Sensitivity measures for different YOULA regulators

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Abstract – *For YOULA-parameterized regulators different sensitivity measures are introduced and compared and the influence of a new observer topology is also treated.*

Keywords: YOULA regulator, YOULA parameterization, sensitivity, observer, state-feedback, model error

1. Introduction

The simple YOULA parameterization [5], [6] is not so widely known than the YOULA-KUCZERA parameterization [4], [5]. The classical YOULA parameterization gives a very simple way for open-loop stable processes when the regulator can be analytically designed by explicit formulas.

The YOULA parameter is, as a matter of fact, a stable (by definition), regular transfer function

$$Q(s) = \frac{C(s)}{1 + C(s)P(s)} \quad \text{or shortly} \quad Q = \frac{C}{1 + CP} \qquad (1)$$

where $C(s)$ is a stabilizing regulator, and $P(s)$ is the transfer function of the stable process.

It follows from the definition of the YOULA parameter that the structure of the realizable and stabilizing regulator in the YOULA-parameterized control loop is fixed:

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} \text{ or shortly } C = \frac{Q}{1 - QP} \qquad (2)$$

The YOULA parameterized control loop is shown in Fig. 1.

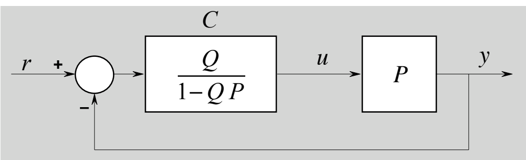

Figure 1. YOULA-parameterized control loop

The YOULA parameterization can be extended for two-degree-of-freedom control systems and applying reference models for the tracking and noise rejection properties of the closed-loop simple design formulae can be developed for the regulator design [1], [2].

2. Uncertainties of process models and closed-loop parameters

The process parameters are never known precisely and the process is subject to change. The environment can change, which can in turn change the parameters of the process in a given region. Negative feedback reduces the sensitivity of the system to parameter changes. Therefore regulator design needs to take possible parameter changes into account. The required behavior of the control loop must be fulfilled not only for the nominal parameters but also for the possible parameter changes.

The knowledge of a process is never exact, independently of the method – whether measurement-based identification (ID) or physico-chemical theoretical considerations – by which its model is determined. The uncertainty of the plant can be expressed by the absolute model error

$$\Delta P = P - \hat{P} \qquad (3)$$

and the relative model error

$$\ell = \frac{\Delta P}{\hat{P}} = \frac{P - \hat{P}}{\hat{P}} \qquad (4)$$

where $\hat{P}$ is the available nominal model used for regulator design and P is the real plant.

Let us now investigate the behavior of the control system if the transfer function of the process changes from the (real) value $P(s)$ to the nominal (model) value $\hat{P}(s)$. The overall transfer function of the open loop is $L = CP$. For small changes in the process

$$\Delta L = \frac{\partial L}{\partial P} \Delta P = C \Delta P \qquad (5)$$

Applying the relative changes we obtain

$$\frac{\Delta L}{L} = \ell_{\mathrm{L}} = \frac{C\Delta P}{CP} = \frac{\Delta P}{P} = \ell \quad ; \quad \Delta P(s) = P(s) - \hat{P}(s) = \Delta \qquad (6)$$

The overall transfer function of the negative feedback closed-loop is

$$T = \frac{CP}{1+CP} \tag{7}$$

For small changes

$$\Delta T = \frac{\partial T}{\partial P}\Delta P = \frac{C}{\left(1+CP\right)^2}\Delta P \tag{8}$$

For relative changes:

$$\frac{\Delta T}{T} = \ell_{\mathrm{T}} = \frac{1}{1+CP}\frac{\Delta P}{P} = S\frac{\Delta P}{P} = S\ell \tag{9}$$

where S is the sensitivity function of the closed-loop

$$S = \frac{\Delta T/T}{\Delta P/P} = \frac{1}{1+CP} \tag{10}$$

Consider the following three simple closed control loops which can be used in model-based regulator design. The first closed-loop can be seen in Fig. 2. Here it is assumed that the regulator C is computed from the theoretical real process P and is placed together with the real process in the closed-loop. Obviously this closed-loop is not realistic and represents an ideal case only.

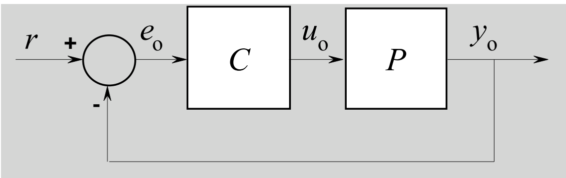

Figure 2. The theoretical closed system

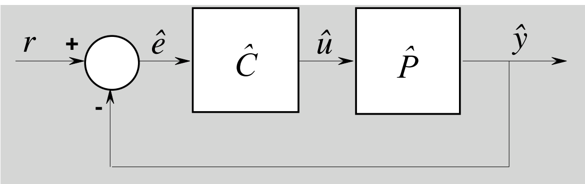

Figure 3. The nominal closed system

The next version can be seen in Fig. 3, and is usually applied in design tasks, namely, when the regulator $\hat{C}$ is determined on the basis of the process model $\hat{P}$ and the whole closed-loop is model-based. This case is usually called the nominal system. This closed-loop depends only on the designer, on the knowledge of the process and the suggested regulator. The scheme can be used in simulation, optimization and design tasks.

The third version of the closed system is what operates in the reality. A model-based regulator is used together with the real process in the closed-loop as in Fig. 4. Usually measurements, verifications and application of identification methods take place in this kinds of closed-loops.

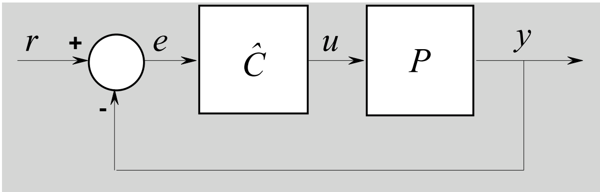

Figure 4. The real closed system appearing in the practice

Table 1. The sensitivity and complementary sensitivity functions of the three systems

System function	ideal	nominal	real
T	$T = \frac{CP}{1+CP}$	$\hat{T} = \frac{\hat{C}\hat{P}}{1+\hat{C}\hat{P}}$	$\tilde{T} = \frac{\hat{C}P}{1+\hat{C}P}$
S	$S = \frac{1}{1+CP}$	$\hat{S} = \frac{1}{1+\hat{C}\hat{P}}$	$\tilde{S} = \frac{1}{1+\hat{C}P}$

The sensitivity and complementary sensitivity functions for the above three closed systems are summarized in Table 1. The computation of each element is very different and they must not be mixed. Obviously, in the ideal case when $\hat{P} = P$ the elements in the same rows are equal.

Table 2. The sensitivity and complementary sensitivity functions for the YOULA-parameterized control loop

System function	ideal	nominal	real
T	QP	$\hat{Q}\hat{P}$	$\frac{\hat{Q}\hat{P}\left(1+\ell\right)}{1+\hat{Q}\hat{P}\ell}$
S	$1-QP$	$1-\hat{Q}\hat{P}$	$\frac{1-\hat{Q}\hat{P}}{1+\hat{Q}\hat{P}\ell}$

It is easy to check that Table 1. for the YOULA-parameterized control loop is changing to Table 2 if $\hat{C} = \hat{C}\left(\hat{P}\right)$ is the model based YOULA regulator. They can be rewritten in another form, too (see Table 3).

Table 3. The other forms of the sensitivity functions for the YOULA-parameterized control loop

System function	ideal	nominal	real
T	$T = QP$	$\hat{T}$	$\hat{T}\frac{1+\ell}{1+\hat{T}\ell}$
S	$S = 1-QP$	$\hat{S} = 1-\hat{Q}\hat{P}$	$\hat{S}\frac{1}{1+\hat{T}\ell}$

Let us investigate how the real system approximates the nominal one, which is always the basis for the design. Compute the relative error

$$\ell_{\tilde{T}} = \frac{\hat{T}-\tilde{T}}{\tilde{T}} = -\frac{\ell}{1+\ell}\left(1-\hat{T}\right) = -\frac{\ell}{1+\ell}\hat{S} \tag{11}$$

This is an excellent property, because $\hat{S}$ attenuates the relative model error at the low frequency domain. Usually the sensitivity function is a high pass filter.

3. Introduction of the observer-based YOULA regulator

It is well known that the model based YOULA-regulator corresponds to the *Internal Model Control Structure* (*IMC*), presented in Fig. 5. The equivalent *IMC* structure based YOULA-regulator performs the feedback from the model error ε_Q.

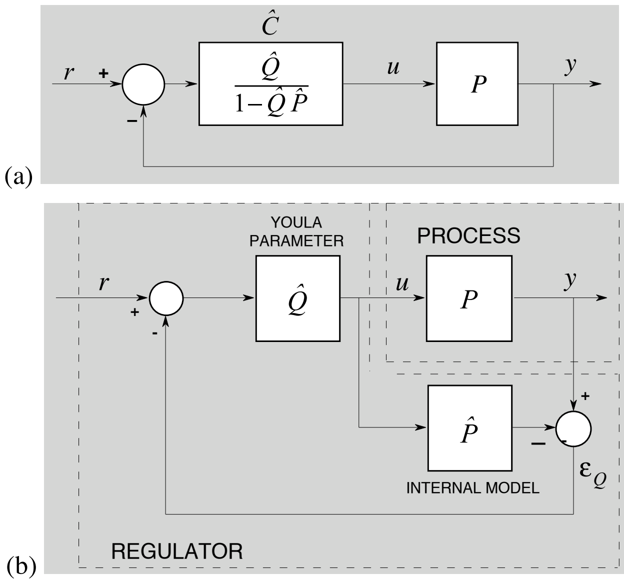

Figure 5. The equivalent *IMC* structure of a YOULA-regulator

Similarly to the classical "*State-Feedback-Observer*" (*SFO*) scheme it is possible to construct an internal closed-loop performing the feedback by $\hat{K}_l$ from ε_l (see Fig. 6, [3]) to reduce the model error using the classical observer principle.

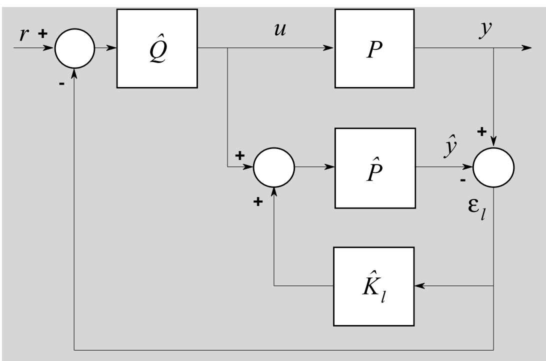

Figure 6. The observer-based *IMC* structure

With straightforward block manipulations the observer based *IMC* topology can be reduced to the two closed-loops system shown in Fig. 7.

The relationship between the two errors in Figs. 5b and 6 is

$$\varepsilon_l = \frac{1}{1+\hat{K}_l\hat{P}}\left(y-\hat{P}u\right) = \frac{1}{1+\hat{L}_l}\varepsilon_Q = \hat{H}\,\varepsilon_Q \quad ; \quad \hat{L}_l = \hat{K}_l\hat{P} \qquad (12)$$

i.e., the observer principle virtually reduces the model error by $\hat{H}$. Here $\hat{L}_l$ is the internal loop transfer function.

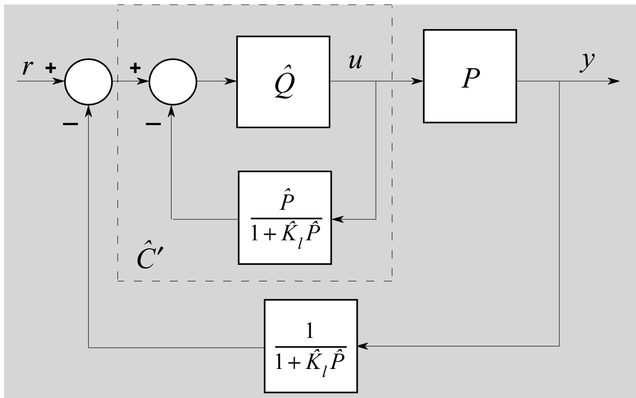

Figure 7. Equivalent closed-loops for the observer-based *IMC* structure

The introduction of the observer feedback changes the YOULA-parameterized regulator to

$$\hat{C}'\left(\hat{P}'\right) = \frac{\hat{Q}}{1-\hat{Q}\dfrac{\hat{P}}{1+\hat{K}_l\hat{P}}} = \frac{\hat{Q}\left(1+\hat{K}_l\hat{P}\right)}{1+\hat{K}_l\hat{P}-\hat{Q}\hat{P}} \qquad (13)$$

The form of $\hat{C}'$ shows that the regulator virtually controls a fictitious plant $\hat{P}'$, which is also demonstrated in Fig. 7. Here the fictitious plant is

$$\hat{P}' = \hat{H}\hat{P} = \frac{\hat{P}}{1+\hat{L}_l} = \frac{\hat{P}}{1+\hat{K}_l\hat{P}} \qquad (14)$$

The error attenuating filter is

$$\hat{H} = \frac{1}{1+\hat{K}_l\hat{P}} = H\frac{1+\ell}{1+H\ell} \quad \text{where} \quad H = \frac{1}{1+\hat{K}_l P} \qquad (15)$$

The nominal complementary sensitivity function in the observer-based *IMC* structure is

$$\hat{T}'_{\text{ry}} = \frac{\hat{C}'\hat{P}}{1+\hat{C}'\hat{P}\hat{H}} = \hat{Q}\hat{P} = \hat{T} \qquad (16)$$

so this is equal to the observer free case.
Compute the complementary sensitivity function of the real loop now

$$\tilde{T}'_{\text{ry}} = \frac{\hat{C}'P}{1+\hat{C}'P\hat{H}} = \frac{\hat{T}(1+\ell)}{1+\hat{T}\hat{H}\ell} \tag{17}$$

and the relative error $\ell'_{\tilde{T}'}$ of $\tilde{T}'_{\text{ry}}$ is, similarly to (11)

$$\ell'_{\tilde{T}'} = \frac{\hat{T}'_{\text{ry}} - \tilde{T}'_{\text{ry}}}{\tilde{T}'_{\text{ry}}} = -\frac{\ell}{1+\ell}\left(1-\hat{T}\hat{H}\right) \tag{18}$$

Compute the complementary sensitivity function of the ideal loop

$$T'_{\text{ry}} = \frac{C'P}{1+C'PH} = QP \tag{19}$$

which follows from (16). The above results are summarized in Table 4. The most important result of this analysis is that the observer-based YOULA regulator gives the same nominal and ideal complementary sensitivity functions as the original YOULA regulator.

Table 4. The complementary sensitivity functions with observer-based YOULA regulator

System function	ideal T'_{ry}	nominal $\hat{T}'_{\text{ry}}$	real $\tilde{T}'_{\text{ry}}$
T'_{ry}	QP	$\hat{Q}\hat{P}$	$\frac{\hat{Q}\hat{P}(1+\ell)}{1+\hat{Q}\hat{P}\hat{H}\ell}$

4. Reference model based YOULA regulator design

The simplest YOULA regulator based on reference model design [1], [2] is

$$\hat{C} = \frac{\hat{Q}}{1-\hat{Q}\hat{P}} = \frac{R_{\text{n}}}{1-R_{\text{n}}}\hat{P}^{-1} \quad ; \quad C = \frac{Q}{1-QP} = \frac{R_{\text{n}}}{1-R_{\text{n}}}P^{-1} \tag{20}$$

where the model based YOULA parameter

$$\tilde{Q} = \hat{Q} = R_{\text{n}}\hat{P}^{-1} \quad ; \quad Q = R_{\text{n}}P^{-1} \tag{21}$$

was applied, because in practical design cases $\tilde{Q} = \hat{Q} \neq Q$. Here R_{n} is the desired reference model for the tracking. Applying this regulator, the Table 1 will be changed to Table 5.

Calculate now the relative design error ℓ_{x} obtained with the different complementary sensitivity functions. The obtained relationships are shown in Table 6, where

$$\ell_{\text{x}} = \frac{R_{\text{n}} - T_{\text{x}}}{T_{\text{x}}} \tag{22}$$

Table 5. The complementary sensitivity functions with YOULA regulator

System function	ideal	nominal	real
T	R_{n}	R_{n}	$\frac{R_{\text{n}}(1+\ell)}{1+R_{\text{n}}\ell}$
Q	$R_{\text{n}}P^{-1}$	$R_{\text{n}}\hat{P}^{-1}$	$R_{\text{n}}\hat{P}^{-1}$

Table 6. The relative design errors with YOULA regulator

System function	ideal	nominal	real
ℓ_{x}	0	0	$-\frac{\ell}{1+\ell}\left(1-R_{\text{n}}\right)$

Here

$$\ell_{\tilde{T}} = -\frac{\ell}{1+\ell}\left(1-R_{\text{n}}\right) = -\frac{\ell}{1+\ell}S_{\text{o}} \tag{23}$$

and S_{o} is the sensitivity function of the ideal system. This is an excellent property, because S_{o} attenuates the relative model error at the low frequency domain, see (11).

The advantage of the reference model based design is that the uncertainty in the YOULA parameter is reduced to uncertainty of the process model only. Therefore the relative design errors for the ideal and nominal system are zero.

It is interesting to investigate how these system functions change using an observer-based YOULA regulator, when

$$\hat{C}'\left(\hat{P}'\right) = \frac{R_{\text{n}}\left(1+\hat{K}_l\hat{P}\right)\hat{P}^{-1}}{1+\hat{K}_l\hat{P}-R_{\text{n}}} = \frac{R_{\text{n}}\left(\hat{P}^{-1}+\hat{K}_l\right)}{1+\hat{K}_l\hat{P}-R_{\text{n}}} \tag{24}$$

The obtained relationships are shown in Table 7.

Table 7. The complementary sensitivity functions with observer-based YOULA regulator

System function	ideal	nominal	real
T	R_{n}	R_{n}	$\frac{R_{\text{n}}(1+\ell)}{1+R_{\text{n}}\hat{H}\ell}$
Q	$R_{\text{n}}P^{-1}$	$R_{\text{n}}\hat{P}^{-1}$	$R_{\text{n}}\hat{P}^{-1}$

Table 8. The relative design errors with observer-based YOULA regulator

System function	ideal	nominal	real
ℓ'_{x}	0	0	$-\frac{\ell}{1+\ell}\left(1-R_{\text{n}}\hat{H}\right)$

Calculate now the relative design errors ℓ'_{x} obtained for observer-based YOULA regulator, which are summarized in Table 8.

5. Sensitivity reduction by different observer regulators

Investigate the sensitivity reductions for three simple regulators for $\hat{K}_l$. First select an integrating (*I*) regulator, when

$$\hat{K}_l = \frac{A_l}{s} \tag{25}$$

The error attenuating filter is

$$\hat{H}(j\omega) = \frac{1}{1+\hat{K}_l\hat{P}} = \frac{1}{1+\frac{A_l}{j\omega}\hat{P}} = \begin{cases} 0 & ; \quad \omega \to 0 \\ 1 & ; \quad \omega \to \infty \end{cases} \tag{26}$$

For a proportional integrating (*PI*) regulator

$$\hat{K}_l = A_l \frac{1+T_l s}{s} \tag{27}$$

and the error attenuating filter is

$$\hat{H}(j\omega) = \frac{1}{1+\hat{K}_l\hat{P}} = \frac{1}{1+A_l\frac{1+T_l j\omega}{j\omega}\hat{P}} = \begin{cases} 0 & ; \quad \omega \to 0 \\ 1 & ; \quad \omega \to \infty \end{cases} \tag{28}$$

The above limit values mean that *I* type observer regulators can provide zero sensitivity at the low frequency domain ($\omega \to 0$), so they can tolerate large errors in the process gain.

For a proportional (*P*) regulator

$$\hat{K}_l = A_l \tag{29}$$

and the error attenuating filter is

$$\hat{H}(j\omega) = \frac{1}{1+\hat{K}_l\hat{P}} = \begin{cases} \dfrac{1}{1+\hat{K}_l\hat{P}(0)} & ; \quad \omega \to 0 \\ \dfrac{1}{1+\hat{K}_l\hat{P}(\omega)} & ; \quad \omega \to \infty \end{cases} \tag{30}$$

This means that zero sensitivity at the low frequency domain ($\omega \to 0$) can be reached by choosing large $\hat{K}_l \to \infty$ observer regulator gain within the stability domain.

For a phase lead/lag regulator

$$\hat{K}_l = A_l \frac{1+T_2 s}{1+T_1 s} \tag{31}$$

and the error attenuating filter is

$$\hat{H}(j\omega) = \frac{1}{1+\hat{K}_l\hat{P}} = \begin{cases} \dfrac{1}{1+A_l\hat{P}(0)} & ; \quad \omega \to 0 \\ \dfrac{1}{1+\frac{T_2}{T_1}A_l\hat{P}(\infty)} & ; \quad \omega \to \infty \end{cases} \tag{32}$$

The above regulator types mean that no classical regulator can drastically reduce the model error in the important medium frequency domain. For such purpose special regulator loop-shaping methodology must be applied.

6. Simulation example

Consider a simple first order process and its model as

$$P = \frac{A}{1+sT} = \frac{1}{1+10s} \quad ; \quad \hat{P} = \frac{\hat{A}}{1+s\hat{T}} = \frac{1.5}{1+20s} \tag{33}$$

Select the design goal to spead up the oparation five times, i.e. select a reference model

$$R_{\mathrm{n}} = \frac{1}{1+sT_{\mathrm{n}}} = \frac{1}{1+2s} \tag{34}$$

The YOULA regulator based on reference model design [1], [2] is

$$\hat{C} = \frac{\hat{Q}}{1-Q\hat{P}} = \frac{R_{\mathrm{n}}}{1-R_{\mathrm{n}}}\hat{P}^{-1} = \frac{1+s\hat{T}}{sT_{\mathrm{n}}} = \frac{1+20s}{2s} \tag{35}$$

where the model based YOULA parameter

$$\hat{Q} = R_{\mathrm{n}}\hat{P}^{-1} = \frac{1}{\hat{A}}\frac{(1+s\hat{T})}{1+sT_{\mathrm{n}}} = \frac{1}{1.5}\frac{1+20s}{1+2s} \tag{36}$$

was applied.

First select a *PI*-type observer regulator

$$\hat{K}_l = A_l \frac{1+T_l s}{s} = 0.01\frac{1+2s}{s} \tag{37}$$

The observer-based YOULA regulator is shown in Fig. 8. This scheme can be further simplified as Fig. 7 shows.

It is interesting to show the step responses of the different elements in this scheme. Fig. 9 shows these functions for the true process, the model, the reference model and the observer-based closed control loop $\tilde{T}'_{\mathrm{ry}}$. The reached error: $R_{\mathrm{n}} - \tilde{T}'_{\mathrm{ry}}$ is also shown in the figure.

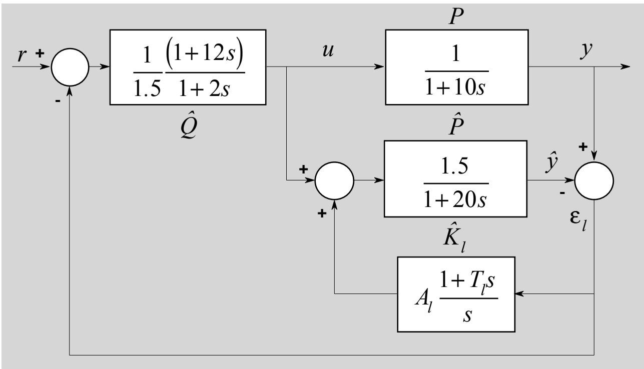

Figure 8. *PI*-type observer-based YOULA regulator

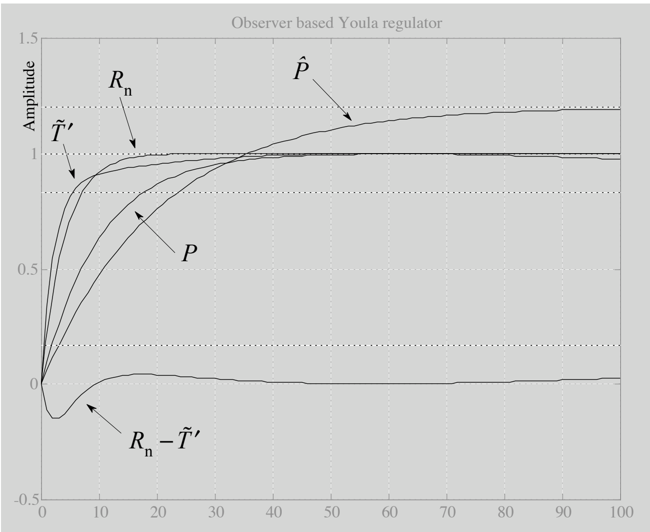

Figure 9. The most important step responses if the reference signal r is a unit step

7. Conclusions

The YOULA parameter based regulator design is an excellent tool for cases when the open-loop process is stable. This approach gives explicit analytical formulas for the design procedure. Unfortunately the different sensitivity measures for such regulators are missing from the control references. This paper tries to eliminate this gap giving a detailed analysis for the relative sensitivity measures of these regulators.

The paper also includes the extension of the observer principle for YOULA regulators reducing the model error similar to the classical state feedback/observer topologies.

The influence of the different observer regulators for the error attenuating filter is also shown.

Finally a simple simulation result is shown where the model error is 100 % in the time constant and 50 % in the gain of the real process. The simulation clearly shows that very good result can be obtained combining the YOULA regulator and the observer principle.

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