtain the expression of the upper LFT.

In Shmulyan [1980] this technique was termed as "transformers". Different choices for the ambient signal space and different choices of the permutation matrix leads to very different transforms. However, all these maps are related to some Möbius or some LFT transforms. The assertion of Theorem 1.1 can be also understand in this framework.

If the $\mathcal{R}_P$ is a graph of an operator, i.e., if $\begin{pmatrix} P_{yw} & P_{yu} \\ 0 & I_u \end{pmatrix}$ is invertible, the representant can be obtained by the Möbius transform defined by $\hat{P}_g=\mathfrak{M}_{\Pi_l}(P_g)$ which is called the Potapov-Ginsbourg transformation of P_g. In this case $\mathfrak{F}_l(P_g,K)=\mathfrak{M}_{\hat{P}_g}(K)$. Note that the actual formula (and the actual existence condition) for the Potapov-Ginsburg transform depends on the choice of the Möbius transform (signal order).

Nested LFTs corresponds to the composition of the associated linear relations. The group structure on the representants is also present, however, the familiar matrix product should be changed to the less accessible Redheffer (star) product, see, e.g., Zhou and Doyle [1999]:

$$A\star B=\begin{pmatrix} \mathfrak{F}_l(A,B_{11}) & A_{12}(I-B_{11}A_{22})^{-1}B_{12} \\ B_{21}(I-A_{22}B_{11})^{-1}A_{21} & \mathfrak{F}_u(B,A_{22}) \end{pmatrix}.$$

Observe that $\hat{\Psi}$ in the generalized small gain setting of Figure 1(a) is a Potapov-Ginsburg transform of Ψ. This transform is always well defined, since 0 should always be in the domain of the corresponding Möbius transform, which implies the invertibility of the corresponding block matrix. On Figure 1(b) the resulting generalized plant is obtained as $\hat{\Psi}\star M$.

To overcome the limitations imposed by the Potapov-Ginsburg transform we need a further step: we relax the desire to obtain the LFT as a single Möbius transform. Instead we consider generalized Möbius transforms and the LFT will be expressed by two transforms of different type.

4. GENERALIZED SCATTERING APPROACH

Consider the matrix G and the feedback connection

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}, \quad u = Ky.$$

For every $(u, y)^T = (Ky, u)^T$ we would like to have $(z, w)^T = (Fw, w)^T$ with an operator F, i.e., to ensure $\mathcal{G}_F^{-1} S = G \mathcal{G}_K^{-1}$ for some nonsingular S. If $(G_{21}K + G_{22})^{-1}$ exists, then the map $z = (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1} w$ defines the right CSD transformation $\mathfrak{C}_G(K)$ of G and K:

$$\mathfrak{C}_G(K) = (G_{12} + G_{11}K)(G_{22} + G_{21}K)^{-1}$$

on $\mathrm{dom}\mathfrak{C}_G = \{K \mid (G_{22} + G_{21}K) \text{ is invertible } \}$.

By reversing the role of the signals, i.e., with $(u, y)^T = (y, Pu)^T$ and $(z, w)^T = (z, Fz)^T$ we have an analogous transform $\mathfrak{C}_{\tilde{G}}$, i.e., $w = (G_{21} + G_{22}P)(G_{11} + G_{12}P)^{-1} z$.

When G is nonsingular we are talking on Möbius transforms, which will be denoted by $\mathfrak{C}_G(K) = \mathfrak{K}_G(K)$ (controller transform) and $\mathfrak{C}_{\tilde{G}}(P) = \mathfrak{P}_G(P)$ (plant transform), respectively.

Analogously, consider the constraint

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \quad u = Ky.$$

When $(\tilde{G}_{11} - K\tilde{G}_{21})^{-1}$ exists, we have

$$0 = (I \ -K) \begin{pmatrix} K \\ I \end{pmatrix} = (I \ -K) \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix},$$

follows $z = -(\tilde{G}_{11} - K\tilde{G}_{21})^{-1}(\tilde{G}_{12} - K\tilde{G}_{22})\, w$. Thus, the left CSD transformation $\mathfrak{C}^d_{\tilde{G}}(K)$ of $\tilde{G}$ and K can be defined as

$$\mathfrak{C}^d_{\tilde{G}}(K) = -(\tilde{G}_{11} - K\tilde{G}_{21})^{-1}(\tilde{G}_{12} - K\tilde{G}_{22}),$$

on $\mathrm{dom}\mathfrak{C}^d_{\tilde{G}} = \{K \mid (\tilde{G}_{11} - K\tilde{G}_{21}) \text{ is invertible } \}$.

We have the chain rule

$$\mathfrak{C}_{G_1}(\mathfrak{C}_{G_2}) = \mathfrak{C}_{G_1G_2}, \quad \mathfrak{C}^d_{\tilde{G}_1}(\mathfrak{C}^d_{\tilde{G}_2}) = \mathfrak{C}^d_{\tilde{G}_2\tilde{G}_1}. \tag{11}$$

Thus, for Möbius transforms, i.e., if $G_1 = G_2^{-1}$ we have $\mathfrak{C}_{G_1G_2}(K) = \mathfrak{C}_I(K) = K$. Moreover, for Möbius transforms the ordinary and the dual variants are tightly coupled according to the identities of the type

$$\begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} T^{-1} T \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{12}$$

While we are not going to list all of them, the relevant ones are shown for further reference ($\hat{K} = \mathfrak{K}_T$ and $\hat{P} = \mathfrak{P}_T$):

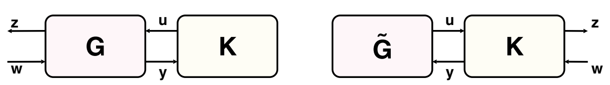
$$\hat{K} = (AK + B)(CK + D)^{-1} = -(E - KG)^{-1}(F - KH), \tag{13}$$

$$\hat{P} = (C + DP)(A + BP)^{-1} = -(H - PF)^{-1}(G - PE), \tag{14}$$

$$D - \hat{P}B = (H - PF)^{-1}. \tag{15}$$

Let us consider now the feedback connection represented by two coupling CSD matrices:

$$\begin{pmatrix} z \\ w \end{pmatrix} = G \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} u \\ y \end{pmatrix} = \tilde{G} \begin{pmatrix} a \\ b \end{pmatrix}, \quad u = Ky.$$



(a) RCSD (b) LCSD

Fig. 4. Chain scattering-matrix description

Then, we have $a = \mathfrak{C}^d_{\tilde{G}}(K)b$ and $z = \mathfrak{C}_G\big(\mathfrak{C}^d_{\tilde{G}}(K)\big)w$, provided, that the corresponding CSDs exists.

The dual variant of the feedback connection can be also represented by two coupling CSDs as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \tilde{G} \begin{pmatrix} z \\ w \end{pmatrix}, \quad \begin{pmatrix} a \\ b \end{pmatrix} = G \begin{pmatrix} u \\ y \end{pmatrix}, \qquad u = Ky.$$

Thus, from $a = \mathfrak{C}_G(K)b$ one has

$$\begin{pmatrix} \mathfrak{C}_G(K) \\ I \end{pmatrix} b = \tilde{G} \begin{pmatrix} z \\ w \end{pmatrix},$$

hence $z = \mathfrak{C}^d_{\tilde{G}}\big(\mathfrak{C}_G(K)\big)w$.

Since the description does not depend on the choice of the basis on this space, for any nonsingular M and $\tilde{M}$ we have

$$\mathfrak{C}_{GM} \circ \mathfrak{C}^d_{\tilde{G}M} = \mathfrak{C}_G \circ \mathfrak{C}^d_{\tilde{G}}, \tag{16}$$

$$\mathfrak{C}^d_{\tilde{M}\tilde{G}} \circ \mathfrak{C}_{\tilde{M}G} = \mathfrak{C}^d_{\tilde{G}} \circ \mathfrak{C}_G. \tag{17}$$

In order to relate these CSDs to LFTs consider

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix}, \quad u = Ky,$$

to obtain, after a suitable rearrangement of the signals, the constraints $u = Ky$ and

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} P_{zu} & P_{zw} \\ 0 & I_w \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}, \quad \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} I_u & 0 \\ P_{yu} & P_{yw} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}.$$

This leads to the right-left CSD formulation of an LFT, $\mathfrak{F}_l(P_g, K) = \mathfrak{C}_R \circ \mathfrak{C}^d_{\tilde{R}}(K)$, with

$$R = \begin{pmatrix} P_{zu} & P_{zw} \\ 0 & I_w \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} I_u & 0 \\ P_{yu} & P_{yw} \end{pmatrix}. \tag{18}$$

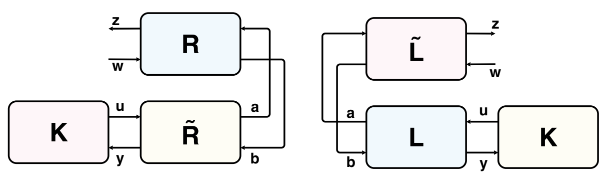
The description

$$\begin{pmatrix} I_z & -P_{zw} & -P_{zu} & 0 \\ 0 & -P_{yw} & -P_{yu} & I_y \end{pmatrix} \begin{pmatrix} z \\ w \\ u \\ y \end{pmatrix} = 0,$$

leads to the left-right CSD formulation of an LFT, $\mathfrak{F}_l(P_g, K) = \mathfrak{C}^d_{\tilde{L}} \circ \mathfrak{C}_L(K)$, with

$$L = \begin{pmatrix} P_{zu} & 0 \\ P_{yu} & -I_y \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} I_z & -P_{zw} \\ 0 & -P_{yw} \end{pmatrix}. \tag{19}$$

Given the special form of the matrices in (18) and in (19) one might wonder how $\mathfrak{C}_G \circ \mathfrak{C}^d_{\tilde{G}}$ or $\mathfrak{C}^d_{\tilde{G}} \circ \mathfrak{C}_G$ can be related to an LFT, in general. The answer is provided by the factorization of the general matrices based on a Schur complement or by Proposition 2.1 and an application of the properties (11) and (16). In the proof of the main result of the paper, presented in the next section, we will show how this technique can be applied.



(a) R-L CSD (b) L-R CSD

Fig. 5. LFT and CSD

5. THE LOOP TRANSFORMATION RESULT

Theorem 5.1. Let us consider the transformation of the standard LFT control loop from Figure 2 defined by an unimodular T which sends K to $\hat{K} = \mathfrak{K}_T(K)$ and we also assume that $P_{yu} \in \mathrm{dom}\mathfrak{P}_T$. Then we have

$$\mathfrak{F}_l(P_g, K) = \mathfrak{F}_l(\hat{P}_g, \hat{K}), \tag{20}$$

where $\hat{P}_g = \begin{pmatrix} \hat{P}_{zw} & \hat{P}_{zu} \\ \hat{P}_{yw} & \hat{P}_{yu} \end{pmatrix} =$

$$\begin{pmatrix} P_{zw} - P_{zu}(A + BP_{yu})^{-1}BP_{yw} & P_{zu}(A + BP_{yu})^{-1} \\ (P_{yu}F - H)^{-1}P_{yw} & (C + DP_{yu})(A + BP_{yu})^{-1} \end{pmatrix}. \tag{21}$$

Moreover, using the notation of (2) and (6),

$$\mathcal{L}(P_g, K) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & FC & -EB \\ 0 & -HC & GB \end{pmatrix} + \\ + \begin{pmatrix} I & 0 & 0 \\ 0 & E & -F \\ 0 & -G & H \end{pmatrix} \mathcal{L}(\hat{P}_g, \hat{K}) \begin{pmatrix} I & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}, \tag{22}$$

i.e., (internal) stability of the corresponding LFT loops are equivalent.

Proof. The starting point of the proof is the formula $\mathcal{L}(P_g, K) = \mathfrak{F}_l(P_{e,p}, K)$ of Theorem 1.1, where $P_{e,p}$ is given by (3). Then we have the right-left CSD formulation $\mathfrak{F}_l(P_{e,p}, K) = \mathfrak{C}_R \circ \mathfrak{C}^d_{\tilde{R}}(K)$, with

$$R = \begin{pmatrix} -P_{zu} & P_{zw} & P_{zu} & 0 \\ -I & 0 & I & 0 \\ P_{yu} & -P_{yw} & -P_{yu} & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{23}$$

$$\tilde{R} = \begin{pmatrix} I_u & 0 & 0 & 0 \\ P_{yu} & -P_{yw} & -P_{yu} & I \end{pmatrix}. \tag{24}$$

In what follows, due to the space limitations, the elementary computations are left out and only the main steps are presented. For convenience let us denote by $\bar{T} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$ and $\bar{T}^{-1} = \begin{pmatrix} E & -F \\ -G & H \end{pmatrix}$. According to (11) we should consider $T\tilde{R}$ for the transformed loop, which is factorized to $\tilde{R}_a M_a$, where

$$\tilde{R}_a = \begin{pmatrix} I_u & 0 & 0 & 0 \\ \hat{P}_{yu} & -XP_{yw} & -XP_{yu} & X \end{pmatrix}$$

with $X = (D - \hat{P}_{yu}B)$ and

$$M_a = \begin{pmatrix} A + BP_{yu} & -BP_{yw} & -BP_{yu} & B \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

According to the identity $\mathfrak{C}_R = \mathfrak{C}_{\mathrm{diag}(I_z, \bar{T}^{-1}, I)}\mathfrak{C}_{R_a}$, with $R_a = \mathrm{diag}(I_z, \bar{T}, I)R$ we have $R_a = R_b M_a$, where

$$R_b = \begin{pmatrix} -\hat{P}_{zu} & \hat{P}_{zw} & \hat{P}_{zu} & 0 \\ -I & 0 & A & 0 \\ \hat{P}_{yu} & -\hat{P}_{yw} & -XP_{yu} - C & X \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

We obtained $\mathfrak{F}_l(P_{e,p}, K) = \mathfrak{C}_{\mathrm{diag}(I_z, \bar{T}^{-1}, I)} \circ \mathfrak{C}_{R_b M_a} \circ \mathfrak{C}^d_{\tilde{R}_a M_a}(\hat{K})$.

Applying property (16) we get the simpler formula $\mathfrak{F}_l(P_{e,p}, K) = \mathfrak{C}_{\mathrm{diag}(I_z, \bar{T}^{-1}, I)} \circ \mathfrak{C}_{R_b} \circ \mathfrak{C}^d_{\tilde{R}_a}(\hat{K})$. A repeated application of this property considering the matrix $M_c = \mathrm{diag}(I_z, I_y, T^{-1}, I)$ leads to

$$\mathfrak{F}_l(P_{e,p}, K) = \mathfrak{C}_{T_a} \circ \mathfrak{C}_{R_c} \circ \mathfrak{C}^d_{\tilde{R}_c}(\hat{K}),$$

where

$$T_a = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & E & -F & 0 & -BG & -BH \\ 0 & -G & H & 0 & -CE & -CF \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & E & F \\ 0 & 0 & 0 & I & G & H \end{pmatrix}$$

$$R_c = \begin{pmatrix} -\hat{P}_{zu} & \hat{P}_{zw} & \hat{P}_{zu} & 0 \\ -I & 0 & I & 0 \\ \hat{P}_{yu} & -\hat{P}_{yw} & -\hat{P}_{yu} & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

$$\tilde{R}_c = \begin{pmatrix} I_u & 0 & 0 & 0 \\ \hat{P}_{yu} & -\hat{P}_{yw} & -\hat{P}_{yu} & I \end{pmatrix},$$

i.e.,

$$\mathfrak{F}_l(P_{e,p}, K) = \mathfrak{C}_{T_a}(\mathfrak{F}_l(\hat{P}_{e,p}, \hat{K})), \tag{25}$$

which is the desired result. Observe that (25) is in the form (5) and note that in the computations we have used the identities (14) and (15).

Remark 5.1. Concerning the assumption that $P_{yu} \in \mathrm{dom}\mathfrak{P}_T$ observe that the existence of $\mathfrak{M}_T(K)$ does not imply, in general, the existence of $\mathfrak{P}_T(P)$. See, e.g., $P = 2$, $K = 3$ and $T = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$. While the assumption on the unimodularity is obvious from the result, it also follows from the natural requirement to have a transformation that maps $\mathcal{L}_2$ to $\mathcal{L}_2$ bijectively.

Remark 5.2. The formula (20) can be computed directly by using the technique of Remark 2.1. Which is not obvious for the first glance, and it is an extension over the previous results, is that this relation does not involve the existence of a Potapov-Ginsburg transform, i.e., invertibility of A. One can check that if A^{-1} exists, the formula for $\hat{P}_g$ is identical with the one obtained as $\hat{P}_g = P_g \star \hat{T}$, where $\hat{T} = \begin{pmatrix} -A^{-1}B & A^{-1} \\ D - CA^{-1}B & CA^{-1} \end{pmatrix}$.

Based on this result one can verify (22) by a direct computation by an intensive application of the identities of type (13), (14) and (15). Those readers who like such adventures are invited to perform these computations in order to appreciate then the efficiency of the proposed technique.

Remark 5.3. As easy examples for the application of the general loop transformation result we list some standard configurations, see, e.g., Green and Limebeer [1995]. Weights corresponds to $T = \begin{pmatrix} W & 0 \\ 0 & I \end{pmatrix}$ with W unimodular. Linear shift corresponds to $T = \begin{pmatrix} I & H \\ 0 & I \end{pmatrix}$, where H is stable. The context of the small-gain theorem provides an indication which choice of H ensures the existence of $\hat{P}$. The choice $T = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ relates incrementally strictly passive systems to contractions. To obtain exactly the form of Theorem 3.5.6 of Green and Limebeer [1995], observe that for contractions (P, K) is stable if and only if $(P, -K)$ is stable.

Remark 5.4. The classical Youla result for LFTs also fits in this framework: one of the key observations is that the LFT loop is stable for a K if and only if the pair $(P_g, \mathrm{diag}\{0, K\})$ is stable. If the loop is stabilizable, by fixing a particular stabilizing K_0 we have a double coprime factorization induced by the stable pair (P_{yu}, K_0) (inner loop): $K_0 = uv^{-1} = \tilde{v}^{-1}\tilde{u}$ and $P_{yu} = nm^{-1} = \tilde{m}^{-1}\tilde{n}$. Considering the unimodular matrix $T = \begin{pmatrix} m & u \\ n & v \end{pmatrix}$ it is immediate that $\hat{P}_{yu} = 0$ and $\hat{K}_0 = 0$. Moreover, it is immediate that the pair $(0, q)$ is stable if and only if q is stable. Thus, applying Theorem 5.1 we obtain all the stabilizing controllers of P_{yu} as $\mathfrak{K}_{T^{-1}}(q)$, i.e., the Youla parametrization.

A more advanced classical application is the derivation of the suboptimal $\mathcal{H}_\infty$ controller set, see, e.g., Tsai and Gu [2014].

Remark 5.5. One can also prove that for LFTs the Youla parametrization provides the same set that internally stabilizes the LFT loop. In order to prove this fact let us start from a double coprime factorization of $\mathrm{diag}\{0, K_0\}$. It turns out that, by inverting the usual roles, we have a dual Youla parametrization of P_g. It follows that P_g should have the following form

$$P_g = \begin{pmatrix} q_{zw} & q_{zu} \\ q_{yw} & 0 \end{pmatrix} \star \begin{pmatrix} -m^{-1}u & m^{-1} \\ \tilde{m}^{-1} & 0 \end{pmatrix} \star \begin{pmatrix} 0 & I \\ I & P_{yu} \end{pmatrix},$$

where q_{zw}, q_{zu}, q_{yw} are stable systems. The resulting closed–loop form for a stabilizing controller is given by

$$\mathfrak{F}_l(P_g, K) = q_{zw} + q_{zu} q q_{yw}, \tag{26}$$

where q is the Youla parameter of K relative to the given double coprime factorization of P_{yu}.

6. CONCLUSION

In robust control problems often it is convenient to perform loop-transformations, i.e., to consider maps between controller sets that are defined by Möbius transformations. These loop-transformations are also intimately related to different factorizations, that simplify the structure of the problem.

Starting from a fundamental observation that relates internal stability of the control loops to weak-stability of certain LFTs, the paper provides an exhaustive answer to the question under which conditions the internal stability of a control loop (LFT) is preserved by performing a loop-transformation. It is shown that Möbius transformations defined by unimodular matrices preserves the internal stability of the loop. It is also given an explicit formula that relates the original and the transformed loop.

The paper also provides an overview on the various transformation techniques and their interrelations, as the Möbius transformation, the LFT and the different generalized chain scattering (CSD) transforms. This knowledge gives a solid base for those approaches that use an input-output framework in combination with the IQC analysis and synthesis techniques for robust stability and performance problems.

The investigation of nontrivial interplay between the IQC technique and a Möbius transformation based approach, e.g., problems revolving around the hard vs. soft IQC equivalence and related factorization questions, goes well beyond the possibilities of this work and are considered in a forthcoming paper.

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