# Performance Improvement from LS to ML Estimation

**I. Vajk, J. Hetthéssy and L. Keviczky**

Department of Automation and Applied Informatics
Budapest University of Technology an Economics
Institute of Computer Science and Automation
Hungarian Academy of Sciences
MTA-BME Control Engineering Research Group
H-1117 Budapest, Magyar Tudósok krt. 2, HUNGARY

**Abstract** – *The paper presents a family of identification methods for linear dynamic systems. The key point is that the structure of the proposed algorithm can be scaled. The trade-off between calculation complexity and estimation performance is investigated. The paper demonstrates the performance improvement from LS to ML estimation. It is also shown that the efficiency of the algorithms significantly depends on the selected sampling time.*

**Keywords:** system identification, subspace method, linear dynamic model, structural total least squares method

## 1. Introduction

Identification of dynamic systems is a deconvolution problem in essence, namely knowing the driving input and the measured output a model of the system is to be set up. Several books and papers give excellent reviews on various identification algorithms (Ljung 1999), (Söderström 1989).

Noises corrupting the signals available for the identification have certainly a strong impact on the identification results. The influence of the noise can be decreased by appropriately averaging or weighting the noisy signals. This way identification can be considered as a science of „averaging and weighting". A well-known example is the fact that the expected value can be estimated by arithmetic mean in case of independent samples of identical distribution (IID samples). For single output systems corrupted by IID noise a frequently used approach is to apply

$$\hat{\theta}_* = \arg\min_\theta \tfrac{1}{2} n^T(\theta) n(\theta) \qquad (1)$$

to minimize the sum of the error squares. In Eq. (1) $\theta$ stands for the vector of parameters to be estimated, while $n(\theta)$ represents an error vector calculated from the noisy observations and the parametric model to be determined. The error components can be weighted. In case of known noise covariance an efficient weighting matrix is the inverse of the covariance matrix *cov(n)*, leading to the estimation strategy

$$\hat{\theta}_* = \arg\min_\theta \tfrac{1}{2} n^T(\theta) cov^{-1}(n) n(\theta) \qquad (2)$$

substantially reducing the impact of the noise effect. Here the noise is assumed to have zero mean and *cov(n)* covariance.

If *cov(n)* is of strip/block structure, then the above estimation can be extended according to

$$\hat{\theta}_* = \arg\min_\theta \tfrac{1}{2} \sum {n_i}^T(\theta) cov^{-1}(n_i) n_i(\theta) \qquad (3)$$

where $n_i \in R^q$, and further on $C_q = cov(n_i)$. Introducing $N_q = [n_1 \quad n_2 \quad \cdots]$ Eq.(3) takes a more compact form by

$$\hat{\theta}_* = \arg\min_\theta \tfrac{1}{2} tr(N_q^T C_q^{-1} N_q) = \arg\min_\theta \tfrac{1}{2} tr(C_q^{-1}(N_q N_q^T)) \qquad (4)$$

Note that the complexity of this problem is less than that of the original problem, consequently less calculation leads to a more precise result. This estimation can serve as an initial estimation for a more complex estimation algorithm deriving even more accurate results.

Considering a general algebraic approach, identification frequently leads to solve an overdetermined set of equations in the form of $F(X) \approx 0$, where the dimension of *F* exceeds the dimension of the *X* vector/matrix to be determined. The simplest models are affine in nature. In this case identification needs to solve an $AX \approx B$ overdetermined set of linear equations, where $X \in R^{m \times q}$ is to be determined given $A \in R^{n \times m}$ and $B \in R^{n \times q}$. Various estimation algorithms apply different interpretations for the notation „$\approx$", resulting in diverse fitting properties. Just to mention a

few variants (Markovsky et al. 2007):

- Least Squares method (LS)

$$\min_{\Delta B,X} \|\Delta B\|_F \quad \text{s.t. } AX = B + \Delta B \tag{5}$$

- Total Least Squares method (TLS)

$$\min_{\Delta A,\Delta B,X} \|[\Delta A\ \Delta B]\|_F \quad \text{s.t. } (A + \Delta A)X = B + \Delta B \tag{6}$$

- Structured Total Least Squares method (STLS)

$$\min_{\Delta p,X} \|\Delta p\|_2 \quad \text{s.t. } A(p + \Delta p)X = B(p + \Delta p) \text{ where } A(p), B(p) \text{ affin} \tag{7}$$

All the above methods aim at identifying models linear in parameters, however, they can be realized via algorithms different in complexity. While the TLS and STLS models are linear separately both in parameters and error terms, they are bilinear in the joint space of the parameters and error terms. As far as the LS method is concerned, to find the optimum needs a matrix inversion, while TLS leads to an eigenvalue/singular value calculation. STLS exhibits a POP type problem, i.e. a global optimum problem. Application of usual numerical optimization techniques (e.g. Levenberg-Marquard algorithm) results in only local optimum points. Excluding special cases identification of linear dynamic systems turns to a POP problem.

All the three problems discussed above can be rewritten as

$$\min_{\Delta p,X} \|\Delta p\|_2 \quad \text{s.t. } G(X)\,\Delta p = r(X) \tag{8}$$

Solutions can be expressed as

$$\min_X r(X)^T \left(G(X)G^T(X)\right)^+ r(X) \tag{9}$$

In all cases, however, the solutions are different from each other in terms of complexity. If the structure of $G(X)$ specifically follows the structure of a block Toeplitz/Hankel matrix, then similarly to (Eq.4) an estimation algorithm by

$$\min_X tr\left(\left(G_q(X)G_q^T(X)\right)^+ r_q(X)r_q^T(X)\right) \tag{10}$$

is obtained. As it has been shown, various conceptual considerations may lead to similar identification algorithms. Later on in this paper it will be shown the application of these techniques to identify dynamic systems.

The rest of the paper is organized as follows. Session 2 is devoted to study the description of linear dynamic systems. Relationships introduced in this Session will be utilized as constraints along the optimization procedures aiming at the identification. Session 3 presents a family of identification methods scalable regarding the calculation complexity. It will be also shown here that using a projection type solution, the solution of the global maximization problem can be avoided. Session 4 presents a simulation example to demonstrate the properties of the scalable identification algorithms, as well as the relation between complexity and effectiveness of the estimation. The paper will be terminated by Conclusion.

## 2. Modeling of dynamic systems

In this session three linear models will be discussed.

*One step model*

Linear dynamic systems are typically modelled in discrete form:

$$y_k = g_k * u_k + h_k * e_k \tag{11}$$

where $y_k$ is the sample of the process output taken at time instant *k*, $u_k$ is the process input applied, $e_k$ is the error term, $g_k$ and $h_k$ are the discrete-time impulse responses of the process and noise models, respectively. The notation $*$ represents convolution. In general, the error term is assumed to be IID with zero mean. Note that the paper discusses SISO systems; however, the results can be extended for MISO or MIMO systems in a straightforward way.

Instead of the nonparametric description by Eq. (11) mostly parametric models are used. State-space descriptions use the following relations:

$$x_{k+1} = A(\theta)x_k + B(\theta)u_k + E(\theta)e_k \tag{12}$$

$$y_k = C(\theta)x_k + D(\theta)u_k + e_k \tag{13}$$

where $x_k \in R^m$ is the state vector of the dynamic system, while *A*, *B*, *C*, *D* and *E* are matrices or vectors with appropriate size involving the system parameters. The above state-space description can be considered as a multi input system with *u* and *e*, and with a single output *y*. It is well known that a state-space description is not unique, it is determined up to a linear matrix transformation. In Eqs. (12-13) $\theta$ denotes the vector involving the parameters to be identified. In most cases the parametrization is affine in nature.

The state-space description by Eqs. (12-13) can be transformed to transfer function form:

$$A(q)y_k = D(q)x_0 + B(q)u_k + C(q)e_k \tag{14}$$

where *A(q)*, *B(q)*, *C(q)* and *D(q)* are polynomials of the shift operator *q*. *A(q)* and *C(q)* are monic polynomials:

$$\begin{gathered} A(q) = \det\left(Iq - A(\theta)\right) \\ B(q) = C(\theta)adj\left(Iq - A(\theta)\right)B(\theta) + \det\left(Iq - A(\theta)\right)D(\theta) \\ C(q) = C(\theta)adj\left(Iq - A(\theta)\right)E(\theta) + \det\left(Iq - A(\theta)\right) \\ D(q) = C(\theta)adj\left(Iq - A(\theta)\right) \end{gathered} \tag{15}$$

*Cumulative model*

If the number of samples is N, then the number of constraints of equation type to be taken into account for the identification is *2N-1*:

$$
\begin{aligned}
&y_1 = Cx_0 + Du_1 + e_1\\
&x_2 = Ax_1 + Bu_1 + Ee_1\\
&y_2 = Cx_2 + Du_2 + e_2 = CAx_0 + CBu_1 + Du_2 + CEe_1 + e_2\\
&x_3 = Ax_2 + Bu_2 + Ee_2\\
&\qquad = A^2x_0 + ABu_1 + Bu_2 + AEe_1 + Ee_2\\
&\dots\\
&y_N = Cx_N + Du_N + e_N =\\
&CA^{N-1}x_0 + CA^{N-2}Bu_1 + \cdots + Du_N + CA^{N-2}Ee_1 + \cdots + e_N
\end{aligned}
\tag{16}
$$

In the above relations the dependency of the matrices and vectors from the parameters has been avoided. The equations can be rearranged into matrix form:

$$y = \Gamma x_0 + H_u u + H_e e \tag{17}$$

where

$$
\begin{aligned}
y &= [y_1 \quad y_2 \quad \dots \quad y_N]^T\\
u &= [u_1 \quad u_2 \quad \dots \quad u_N]^T\\
e &= [e_1 \quad e_2 \quad \dots \quad e_N]^T
\end{aligned}
$$

$$\Gamma = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{N-1} \end{bmatrix} \tag{18}$$

$$H_u = \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \dots & \dots & \dots & \dots \\ CA^{N-2}B & CA^{N-3}B & \dots & D \end{bmatrix} \tag{19}$$

$$H_e = \begin{bmatrix} 1 & 0 & \dots & 0 \\ CE & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ CA^{N-2}E & CA^{N-3}E & \dots & 1 \end{bmatrix} \tag{20}$$

The relation (17) clearly shows that in case of linear dynamic systems both $\Gamma x_0$ and $H_e e$ are bilinear functions of the unknown variables. Eq. (17) can further be transformed into

$$y = [\Gamma(\mathrm{A},\mathrm{C}) \quad G_u(A,C,u)] \begin{bmatrix} x_0 \\ B \\ D \end{bmatrix} + G_e(A,C,e) \begin{bmatrix} E \\ 1 \end{bmatrix} \tag{21}$$

to indicate that the model is linear in $x_0$, $B$, $D$ and $E$.

Multiply Eq. (17) with the following square Toeplitz matrix from the left:

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & 1 & 0 & \dots & 0 & 0 \\ a_2 & a_1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & a_1 & 1 \end{bmatrix} \tag{22}$$

where $a_i$ is the i-th coefficient of the polynomial $A(q)$. Then the multiplication gives:

$$\mathcal{A}y = \mathcal{A}\Gamma x_0 + \mathcal{A}H_u u + \mathcal{A}H_e e \tag{23}$$

According to the Caley-Hamilton theorem all square matrices satisfy their own characteristic polynomial. This allows us to rewrite Eq. (23) as

$$\mathcal{A}y = \mathrm{z} + \mathcal{B}u + \mathcal{C}e \tag{24}$$

where $\mathcal{B}$ and $\mathcal{C}$ are both Toeplitz matrices identically structured as $\mathcal{A}$, except the elements of the $\mathcal{B}$ and $\mathcal{C}$ matrices are constructed by the polynomials $B(q)$ and $C(q)$, respectively. Excluding the first *m* entries, vector z has only entries with value 0. Formally $\mathrm{z} = \mathcal{Z}z_0$ can be written, where

$$\mathcal{Z} = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{25}$$

*Hankel generalized model*

Construct Hankel matrices using the vectors *y*, *u* and *e*:

$$
\begin{aligned}
Y_q &= Hankel(y,q)\\
U_q &= Hankel(u,q)\\
E_q &= Hankel(e,q)
\end{aligned}
$$

where

$$Hankel(x,q) = \frac{1}{\sqrt{N-q+1}} \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_{N-q+1} \\ x_2 & x_3 & x_4 & \dots & x_{N-q+2} \\ \dots & \dots & \dots & \dots & \dots \\ x_q & x_{q+1} & x_{q+2} & \dots & x_N \end{bmatrix} \tag{26}$$

Signals just introduced allow us to write expressions similar to Eq. (17) and Eq. (24):

$$Y_q = \Gamma_q X_{0q} + H_{u,q} U_q + H_{e,q} E_q \tag{27}$$

$$\mathcal{A}_q Y_q = \mathcal{Z}_q Z_{0q} + \mathcal{B}_q U_q + \mathcal{C}_q E_q \tag{28}$$

Matrices $\Gamma_q$, $H_{u,q}$, $H_{e,q}$, $\mathcal{Z}_q$, $\mathcal{A}_q$, $\mathcal{B}_q$ and $\mathcal{C}_q$ in the above equations have identical structure with matrices $\Gamma$, $H_u$, $H_e$, $\mathcal{Z}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ introduced earlier. The only difference is manifested in the size of the matrices. Further on *N* is to be replaced by $q$ in the relations. As matrix $\mathcal{Z}_q$ involves 0 elements excluding its first *m* rows, omitting the first *m* rows Eq. (28) can be reformulated as

$$\tilde{\mathcal{A}}_q Y_q = \tilde{\mathcal{B}}_q U_q + \tilde{\mathcal{C}}_q E_q \tag{29}$$

Here the symbol $\tilde{}$ denotes matrices obtained after omitting the first $m$ rows. When reformulating, $\tilde{\mathcal{A}}_q\Gamma_q = 0$ has been taken into account. Note that matrix $[\tilde{\mathcal{A}}_q^T \quad \Gamma_q]$ is non-singular.

In case of output error noise a model is looked for, where $H_e$ is a unity matrix and $\mathcal{C}_q = \mathcal{A}_q$.

### 3. A family of identification methods

From mathematical point of view identification of dynamic systems can be considered as a constrained optimization problem. Using state-space system description the object of the optimization is

$$\min_{\theta,X_{0q}} \|E_q\|_F \text{ s.t. } Y_q = \Gamma_q X_{0q} + H_{u,q}U_q + H_{e,q}E_q \quad (30)$$

If transfer functions are used for system description the optimization takes the following form:

$$\min_{\theta,Z_{0q}} \|E_q\|_F \text{ s.t. } \mathcal{A}_q Y_q = \mathcal{Z}_q Z_{0q} + \mathcal{B}_q U_q + \mathcal{C}_q E_q \quad (31)$$

i.e. the constraints follows Eq. (28). The most frequently used norm in the objective functions to characterize the measure of the fitting is the Frobenius norm. Other norms can also be used, however, easy mathematical handling of the expressions required by Frobenius norms makes it a popular choice. The outlined method is scalable, it will be shown that increasing size $q$ the estimation can be improved, but at the same time the complexity required by the necessary calculations will grow. The maximum value of $q$ is limited by the number of the measured input/output pairs. The minimum value of $q$ is determined by the dimension of the state-space dimension of the system to be identified: $q_{min} = m + 1$.

In the sequel identification of dynamic systems involving output error type noise will be discussed. The method to be developed can be generalized for ARMAX models, as well.

In the course of the optimization procedure projections will be used several times. In this respect the following notations will be used: $P_A = A(A^TA)^+A^T$, and $P_A^\perp = I - A(A^TA)^+A^T$. respectively, where $^+$ denotes Moore-Penrose pseudo inverse.

The fitting error by $\|E_q\|_F$ depends on the initial value of the variables and on the parameters to be estimated. First find the minimum of the objective function with respect to the initial values.

*Minimization w.r.t. the initial values*

According to the derived relationships the initial values are linear in the constraints, thus

$$\min_{X_{0q}} \|E_q\|_F \text{ s.t. } Y_q = \Gamma_q X_{0q} + H_{u,q}U_q + E_q \quad (32)$$

can be obtained by the following projection:

$$\min_{X_{0q}} \|E_q\|_F = \left\|P_{\Gamma_q}^\perp E_q\right\|_F = \left\|P_{\Gamma_q}^\perp (Y_q - H_{u,q}U_q)\right\|_F \quad (33)$$

Now find the QR factorized form of the Hankel matrices built by the measured signals:

$$\begin{bmatrix} U_q \\ Y_q \end{bmatrix} = R_q Q_q = \begin{bmatrix} R_{u,q} & 0 \\ R_{yu,q} & R_{y,q} \end{bmatrix} Q_q \quad (34)$$

In the above decomposition $R_q$ is a lower triangular matrix. Using the QR factorization the objective function takes the following form:

$$\min_{X_{0q}} \|E_q\|_F = \left\|P_{\Gamma_q}^\perp [-H_{u,q} \quad I] R_q\right\|_F \quad (35)$$

Note that in Eq. (33) the Frobenius norm should be calculated with matrix size $2q \times (N - q + 1)$, while in Eq. (35) the norm calculation should be calculated with matrix size $2q \times 2q$. The adantages of introducing the QR factorization are as follows:

1. QR factorization ensures more favourable numerical conditions than the original problem.
2. While the objective function should be evaluated several times in the course of the optimization, the $R_q$ matrix derived from the QR factorization should just once be determined.

Finding the optimal value for the initial condition using transfer function approach also requires a minimization procedure subject to equation constraint:

$$\min_{Z_{0q}} \|E_q\|_F \text{ s.t. } \mathcal{A}_q Y_q = \mathcal{Z}_q Z_{0q} + \mathcal{B}_q U_q + \mathcal{A}_q E_q \quad (36)$$

Minimization can be performed via projection in this case, as well and the procedure leads to

$$\min_{Z_{0q}} \|E_q\|_F = \left\|P_{\tilde{\mathcal{A}}_q^T} E_q\right\|_F = \left\|\tilde{\mathcal{A}}_q^+ (\tilde{\mathcal{A}}_q Y_q - \tilde{\mathcal{B}}_q U_q)\right\|_F \quad (37)$$

Using QR factorization the value of the minimized objective function becomes

$$\min_{Z_{0q}} \|E_q\|_F = \left\|\tilde{\mathcal{A}}_q^+ [-\tilde{\mathcal{B}}_q \quad \tilde{\mathcal{A}}_q] R_q\right\|_F \quad (38)$$

Eq. (35) clearly shows that in case of transfer function approach the minimum is only a function of the matrices with reduced size (denoted by $\tilde{}$ ). This means that the (31) problem reduces to

$$\min_{\theta,Z_{0q}} \|E_q\|_F = \min_{\theta} \|E_q\|_F \text{ s.t. } \tilde{\mathcal{A}}_q Y_q = \tilde{\mathcal{B}}_q U_q + \tilde{\mathcal{A}}_q E_q \quad (39)$$

Functions given by Eqs. (35) and (38) depend only on the parameters stored in θ. The value of the objective function can be calculated in a relatively cheap way. The low and restricted dimension of the parameter vector offers the way to find the minimum via numerical optimum seeking

procedures. As the identification problem exhibits a global optimization problem, local optimum seeking procedures (e.g. MATLAB fminsearch rutin) need to be supported by appropriate initial value selection.

*Subspace estimation*

Beyond the direct minimization of the objective function the subspace based identification offers another alternative. Here the minimization of the fitting error is accomplished in two steps:

1. The first step can be interpreted as a data compression. The goal here is to generate the best fitting subspace via the following minimization problem:

$$\Gamma_{opt}, H_{opt} = arg\ \min_{\Gamma_q, H_{u,q}} \left\| P_{\Gamma_q}^{\perp} [-H_{u,q} \quad I] R_q \right\|_F \quad (40)$$

   subject to the following constraint

$$rang(\Gamma_q) \leq m. \quad (41)$$

2. The second step is a parameter estimation performed in the reduced $\Gamma_{opt}$ and $H_{opt}$ or in their complement subspace:

$$\Gamma_{opt} \sim \Gamma_q(\theta) \text{ and } H_{opt} \sim H_{u,q}(\theta) \quad (42)$$

A definite advantage of the method is that if appropriate special parametrization is used, no iterations are necessary for the parameter estimation. A large corpus of theoretical and practical results about the subspace identification is available (Pillonetto et al. 2015; Qin 2006; Van Overschee et al. 1996; Vajk 2003).

In the optimization problem exhibited by Eq. (40) it is assumed that $H_{u,q}$ has arbitrary entries and no structural restrictions exist. Under such assumptions the objective function is linear in $H_{u,q}$, thus the minimum by $H_{u,q}$ can be calculated by the following projection:

$$\min_{H_{u,q}} \left\| P_{\Gamma_q}^{\perp} [-H_{u,q} \quad I] R_q \right\|_F = \left\| P_{\Gamma_q}^{\perp} Y_q P_{U_q^T} \right\|_F = \left\| P_{\Gamma_q}^{\perp} R_{y,q} \right\|_F \quad (43)$$

Now determine the minimum of the objective function according to $\Gamma_q$. It is still assumed that there is no structural restriction regarding the selection of $\Gamma_q$. Take $\Gamma_q$ as a left singular vector of matrix $R_{y,q}$, then

$$\left\| P_{\Gamma_q}^{\perp} R_{y,q} \right\|_F^2 = \sum_{i \in I_{\bar{\Gamma}}} \sigma_i^2 \quad (44)$$

where $I_{\bar{\Gamma}}$ forms a set of left singular vectors not belonging to $\Gamma_q$. Consequently, $\left\| P_{\Gamma_q}^{\perp} R_{y,q} \right\|_F^2$ is minimized if $\Gamma_q$ is set up by left singular vectors of dominant singular values.

So far it has been shown that minimizing $H_{u,q}$ leads to LS, while minimizing $\Gamma_q$ leads to TLS parameter estimation. When estimating matrices $\Gamma_q$ and $H_{u,q}$, respectively, the internal structure of the matrices have not been taken into account. Structural restrictions are only utilized when $\Gamma_q$ and $H_{u,q}$ are estimated.

Similar results are obtained if the Frobenius norm of $E_q = Y_q - \hat{Y}_q$ is minimized first by $H_{u,q}$ without utilizing structural restrictions:

$$\min_{H_{u,q}} \left\| Y_q - \hat{Y}_q \right\|_F^2 = \left\| Y_{qP} - \hat{Y}_{qP} \right\|_F^2 \quad (45)$$

where $Y_{qP} = Y_q P_{U_q^T}$ and $\hat{Y}_{qP} = \Gamma_q X_q P_{U_q^T}$; then the optimization

$$\min_{\hat{Y}_{qP}} \left\| Y_{qP} - \hat{Y}_{qP} \right\|_F^2 \quad (46)$$

subject to $rang(\hat{Y}_{qP}) \leq m$ is performed.

The optimization problem by Eq. (38) composed with transfer functions

$$\min_{\widetilde{\mathcal{B}}_q} \left\| \mathcal{A}_q^{+} [-\widetilde{\mathcal{B}}_q \quad \mathcal{A}_q] R_q \right\|_F \quad (47)$$

will also lead to an LS problem if no structural restrictions are introduced for $\widetilde{\mathcal{B}}_q$. The minimum can be calculated as follows:

$$\min_{\widetilde{\mathcal{B}}_q} \left\| \mathcal{A}_q^{+} [-\widetilde{\mathcal{B}}_q \quad \mathcal{A}_q] R_q \right\|_F = \left\| P_{\mathcal{A}_q^T} R_{y,q} \right\|_F \quad (48)$$

Minimization of this expression with respect to $\mathcal{A}_q^T$ can also be performed by the singular decomposition of $R_{y,q}$. Singular vectors belonging to the *(q-m)* non-dominant singular values result in the minimum of the function.

The left singular vectors belonging to the *m* highest singular values can produce the $\Gamma_{opt}$ matrix, which can also be considered as an estimation of the extended observability matrix. The $\mathcal{A}_{opt}^T$ matrix is constructed by the rest of the left singular vectors. The above derivation of the matrices implies that the product of the two matrices is zero. Apart from special singular value arrangements the two subspaces can be derived unambiguously. Derivation of $\Gamma_{opt}$ and $\mathcal{A}_{opt}^T$ shows that these matrices are not unique. Assume $T_\Gamma$ and $T_A$ are invertible matrices of appropriate size. Then if $\Gamma_{opt}$ is a solution of the problem, then $\Gamma_{opt} T_\Gamma$ is a solution, too. Similarly, if $\mathcal{A}_{opt}^T$ is a solution, then $\mathcal{A}_{opt}^T T_A^T$ delivers the same value for the objective function.

*Parameter estimation in the subspaces*

Depending on the parametrization of the system description applied, various strategies can be used for the parameter estimation. In known subspaces, e.g. $\Gamma_{opt} T_\Gamma \approx \Gamma_q(\theta)$ or $T_A\, \mathcal{A}_{opt} \approx \mathcal{A}_q\,(\theta)$ can be used for the parameter estimation.

Just as an example, utilizing the special built up of the extended observability matrix the *A* and *C* matrices can be estimated. The first row of the observability matrix can be

taken as C. Introduce matrices $U_1$ and $U_q$ by simply leaving the first and last row in $\Gamma_{opt}$. The special built up of the observability matrix allows us to write $U_q \approx U_1 A$. Thus the state matrix can be estimated as $\hat{A} = U_1^+ U_q$.

Now as we know A and C finding the B and D parameters in the state model leads to an LS problem minimizing the objective function by Eq.(35), as the error is linear in parameters and the objective function is quadratic. As far as the transfer function description is concerned, minimizing the objective function by Eq. (38) w.r.t. the coefficients of the *B(q)* polynomial is another LS problem to be solved given the *A(q)* polynomial.

## 4. Performance improvement

The identification methods discussed in the paper have been investigated in a Monte Carlo simulation study using the following linear, continuous-time, second order process:

$$y = \frac{1}{1/9sec^{-2}\ s^2 + 0.2sec^{-1}\ s + 1}\ u \qquad (49)$$

Also, assume additive white noise with variance of 0.05 acting on the output. The total observation time is 36 sec. Numbers of samples processed by the identification algorithms are 500, 1000, 2000 and 4000, respectively. As far as the excitation is concerned, the input is lowered from 1 to -1, then raised from -1 to 1 periodically in every 6 sec.

Using the same simulation environment several identification runs over the same samples have been evaluated. For each identification run the $q$ scaling factor (essentially the size of the Hankel matrices used) sweeps the range from $q=m+1=3$ to $q=30$. As an overall measure for the effectiveness of the parameter estimation the empirical standard deviation have been calculated:

$$s_k(q) = \sqrt{\frac{1}{L}\sum_{j=1}^{L}(\hat{a}_{k,q,j} - a_k)^2} \qquad (50)$$

where $j$ denotes the index of the simulation run, $L=100$ stands for the number of simulation runs performed, while $a_k$ $(k=1,2)$ are the discrete time system coefficients.

Results of the simulation study related to the $a_1$ coefficient are shown in Fig. 1. Similar results have been obtained for $a_2$. The results are in harmony with the expectations, namely the more information is used by a parameter estimation procedure, the less variance in the estimation can be achieved. Consequently, the TLS algorithm working with the smallest amount of information delivers the most modest results. On the other hand, $q=N$ requires the highest amount of calculation and results in an estimation with the smallest variance. If the noise distribution is of normal iid, then the minimization of the functions (30) and (31) gives maximum likelihood estimation. Fig. 1 shows that the useful choice of the tuning parameter $q$ is significantly affected by the dynamics of the investigated system. The efficiency of the estimation is inversely proportional to the used sampling time.

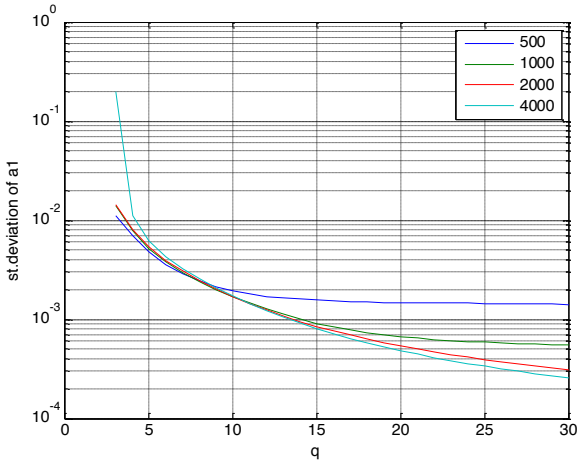


Fig.1. Variation of the standard deviation of the $a_1$ parameter estimation with respect to the size of the Hankel matrices applied and the number of the samples used for identification

## 5. Conclusions

The paper is about the identification of dynamic systems. An effective way has been shown to solve the related deconvolution problem. The presented family of estimation methods allows the introduction of a trade-off between the computational complexity and the estimation performance. Details discussed in this paper have been restricted to involve only SISO systems with output error model. The presented method can be generalized to ARMAX MIMO systems in a direct way.

## 6. References


Ljung, L. (1999) System Identification: Theory for the User. Prentice-Hall, Inc.

Markovsky, I., Van Huffel S. (2007) Overview of total least-squares methods. Signal Processing Vol.87 (10) pp. 2283-2302.

Pillonetto, G., Dinuzzo,F., Chen,T., De Nicolao, G., Ljung, L. (2015) Kernel methods in system identification, machine learning and function estimation: A survey. Automatica Vol.50(3) pp. 657-682.

Qin,S.J (2006) An overview of subspace identification. Computers and Chemical Engineering 30, pp 1502-1513.

Söderström,T. (1989) System identification. Prentice Hall International (UK) Ltd.

Vajk, I. (2003) Identification methods in a unified framework. 13th IFAC Symposium on system identification, Rotterdam, The Netherlands. pp. 2010-2015.

Van Overschee,P, De Moor, B. (1996) Subspace Identification for Linear Systems: Theory, Implementation, Applications. Kluwer Academic Pubishers.