Analysis of input delay systems using integral quadratic constraint

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Abstract: The $\mathcal{L}_2$-gain computation of a linear time-invariant system with state and input delay is discussed. The input and the state delay are handled separately by using dissipation inequality involving a Lyapunov-Krasovskii functional and integral quadratic constraints. A conic combination of IQCs is proposed for characterizing the input delay, where the coefficients are linear time-invariant systems. The numerical example (a vehicle platoon) confirm that using this dissipativity approach a more effective method for $\mathcal{L}_2$-gain computation is obtained.

1 Introduction

Dynamic systems with both state and input delay emerge for example in distributed systems and in large scale systems. The problem of induced $\mathcal{L}_2$-gain computation of systems with input delay can be resolved in many special cases.

If only delayed input acts on the system, then it can be handled as considering this as another input without delay. Delay on the control input transforms to state delay when closing the loop (Fridman and Shaked, 2004). The problem arise when the delayed and actual disturbance input acts simultaneously on the system.

In (Cheng et al., 2012), the actual input and the delayed input were considered as two independent inputs, which results in an overestimation of the $\mathcal{L}_2$-gain, due to disregarding the relation between them. The other paper, which examined the effects of the input delay, is (Rödönyi and Varga, 2015). Four different methods were considered to compute the $\mathcal{L}_2$-gain for state and input delay system. The best of these methods according to the numerical results in time-invariant and also in time-varying delay cases is the augmentation of the system with additional dynamics. With this method the input delay is transformed to state delay that can be handled for example by Lyapunov-Krasovskii functionals (LKFs).

Another method was examined in (Rödönyi and Varga, 2015), where integral quadratic constrains (IQCs) was used to describe the input delay in the system. A conic combination of two IQCs was used with constant coefficients.

It is shown in this paper that the upper bound of the $\mathcal{L}_2$-gain can be improved further as compared to the method of additional dynamics by applying dynamic coefficients in the IQC approach.

The structure of the paper is the following: First the system in consideration is described in Section 2 together with the emerging problem. In Section 3 some preliminary tools are presented together with a lower bound computation method and additional dynamics approach. In Section 4 the new method is presented to compute the $\mathcal{L}_2$-gain in case of input and state delay using Lyapunov-Krasovskii functional and integral quadratic constraints in the time-domain. This method is compared with the other two methods in Section 5 on an example of vehicle platoon. In Section 6 a few conclusion are drawn.

Notations. Matrix inequality $M > 0$ ($M \geq 0$) denotes that M is symmetric and positive (semi-) definite, i.e. all of its eigenvalues are positive (or zero). Negative (semi-) definiteness is denoted by $M < 0$ ($M \leq 0$). The transpose and conjugate transpose of a matrix M is denoted by M^T and M^*, respectively. $\bar{\sigma}(M)$ denotes the maximum singular value of matrix M. The upper linear fractional transformation is defined by $\mathcal{F}_U(M,\Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$, where $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. $\mathcal{L}_2^n$ denotes the space of square integrable signals with norm defined by $\|x\|_2 = \left(\int_0^\infty \|x(t)\|^2 dt\right)^{1/2}$, where $\|x(t)\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

2 Problem formulation and sketch of the solution

A linear input and state delayed system denoted by Ω is described in this paper by the following form

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + Bd(t) + B_h d(t-h) \quad (1a)$$
$$y(t) = Cx(t), \quad (1b)$$

where $x \in \mathbb{R}^{n_x}$, $d \in \mathbb{R}^{n_d}$ and $y \in \mathbb{R}^{n_y}$ is the state, disturbance and output of the system, respectively. A, A_h, B, B_h and C are constant matrices with appropriate dimensions. Here only time-invariant delay is considered, therefore h is constant. For the sake of simplifying the discussion a single delay is considered, but the method can be generalized to handle multiple delays and different state and input delays. The initial condition for the Ω system is the following

$$x(t) = \phi(t), \quad t \in [-h, 0], \quad (2)$$

where $\phi : [-h,0] \to \mathbb{R}^{n_x}$ is a given continuous function. Let $x_t(\xi)$ denote $x(t+\xi)$ for $\xi \in [-h,0]$.

The goal of the paper is to compute the $\mathcal{L}_2$-gain of system Ω defined as

$$\|\Omega\|_\infty = sup_{0 \neq d \in L_2^{n_d}, \phi=0} \frac{\|y\|_2}{\|d\|_2}. \quad (3)$$

2.1 Sketch of the solution

To analyse delayed systems different methods exist like Lyapunov-Krasovskii functionals, Razumikhin theorem, integral quadratic constraints approach and frequency-domain methods. The advantage of using the complete Lyapunov-Krasovskii functional is that it gives sufficient and necessary condition of stability in case of constant delays.

The combination of Lyapunov-Krasovskii functional and integral quadratic constraint approach is used: LKF for the state delay and IQC for the input delay.

Let $S_h(d) := d(t-h) - d(t)$ denote the difference between the delayed input and the input. Then $d(t-h)$ is replaced in (1) by $S_h(d) + d(t)$ and the system is reformulated by the linear fractional transformation (LFT) form $\mathcal{F}_U(G, S_h)$, where system G is the following

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + (B+B_h)d(t) + B_h w(t), \quad (4a)$$

$$\begin{bmatrix} v(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ C \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d(t) \\ w(t) \end{bmatrix} \quad (4b)$$

and $w = S_h(v)$. The S_h perturbation term is described by integral quadratic constraint. The G plant contains state delay, the $\mathcal{L}_2$-gain of such a system can be computed using complete Lyapunov-Krasovskii functionals. Adding the time-domain IQC to the derivative of the LKF and the $\mathcal{L}_2$-gain condition the $\mathcal{L}_2$-gain of the Ω system can be computed. The exact formulation of this $\mathcal{L}_2$-gain bound computation technique will be discussed in the following sections.

3 Preliminaries

In this section the complete Lyapunov-Krasovskii functional is presented for establishing stability of a time-delay system. Then two $\mathcal{L}_2$-gain computation methods are described for input delayed systems. One of them is a frequency-domain formula and the other one is a time-domain method described in (Rödönyi and Varga, 2015).

The focus of this paper is to give an efficient time-domain method for $\mathcal{L}_2$-gain computation of system Ω using integral quadratic constraints. An introduction to IQCs is given in Section 3.4.

3.1 Lyapunov-Krasovskii functional

One method to establish stability of a time-delay system is to use Lyapunov-Krasovskii functionals. In the literature different LKFs are proposed for time-invariant delay. Here the so called complete LKF will be used presented in (Gu et al., 2003):

$$\begin{aligned} V(x_t) \quad = \quad & x^T(t)Px(t) + 2x^T(t)\int_{-h}^{0} Q(\xi)x(t+\xi)d\xi \\ & + \int_{-h}^{0}\int_{-h}^{0} x^T(t+\xi)R(\xi,\eta)x(t+\eta)d\eta d\xi \\ & + \int_{-h}^{0} x^T(t+\xi)S(\xi)x(t+\xi)d\xi. \quad (5) \end{aligned}$$

The sufficient and necessary conditions for asymptotic stability of system Ω are that $P = P^T \in \mathbb{R}^{n_x \times n_x}$, for all $-h \leq \xi, \eta \leq 0$, $Q(\xi) \in \mathbb{R}^{n_x \times n_x}$, $R(\xi,\eta) = R^T(\xi,\eta) \in \mathbb{R}^{n_x \times n_x}$, $S(\xi) = S^T(\xi) \in \mathbb{R}^{n_x \times n_x}$ and

$$V(x_t) \geq \varepsilon \|x(t)\|^2 \quad (6)$$
$$\dot{V}(x_t) \leq -\varepsilon \|x(t)\|^2, \quad (7)$$

for some $\varepsilon > 0$. In this LKF the variables Q, R and S are matrix functions, which are approximated by piece-wise linear functions in the analysis.

The domain of this matrix functions $[-h,0]$ (or $[-h,0]\times[-h,0]$) are divided into N (or N by N) segments. Each segment indexed by p or (p,q) can be described with the help of matrix parameters Q_p, S_p, $R_{pq} = R_{qp}^T$, $p,q = 0,1,2,\ldots,N$ so that for $0 \leq \alpha \leq 1$

and $0 \leq \beta \leq 1$

$$\begin{aligned} Q(-pl+\alpha l) &= (1-\alpha)Q_p + \alpha Q_{p-1} \\ S(-pl+\alpha l) &= (1-\alpha)S_p + \alpha S_{p-1} \end{aligned}$$

and

$$R(-pl+\alpha l, -ql+\beta l) =$$
$$\begin{cases} (1-\alpha)R_{pq} + \beta R_{p-1,q-1} + (\alpha-\beta)R_{p-1,q} & \alpha \geq \beta \\ (1-\beta)R_{pq} + \alpha R_{p-1,q-1} + (\beta-\alpha)R_{p,q-1} & \alpha < \beta \end{cases}$$

Using this technique the stability conditions can be described in a linear matrix inequality (LMI) form. This method is known as discretized complete LKF and is described in (Gu, 1997).

3.2 A lower bound computation

In case of time-invariant delay the $\mathcal{L}_2$-gain of system Ω can be computed exactly in the frequency-domain:

$$\|\Omega\|_\infty = \max_\omega \bar{\sigma}\left(C(j\omega I - A - A_h e^{-j\omega h})^{-1} \times (B + B_h e^{-j\omega h})\right). \quad (8)$$

However numerically this maximum can not be calculated in case of lightly damped modes. The $\mathcal{L}_2$-gain was computed on a grid of the frequency interval according to (8), which gives a lower bound of the gain.

The other methods will give an upper bound on the $\mathcal{L}_2$-gain of the Ω system. Those will be compared with this method.

3.3 Additional dynamics

This method was proposed in (Rödönyi and Varga, 2015), where the input delay was transformed to state delay using a low-pass filter. Assume, that the input $d(t)$ is band limited. Let W_d be the low-pass filter with $\|W_d\|_\infty = 1$ and

$$\begin{aligned} \dot{x}_d(t) &= A_d x_d(t) + B_d d(t), \\ d_d(t) &= C_d x_d(t). \end{aligned} \quad (9)$$

Using this filter the Ω system can be augmented with the following

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_h x(t-h) + Bd(t) + B_h d_d(t-h) \\ &y(t) = Cx(t). \end{aligned} \quad (10)$$

Then the $\mathcal{L}_2$-gain of the Ω system can be computed using LKFs on system (9) and (10).

Using a low-pass filter the high-frequency components of d are filtered out, therefore the filter need to be chosen carefully.

3.4 Integral quadratic constraint

In system analysis a very powerful tool to describe the robustness in the system is to use integral quadratic constraints.

Definition 1 ((Megretski and Rantzer, 1997)). *Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(n_v+n_w)\times(n_v+n_w)}$ be a Hermitian-valued function. Two signals $v \in \mathcal{L}_2^{n_v}[0,\infty)$ and $w \in \mathcal{L}_2^{n_w}[0,\infty)$ satisfy the IQC defined by Π if*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0, \quad (11)$$

where $\hat{v}(j\omega)$ and $\hat{w}(j\omega)$ are Fourier transforms of v and w, respectively. A bounded, causal operator $\Delta : \mathcal{L}_{2e}^{n_v}[0,\infty) \to \mathcal{L}_{2e}^{n_w}[0,\infty)$ satisfies the IQC defined by Π, if (11) holds for all $v \in \mathcal{L}_2^{n_v}[0,\infty)$ and $w = \Delta(v)$.

The input delay of the system can be described via IQCs. To this end, the system has to be given by the interconnection of a plant and a perturbation term, S_h the deviation between the delayed and the undelayed signal as $S_h(v) := v(t-h) - v(t)$.

To describe the constant time-delay term S_h three IQCs were proposed in (Pfifer and Seiler, 2015)

$$\Pi_1 = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad (12)$$

$$\Pi_2(j\omega) = \left|\frac{j\omega+1}{10j\omega+1}\right|^2 \begin{bmatrix} |\zeta_2(j\omega)|^2 & 0 \\ 0 & -1 \end{bmatrix}, \quad (13)$$

$$\Pi_3(j\omega) = \begin{bmatrix} 0 & \zeta_3^*(j\omega) \\ \zeta_3(j\omega) & -1 \end{bmatrix}, \quad (14)$$

where

$$\zeta_2(j\omega) := 2\frac{(j\omega h)^2 + 3.5j\omega h + 10^{-6}}{(j\omega h)^2 + 4.5j\omega h + 7.1}, \quad (15)$$

$$\zeta_3(j\omega) := \frac{-2.19(j\omega h)^2 + 9.02j\omega h + 0.089}{(j\omega h)^2 - 5.64j\omega h - 17}. \quad (16)$$

A combination of IQCs still an IQC, if an operator Δ satisfies the IQCs Π_i, $i = 1,2,\ldots,M$, then it also satisfies the IQC $\Pi_d(j\omega) = \sum_{i=1}^{M} \lambda_i(j\omega)\Pi_i(j\omega)$, where $\lambda_i > 0$. Usually combined IQC is used in numerical examples, therefore here these IQCs are studied. The goal is to preserve the dynamics of the combination coefficients λ_i in the time-domain.

For time-domain system analysis an equivalent representation of the general IQC (11) is required. The multiplier Π in IQC (11) is factorized as $\Pi(j\omega) = \Psi^*(j\omega)M\Psi(j\omega)$, where $M = M^T \in \mathbb{R}^{n_z\times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z\times(n_v+n_w)}$. (The factorization method is described according to (Pfifer and Seiler, 2015), where also a detailed description can be found.) Let z be the output of the system Ψ, namely $z := \Psi \begin{bmatrix} v \\ w \end{bmatrix}$. Using the Parseval's theorem the frequency-domain IQC is equivalent with the following expression in the time-domain:

$$\int_0^{\infty} z(t)^T M z(t) dt \geq 0. \tag{17}$$

For general IQCs the constraint (17) holds only over infinite time, for hard IQCs a restrictive constraint of this holds.

Definition 2 ((Megretski et al., 2010)). *Let Π factorized as $\Psi^* M \Psi$ with Ψ stable. Then (Ψ, M) is a hard IQC factorization of Π if for any bounded, causal operator Δ satisfying the IQC defined by Π the following inequality holds*

$$\int_0^{T} z(t)^T M z(t) dt \geq 0 \tag{18}$$

for all $T \geq 0$, $v \in L_2^{n_v}[0,\infty)$, $w = \Delta(v)$ and $z = \Psi \begin{bmatrix} v \\ w \end{bmatrix}$.

Let $\Pi = \Pi^*$ and partition as $\begin{bmatrix} \Pi_{11} & \Pi_{21}^* \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$. According to (Seiler, 2015, Theorem 4) if $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ then Π has J-spectral factorization (Ψ, M), which is a hard factorization. A factorization (Ψ, M) is a J-spectral factorization if $M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $\Psi, \Psi^{-1} \in \mathbb{RH}_{\infty}^{n_z \times (n_v + n_w)}$. Here the J-spectral factorization from (Pfifer and Seiler, 2015) is used, which provide a square, stable and minimum phase Ψ.

Appending this Ψ to S_h the plant of the interconnected system reveals the following:

$$\dot{\hat{x}} = \hat{A}\hat{x}(t) + \hat{A}_h \hat{x}(t-h) + \hat{B} \begin{bmatrix} w(t) \\ d(t) \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \hat{C}\hat{x}(t) + \hat{D} \begin{bmatrix} w(t) \\ d(t) \end{bmatrix}, \tag{20}$$

where $\hat{x} := \begin{bmatrix} x \\ x_{\Psi} \end{bmatrix}$ are the extended state, x_{Ψ} is the state vector of the Ψ system. The exact formula about the computation of the constant state matrices is omitted, however it can be derived easily from the description of plant G (4).

Using a slightly modified version of Theorem 3 from (Pfifer and Seiler, 2015) the $\mathcal{L}_2$-gain for system $\mathcal{F}_U(G, S_h)$ can be computed.

Theorem 1. *Assume $\mathcal{F}_U(G, S_h)$ is well-posed and S_h satisfies the hard IQC defined by (Ψ, M). Then $\|\mathcal{F}_U(G, S_h)\|_{\infty} \leq \gamma$ if there exists a $\lambda > 0$ and a bounded quadratic Lyapunov-Krasovskii functional $V(\hat{x}_t)$ such that for some $\varepsilon > 0$*

- $V(\hat{x}_t) \geq \varepsilon \|\hat{x}(t)\|^2$,
- *the following inequality holds*

$$\lambda z^T M z + \dot{V}(\hat{x}_t) - \gamma^2 d^T d + y^T y \leq - \varepsilon \|\hat{x}(t)\|^2 - \varepsilon \|d(t)\|^2. \tag{21}$$

Proof 1. *Integrate the inequality (21) from $t = 0$ to $t = T$ with the initial condition $\phi(t) = 0$ $t \in [-h, 0]$*

$$\lambda \int_0^T z(\tau)^T M z(\tau) d\tau + V(\hat{x}_T) - \gamma^2 \int_0^T d(\tau)^T d(\tau) d\tau + \int_0^T y^T(\tau) y(\tau) d\tau \leq - \varepsilon \int_0^T (\|\hat{x}(\tau)\|^2 + \|d(\tau)\|^2) d\tau \leq 0.$$

Using that the IQC and the LKF V are non-negative this inequality is equivalent to $\|\mathcal{F}_U(G, S_h)\| \leq \gamma$.

The inequality (21) gives a linear matrix inequality (LMI), if the LKF condition for stability can be formulated as an LMI.

In case of combined IQCs using Theorem 1 the different factorized IQCs are adding to inequality (21) with different λ_i coefficient. However in the frequency-domain these coefficients were frequency-dependent. This dynamics in the time-domain now is omitted.

Arise the question, that why not factorize the λ_i coefficients together with IQC Π_i using J-spectral factorization. For J-spectral factorization the exact dynamics of the IQC is necessary, which is not the case with $\lambda_i \Pi_i$. Therefore a different factorization is necessary for the λ_i coefficients. In the next section a method will be shown to preserve the dynamics of the λ_i coefficients in the time-domain.

4 Main results

4.1 IQC factorization preserving the λ dynamics

The combined IQC in Section 3.4 omit the dynamics of the λ coefficient in time-domain, therefore a new factorization method is presented to preserve this dynamics.

A factorization of the dynamical λ is necessary in similar forms as the IQCs are factorized:

$$\lambda_i(j\omega) = \Theta_i(j\omega)^* \Lambda_i \Theta_i(j\omega) \tag{22}$$

where $i = 1, 2, \ldots, M$. One possible factorization proposed in (Veenman and Scherer, 2014) is the following

$$\Theta_i(j\omega) = \begin{bmatrix} 1 & \frac{1}{j\omega - \rho_i} & \cdots & \frac{1}{(j\omega - \rho_i)^{\nu_i}} \end{bmatrix}^T, \tag{23}$$

$\rho_i < 0, \nu_i \in \mathbb{N}$ fixed parameters and $0 < \Lambda_i \in \mathbb{R}^{(\nu_i+1)\times(\nu_i+1)}$ constant real symmetric matrix. This Θ_i is a basis function, using a fixed pole location (ρ_i constant), then by increasing the dimension of the basis function (ν_i) a better approximation of the λ_i coefficient can be established.

Assume that the dimension of v is 1. Using J-spectral factorization of the IQC ($\Psi_i, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$) and the factorization of λ_i as Equation (22) the factorized combined IQC reveals

$$\Pi_d(j\omega) = \sum_{i=1}^{N} \left(\begin{bmatrix} \Theta_i(j\omega) & 0 \\ 0 & \Theta_i(j\omega) \end{bmatrix} \Psi_i(j\omega) \right)^* \times \begin{bmatrix} \Lambda_i & 0 \\ 0 & -\Lambda_i \end{bmatrix} \begin{bmatrix} \Theta_i(j\omega) & 0 \\ 0 & \Theta_i(j\omega) \end{bmatrix} \Psi_i(j\omega). \tag{24}$$

In case of a frequency independent IQC only the λ coefficient is factorized

$$\Pi_i(j\omega) = [*]^* \begin{bmatrix} \Pi_{11}\Lambda_i & \Pi_{21}^*\Lambda_i \\ \Pi_{21}\Lambda_i & \Pi_{22}\Lambda_i \end{bmatrix} \times \begin{bmatrix} \Theta_i(j\omega) & 0 \\ 0 & \Theta_i(j\omega) \end{bmatrix}. \tag{25}$$

4.2 LMI formulation of $\mathcal{L}_2$-gain computation

Using this new factorized IQC and the discretized complete LKF the inequality (21) in Theorem 1 can be formulated as an LMI. For brevity in this section only one IQC multiplier is considered in form $\lambda\Pi$ as it would be in case of combined multipliers. The LMI formulation of the problem can be easily extended for more IQCs.

Let the state-space form of the connected system $\Gamma := \begin{bmatrix} \Theta & 0 \\ 0 & \Theta \end{bmatrix} \Psi$ be given as

$$\dot{x}_\Gamma(t) = A_\Gamma x_\Gamma(t) + B_{\Gamma 1} d(t) + B_{\Gamma 2} w(t)$$
$$z(t) = C_\Gamma x_\Gamma(t) + D_{\Gamma 1} d(t) + D_{\Gamma 2} w(t),$$

and the dimension of the x_Γ denoted by n_Γ. The constant state-space matrices can be computed using the J-spectral factorization of the Π multiplier and the λ factorization as in (23).

The state-space form of the connected system of Γ and G (4) is the following

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_\Gamma(t) \end{bmatrix} = A_c \begin{bmatrix} x(t) \\ x_\Gamma(t) \end{bmatrix} + A_{ch} \begin{bmatrix} x(t-h) \\ x_\Gamma(t-h) \end{bmatrix} + + \begin{bmatrix} B_{c1} & B_{c2} \end{bmatrix} \begin{bmatrix} w(t) \\ d(t) \end{bmatrix}, \tag{26}$$

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_\Gamma \\ C_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_\Gamma(t) \end{bmatrix} + \begin{bmatrix} D_\Gamma \\ D_c \end{bmatrix} \begin{bmatrix} w(t) \\ d(t) \end{bmatrix}, \tag{27}$$

where

$$A_c = \begin{bmatrix} A & 0 \\ 0 & A_\Gamma \end{bmatrix}, \quad A_{ch} = \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix},$$
$$B_{c1} = \begin{bmatrix} B_h \\ B_{\Gamma 2} \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} B + B_h \\ B_{\Gamma 1} \end{bmatrix},$$
$$C_\Gamma = \begin{bmatrix} 0 & C_\Gamma \end{bmatrix}, \quad C_c = \begin{bmatrix} C & 0 \end{bmatrix},$$
$$D_\Gamma = \begin{bmatrix} D_{\Gamma 2} & D_{\Gamma 1} \end{bmatrix}, \quad D_c = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Before presenting the LMI formulation of the $\mathcal{L}_2$-gain computation some notations must be introduced.

$$\begin{aligned}
\bar{Q} &= \begin{bmatrix} Q_0 & Q_1 & \dots & Q_N \end{bmatrix} \\
\bar{S} &= \frac{1}{l} \operatorname{diag}(S_0\, S_1\, \dots\, S_N) \\
\bar{R} &= \begin{bmatrix} R_{00} & R_{10}^T & \dots & R_{N0}^T \\ R_{10} & R_{11} & \dots & R_{N1}^T \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \dots & R_{NN} \end{bmatrix} \\
\Delta &= \begin{bmatrix} \Delta_{11} & * & * \\ Q_N^T - A_{ch}^T P & S_N & * \\ -B_c^T P & 0 & \gamma^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} - \\
&\quad - \begin{bmatrix} C_c \\ 0 \\ D_c \end{bmatrix} [*]^T - \begin{bmatrix} C_\Gamma \\ 0 \\ D_\Gamma \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} [*]^T \\
\Delta_{11} &= -PA_c - A_c^T P - Q_0 - Q_0^T - S_0 - C_c^T C_c \\
S_d &= \operatorname{diag}\{S_0 - S_1, S_1 - S_2, \dots, S_{N-1} - S_N\} \\
R_d &= \begin{bmatrix} R_{d11} & R_{d12} & \dots & R_{d1N} \\ R_{d21} & R_{d22} & \dots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \dots & R_{dNN} \end{bmatrix} \\
R_{dpq} &= l(R_{p-1,q-1} - R_{pq})
\end{aligned}$$

$$
\begin{aligned}
D^s &= \begin{bmatrix} D^s_{01} & D^s_{02} & \dots & D^s_{0N} \\ D^s_{11} & D^s_{12} & \dots & D^s_{1N} \\ D^s_{w1} & D^s_{w2} & \dots & D^s_{wN} \end{bmatrix} \\
D^s_{0p} &= \frac{l}{2}A_c^T(Q_{p-1}+Q_p)+\frac{l}{2}(R_{0,p-1}+R_{0p}) \\
&\quad -(Q_{p-1}-Q_p) \\
D^s_{1p} &= \frac{l}{2}A_{ch}^T(Q_{p-1}+Q_p)-\frac{l}{2}(R_{N,p-1}+R_{Np}) \\
D^s_{wp} &= \frac{l}{2}B_c^T(Q_{p-1}+Q_p) \\
D^a &= \begin{bmatrix} D^a_{01} & D^a_{02} & \dots & D^a_{0N} \\ D^a_{11} & D^a_{12} & \dots & D^a_{1N} \\ D^a_{w1} & D^a_{w2} & \dots & D^a_{wN} \end{bmatrix} \\
D^a_{0p} &= -\frac{l}{2}A_c^T(Q_{p-1}-Q_p)-\frac{l}{2}(R_{0,p-1}-R_{0p}) \\
D^a_{1p} &= -\frac{l}{2}A_{ch}^T(Q_{p-1}-Q_p)+\frac{l}{2}(R_{N,p-1}-R_{Np}) \\
D^a_{wp} &= -\frac{l}{2}B_c^T(Q_{p-1}-Q_p)
\end{aligned}
$$

Theorem 2. *Assume $\mathcal{F}_U(G,S_h)$ is well-posed and S_h satisfies the hard IQC defined by (Ψ,M). Then $\|\mathcal{F}_U(G,S_h)\|_\infty \leq \gamma$ if there exists $0<\Lambda\in\mathbb{R}^{(\nu+1)\times(\nu+1)}$ and $\{P=P^T,Q_p,S_p,R_{pq}=R_{qp}^T\}\in\mathbb{R}^{(n_x+n_\Gamma)\times(n_x+n_\Gamma)}, p=0,1,\dots,N, q=0,1,\dots,N$ such that the following holds:*

$$
\begin{bmatrix} P & \bar{Q} \\ \bar{Q}^T & \bar{R}+\bar{S} \end{bmatrix} > 0 \tag{28a}
$$

$$
\begin{bmatrix} \Delta & * & * \\ -D^{sT} & R_d+S_d & * \\ -D^{aT} & 0 & 3S_d \end{bmatrix} > 0. \tag{28b}
$$

The LMI formulation of the $\mathcal{L}_2$-gain computation in case of state delay system using the discretized complete LKF is described in (Gu et al., 2003, Prop. 8.5). Only the Δ matrix has to be modified to get the LMI formulation of (21).

5 Numerical example: vehicle platoon

A vehicle platoon model is used as a numerical example. Due to imperfect inter-vehicle communication the system contains input and state delays. Computing the $\mathcal{L}_2$-gain of this system not only stability can established, but also the effects of the communication caused delays are illustrated.

5.1 Vehicle platoon model

The first vehicle in the platoon is driven by a human driver (indexed by 0), and the followers motion determined by on-board controller (indexed by $1,2\dots,n$). The longitudinal dynamics of the ith vehicle is the following:

$$
\dot{p}_i(t) = v_i(t), \tag{29a}
$$

$$
\dot{v}_i(t) = q_i(t)+d_i(t), \tag{29b}
$$

$$
\dot{q}_i(t) = -\frac{1}{\tau_i}q_i(t)+\frac{g_i}{\tau_i}u_i(t), \tag{29c}
$$

where p_i, v_i denote position and velocity, d_i is a disturbance representing both outer effects and modelling error, q_i is an internal state such that the acceleration of the vehicle is $a_i(t)=q_i(t)+d_i(t)$. $\mathcal{L}_2$-gain calculations will be carried out on a homogeneous platoon (the vehicles parameter are the same) with parameters $\tau_i=0.7$ and $g_i=1$ $i=0,1,\dots,n$. u_0 is the signal generated by the pedal signal of the first vehicle driver, $u_i, i=1,2,\dots,n$ is the acceleration demand generated by the controllers.

A leader and predecessor follower control architecture is used with constant spacing policy proposed in (Swaroop and Hedrick, 1999). This means that the controller uses information about the predecessor and also about the leader vehicle, therefore inter-vehicle communication is necessary. This communication can be imperfect causing delays in the system description. Taking this inter-vehicle communication delays into account, the controllers can be described by the following equations

$$
\begin{aligned}
&u_1(t) = -k_1\delta_1(t)-k_2e_1(t)+a_0(t-h) \\
&u_i(t) = -k_{1\beta}\delta_i(t)-k_{2\beta}e_i(t)+k_{a0}a_0(t-h) \\
&+k_{a1}a_{i-1}(t-h)-k_{1\alpha}(v_i(t-h)-v_0(t-h)) \\
&-k_{2\alpha}(p_i(t-h)-p_0(t-h)), \quad i=2,...,n
\end{aligned}
$$

where $\delta_i \triangleq v_i - v_{i-1}$ and $e_i \triangleq p_i - p_{i-1} + L_i$ are the relative speed and spacing error, respectively. The prescribed spacing L_i can be set to zero in the analysis without loss of generality. The k_* are constant parameters of the controllers. The aim of the paper is analysis, therefore the numerical values of these parameters, which will be used from (Rödönyi et al., 2012) are $k_1=0.7, k_2=0.1127, k_{1\alpha}=0.4642, k_{2\alpha}=0.0564, k_{1\beta}=0.2358, k_{2\beta}=0.0564, k_{a1}=0.0449, k_{a0}=0.9551$.

In the analysis only SISO systems are considered, namely $d_0\mapsto e_1$ (the effect of the lead vehicle disturbance on the first spacing error) and $d_1\mapsto e_2$ (the effect of the first vehicle disturbance on the second spacing error).

The first system contains only input delay and the state-space matrices as in Ω system with state vector $x=[e_1,\delta_1,q_1]^T$ are the following:

$$
A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.16 & -1 & -1.43 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},
$$

$$B_h = \begin{bmatrix} 0 \\ 0 \\ 1.43 \end{bmatrix},\ A_h = 0,\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The system $d_1 \mapsto e_2$ has both state and input delay, the state-space matrices using the state vector $x = [e_1, \delta_1, q_1, e_2, \delta_2, q_2]^T$ are the following:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.16 & -1 & -1.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.08 & -0.34 & -1.43 \end{bmatrix},$$

$$A_h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.08 & -0.66 & 0.06 & -0.08 & -0.66 & 0 \end{bmatrix},$$

$$B = [0,1,0,0,-1,0]^T,\ B_h = [0,0,0,0,0,0.06]^T,$$
$$C = [0,0,0,1,0,0].$$

5.2 $\mathcal{L}_2$-gain of a vehicle platoon

A linear combination of the multipliers will be used as

$$\Pi_d(j\omega) = \lambda_1(j\omega)\Pi_1 + \lambda_2(j\omega)\Pi_i(j\omega), \qquad (31)$$

where Π_i can be Π_2 or Π_3, and $\lambda_1, \lambda_2 : j\mathbb{R} \to \mathbb{R}$ that satisfies $\lambda_i(j\omega) > 0, \forall\omega$.

Before the numerical results notations are necessary for the different $\mathcal{L}_2$-gain computation method:

- **LB** the lower bound computation method described in Section 3.2.
- **AD** the additional dynamics method described in Section 3.3 with filter $W_d = \frac{1}{\tau_d s+1}$. The τ_d time-constant of the filter has to be chosen carefully, because higher time constant can cause increased gain, smaller time-constant can cause numerical problems by solving the LMI. Here $\tau_d = 0.05$ is used.
- **TD** the time-domain IQC method with constant λ_i using IQCs Π_2 or Π_3 in the combined Π_d.
- **TDλ** the time-domain IQC method with dynamic λ_i using IQCs Π_2 or Π_3 in the combined Π_d. At the factorization of λ_i the $\rho_i = -1$ parameter is used and the ν_i is increased from 0 until a tight bound with the lower bound is established. In the tables the results using $\nu_i = 2$ are presented.

In Table 1 the numerical results are shown in case of system $d_0 \mapsto e_1$ for the different $\mathcal{L}_2$-gain computation methods. This system does not contain state delay, therefore a simple quadratic storage function $V = x^T P x$ is used instead of an LKF. By the time-domain IQC method with constant λ coefficient (TD-2,3) a highly overestimated $\mathcal{L}_2$-gain is computed compared to the lower bound (LB).

The suggestion is that preserving the dynamics of the λ coefficient a lower upper bound of the $\mathcal{L}_2$-gain can be calculated. Using the factorization method described in Section 4 (TDλ-2,3), nearly the same results are received as the lower bound (LB), meanly a good approximation of the $\mathcal{L}_2$-gain are computed. Two different IQCs are considered Π_2 and Π_3. Better numerical results were established using Π_2 than Π_3.

In an earlier paper (Rödönyi and Varga, 2015) different methods were suggested for $\mathcal{L}_2$-gain computation, and the additional dynamics method (Section 3.3) gave the best numerical results. However here with time-domain IQC preserving the λ dynamics less conservative norms are established.

Table 1: $\mathcal{L}_2$-gain of system $(d_0 \mapsto e_1)$ with time-invariant delay (only input delay).

h	0.05	0.1	0.25	0.5	0.8
LB	1.27	1.35	1.6	2.02	2.51
AD	1.35	1.44	1.69	2.1	2.6
TD-2	3.55	3.56	3.56	3.58	3.63
TDλ-2	1.27	1.35	1.6	2.02	2.51
TD-3	14.06	14.06	14.06	14.06	14.07
TDλ-3	1.28	1.36	1.63	2.07	2.6

In Table 2 the numerical results in case of system $d_1 \mapsto e_2$ are shown. This system contains also state delay, therefore the discretized complete LKF is used from Section 3.1. The different methods gave similar results as in Table 1: the time-domain IQC method using constant λ coefficients (TD-2,3) overestimates the $\mathcal{L}_2$-gain. However if the λ dynamics are preserved (TDλ-2,3) nearly the same gain can be computed as the lower bound (LB) by every delay value.

Table 2: $\mathcal{L}_2$-gain of system $(d_1 \mapsto e_2)$ with time-invariant delay (state and input delay).

h	0.05	0.1	0.25	0.5	0.8
LB	4.44	4.44	4.44	4.44	4.44
AD	4.44	4.44	4.44	4.44	4.44
TD-2	4.52	4.52	4.52	4.52	4.52
TDλ-2	4.44	4.44	4.44	4.44	4.44
TD-3	4.52	4.52	4.52	4.52	4.52
TDλ-3	4.44	4.44	4.44	4.44	4.44

6 Conclusion

A time-domain method for computing upper bound of the $\mathcal{L}_2$-gain of state and input delay systems is presented using a dissipation inequality involving Lyapunov-Krasovskii functionals and conic combination of integral quadratic constraints. The coefficients of the combination of IQCs are proposed to be dynamic systems.

It was shown by a numerical example that the upper bound is very tight, and nearly coincides with the lower bound. As a numerical example a vehicle platoon was examined with leader and predecessor following control architecture and constant spacing policy.

Future works involves the construction of controller synthesis based on this time-domain method. Further extension of this time-domain method will be to consider also uncertainties in the system.

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