Identifying system poles by applying hyperbolic geometry in the unit disc

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Abstract – *Starting from concept of rational orthogonal bases applied in system identification a new idea has been developed to find the poles of a linear dynamical system without using further assumptions on system structure. The solution arises from the hyperbolic geometrical properties of the representations of discrete-time linear systems in the unit circle. Using a hyperbolic distance in combination with Laguerre representations of the system gives the possibility to find some poles, and the iterative use of this procedure results in finding all of them. Finding the poles of a system in in this way gives a solution of the nonparametric identification of systems.*

Keywords: Linear systems, rational representations, system identification, hyperbolic geometry, hyperbolic distance.

1. Introduction

System modeling and identification is an area of great significance in the technical sciences as they can be considered as prerequisites for detection and control problems arising in several technical fields, e.g. in industry, power production, vehicles, biomedical systems, etc. Hence system identification became an important part of research activities of the Systems and Control Laboratory of the Institute of Computer Science and Automation for many decades since the 70's of the last century. Conventional spectrum–based identification methods (using Fourier analysis by FFT) as well as linear model–based methods (using AR, ARMA models and their variants) were successfully applied in the failure detection and diagnostic systems in the reactor and primary circuit of nuclear power plants [7], [2]. Later AR and ARMA model based methods became well–known and popular identification tools by including them in various forms in the System Identification Toolbox of Matlab® [9], [8]. However, it it has become clear that these methods are in nature parametric ones, i.e. they need strong *a priori* assumptions on the system structure.

In the 90's a new idea emerged in system identification, namely the use of nonstandard – rational – orthogonal bases associated with the H^2 and H^∞ spaces on the unit disc – in opposition with the conventional methods of system representations associated with the standard trigonometric basis in the space H^2. Researchers of the Systems and Control Laboratory played significant role in building up the theory of rational orthogonal bases (ROBs), and in constructing methods and algorithms that can be applied in system identification. A series of papers [10], [15], [31], [32], [5], [11], [16], [20], [17], [4] and theses [3], [30], [19] appeared on this topic , and the main result has been published in a monograph [6]. The ROBs can perfectly be used to identify systems in the case when *a priori* information is available with respect to the locations of system poles. Inaccurate knowledge of the pole-positions results in infinite series representations, while approximate knowledge hopefully results in fast decay of the representation coefficients. Nevertheless there were ideas to refine the identified pole positions on the basis of measurements [25], it was obvious that the problem of nonparametric identification has not been solved.

Rational orthogonal bases, namely the Blaschke function and product that can be considered key notions in this theory, offered a way to proceed. The Blaschke function by forming a group in the space $H^2(\mathbb{D})$ (where $\mathbb{D}$ denotes the unit disc in the complex plane) with respect to function composition, hence – similarly to affine or Heisenberg group in the space $L^2(\mathbb{R}$ – can be used to form *wavelet* constructions. Since The Blaschke function realizes a hyperbolic transform in the unit disc, the associated wavelet can also be considered as *hyperbolic* one. An introduction to hyperbolic wavelets can be found in [12], [13], [14], while the opportunities of applying them in system identification has been covered in [21], [27], [23]. It has been shown in these publications that the Laguerre representations play significant role in this theory, as the wavelets generated by them can analytically be expressed, hence special attention was paid to them hereafter.

In this paper a new approach of system identification will be introduced that is based on Laguerre representations of discrete–time signals in the space $H^2(\mathbb{D})$, and largely exploits the hyperbolic geometry generated by the Blaschke transform implied by them [18], [28], [22], [29]. On the basis of these means a method has been constructed that is able to identify the poles of the system associated with the signals. This method does not use any *a priori* assumption on the system concerned beyond the notion of poles within the linear system theory, hence it can be considered as a *nonparametric* identification method. The method is based on discrete time– or frequency–domain measurements, and efficient algorithms have been developed for practical applications.

2. The discrete Laguerre system

The discrete Laguerre system corresponding to the parameter $b \in \mathbb{D}$ is defined on the closed unit disc $\overline{\mathbb{D}}$ as

$$L_n^b(z) := \frac{\sqrt{1-|b|^2}}{1-\overline{b}z}\left(\frac{z-b}{1-\overline{b}z}\right)^n \qquad (1)$$

where $n \in \mathbb{N}$ and the so–called Laguerre–parameter b can be considered as an *inverse pole* ($b = 1/\overline{p}$) of the function. This system forms an orthonormal basis in the Hardy space $H^2(\mathbb{D})$, see e.g. [17], any function $f \in H^2(\mathbb{D})$ can be expressed by the representation:

$$f(z) = \sum_{n=0}^{\infty} l_n L_n^b(z), \qquad (2)$$

where the Laguerre–Fourier coefficients $\{l_n\}$ can be computed by using the inner-product of the Hilbert space $H^2(\mathbb{D})$:

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{\mathrm{i}t})\overline{g}(e^{\mathrm{i}t})\, dt \quad (f, g \in H^2(\mathbb{T})), \qquad (3)$$

as $l_n = \langle f, L_n^b(z)\rangle$.

The convergence of the coefficients of the Laguerre representations can be associated with a hyperbolic metric valid in the unit disc according to the the Poincareé disc model of the hyperbolic geometry. Observe that the common term in (1) is the so–called Blaschke function

$$B_b(z) := \frac{z-b}{1-\overline{b}z}. \qquad (4)$$

Some important features of the Blaschke function are mentioned as follows:

• B_b is an $1-1$ map on the unit circle $\mathbb{T}$ and the open unit disc $\mathbb{D}$, respectively.

• $B_b(z)$ is an inner function in the space $H^2(\mathbb{D})$, i.e., $|B_b(e^{\mathrm{i}t})| = 1$ ($t \in [-\pi, \pi]$).

• The Blaschke functions B_b are isometries with respect to the metric

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1-\overline{z}_1 z_2|} = |B_{z_1}(z_2)| \qquad (5)$$

$(B_{z_1} := B_{(z_1,1)}, z_1, z_2 \in \mathbb{D})$,

that is called — following Poincaré — *pseudo–hyperbolic* metric (see, e.g., [1] for details), i.e.,

$$\rho(B_b(z_1), B_b(z_2)) = \rho(z_1, z_2) \qquad (6)$$

$$(b \in \mathbb{D}, z_1, z_2 \in \mathbb{D}).$$

Using the concept of the Blaschke–function and Blaschke–group with this metric a hyperbolic-type geometry can be built in the unit–disc that is conform with the Poincaré unit–disc model of the hyperbolic geometry. Hence the Blaschke–group can also be referred as *hyperbolic* group.

Using the hyperbolic distances associated with the selected Laguerre parameter and the poles of the system, some poles can be identified. By using an iterative procedure on the basis of the poles already found, all the poles that significantly affects the system behavior can be identified. Identification of the poles gives the opportunity to represent the system in a rational orthogonal basis that leads to total identification of the system (see, e.g., [26]).

3. Basic idea to find poles

Let us consider a transfer function $f \in H^2(\mathbb{D})$ having a single pole (inverse pole $a \in \mathbb{D}$), i.e.,

$$f(z) = \frac{A}{1-\overline{a}z}. \qquad (7)$$

Then, taking a Laguerre representation that corresponds to a parameter b, its n^{th} Laguerre–coefficient can be expressed as

$$l_n = \sqrt{1-|b|^2}\,\frac{(\overline{a}-\overline{b})^n}{(1-\overline{a}b)^{n+1}}. \qquad (8)$$

Observe that these Laguerre coefficients form a geometric sequence with quotient

$$q = \frac{\overline{a}-\overline{b}}{1-\overline{a}b}. \qquad (9)$$

The modulus of the quotient (9) is equal to the hyperbolic distance between the inverse pole of the function (a) and the selected Laguerre parameter (b). Hence the convergence of the Laguerre coefficients — and consequently the convergence of the Laguerre representation belonging to the function f — depends on this hyperbolic distance.

This fact can be used to identify the inverse pole. It is clear from (9) that $\overline{q} = B_b(a)$, hence the quotient value q implies directly the inverse pole location a as $a = B_{b^{-1}}(\overline{q})$.

The idea can be extended to multiple poles. Qualitatively, the Laguerre–representation of a function in partial fraction form can be expressed as a sum of terms corresponding to (8) — each belonging to any element of the set of poles. In the convergence of this sum the term with the maximal modulus dominates, since terms with smaller modulus decay faster. The term of maximal modulus represents maximal hyperbolic distance between the Laguerre–parameter b and the inverse pole a_k associated with this term. Hence the maximum of the hyperbolic distances between a selected parameter $b \in \mathbb{D}$ and the inverse poles a_k ($k \in \mathbb{N}$) of the function f divides the unit disc into mutually disjoint regions.

4. Identifiability domains

According to [18] these regions can be constructed as follows: by fixing the inverse poles $a_1, a_2 \ldots, a_K \in \mathbb{D}$ of function f and applying the hyperbolic distance as defined by (5) the following subsets of $\mathbb{D}$ can be defined:

$$D_i: \quad = \{b \in \mathbb{D}: \rho(b, a_i) > \max_{1\le j\le K, i\neq j} \rho(b, a_j)\},$$
$$D_0: \quad = \bigcup_{j=1}^{K} D_i \quad (i = 1,2,\dots,K). \tag{10}$$

The limits of these subsets can be considered as the sets

$$\mathcal{L}_{ij} := \{b \in \mathbb{D}: \rho(a_i, b) = \rho(a_j, b)\}, \tag{11}$$

that are the hyperbolic perpendicular bisectors associated with the points a_i, a_j that divides $\mathbb{D}$ in two hyperbolic half-planes. Let the following notations be introduced:

$$D_{ij} \quad := \{b \in \mathbb{D}: \rho(a_i, b) > \rho(a_j, b)\}$$
$$D_{ji} \quad := \{b \in \mathbb{D}: \rho(a_i, b) < \rho(a_j, b)\}. \tag{12}$$

The sets D_i-s will be called the identifiability domains of the poles. These can be generated as an intersection of the hyperbolic half-planes, i.e. according to the definitions in (10) and (12)

$$D_i = \bigcap_{k=1, k\neq i}^{K} D_{ik} \quad (i = 1,2,\cdots,K). \tag{13}$$

As a consequence the sets D_i are hyperbolically convex regions, i.e., any hyperbolic line segment connecting two points belonging to any D_i is located as a whole in the same region.

The rate of convergence associated with the Laguerre–representation of a function $f \in H^2(\mathbb{D})$ can be expressed on the basis of the quotient criterion applied to the series. By selecting any point b of D_k as the Laguerre–parameter the limit

$$(Qf)(b) := \lim_{n\to\infty} \frac{l_{n+1}^b}{l_n^b} \tag{14}$$

does exist, and by applying an inverse Blaschke–transform associated with parameter b on the result, one of the poles of function f — namely a_k — can be reconstructed, and finally, the following theorem can be set up, see [18]:

Theorem 1 *For any rational function f of form* (7) *in any point b of D_0 the limit*

$$(Qf)(b) := \lim_{n\to\infty} \frac{\langle L_{n+1}^b, f\rangle}{\langle L_n^b, f\rangle}$$

exists, and $(Qf)(b) = B_b(a_i), \quad b \in D_i \quad (i = 1,2,\dots,K)$.

For multiple poles one has the following estimation of the speed of convergence:

$$\left|\frac{l_{n+1}^b}{l_n^b} - B_b(a_i)\right| = O(q_i^n) \ \ (n \in \mathbb{N}, b \in D_i, q_i < 1), \text{ see [18].}$$

From Theorem 1 it follows that for any $b \in D_i$

$$B_b^{-1}((Qf)(b)) = a_i \quad (b \in D_i, i = 1,2,\cdots,K), \tag{15}$$

i.e., (15) reconstructs all the poles with region $D_i \neq \emptyset$ belonging to them.

It may occur that there is no region $D_i \neq \emptyset$ that belongs to particular poles – in these cases these poles remain hidden. A method of finding the hidden poles by using an iterative process will be introduced below.

5. Recursion to find all the poles

Finding hidden poles, i.e., poles with empty identifiability domain can be based on an important property of representations in rational orthogonal bases. Building an orthogonal basis on the set of already known poles results in a representation where the concerned poles affect only finite number of coefficients while the unknown poles result in infinite representation (see e.g. [26]). Hence only the unknown poles play role in the convergence of the representation.

In this construction a new structure of identifiability domains arise, which bring some so far hidden poles in the focus that become thereby identifiable. Iteratively repeating this procedure can result in finding the complete set of poles of f.

The proposed procedure uses the finite Malmquist–Takenaka system. Starting from a finite set $\{a_n\}$ of inverted poles it is defined as:

$$\phi_n(z) := \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z} \prod_{k=0}^{n-1} B_{a_k}(z). \tag{16}$$

Let the subspace spanned by these functions be $H_N^2(\mathbb{D})$. Then, for any $f \in H_N^2(\mathbb{D})$ we have

$$f(z) = \sum_{n=0}^{N} c_n \phi_n(z),$$

where $c_n = \langle f, \phi_n\rangle$ are the Malmquist–Takenaka–Fourier (MT-Fourier) coefficients.

Having the m^{th} partial sum S_m of the f, i.e.,

$$S_m(z) = \sum_{n=0}^{m-1} c_n \phi_n(z),$$

the error $f - S_m$ can be written in the form

$$f(z) - S_m(z) = \sum_{n=m}^{N} c_n \phi_n(z) = A_{m-1}(z) \cdot$$
$$\cdot \left(c_m \frac{\sqrt{1-|a_m|^2}}{1-\overline{a_m}z} + c_{m+1} \frac{\sqrt{1-|a_{m+1}|^2}}{1-\overline{a_{m+1}}z} B_{a_m}(z) + \cdots \right),$$

where $A_{m-1}(z) = \prod_{k=0}^{m-1} B_{a_k}(z)$ is the Blaschke–product associated with the first $m-1$ inverted poles. Thus

$$R_m(z) \doteq (f(z) - S_m(z))\overline{A_{m-1}(z)}$$

depends only on poles with indices $n \geq m$.

If the first $m-1$ poles of the function f are already known, realizing the pole–identification procedure on the function $R_m(z)$ results in finding further poles including those that were hidden in the previous step. Repeated use of this procedure can result in finding all the poles of the function.

6. Algorithmic issues

The algorithms required to realize the identification process can be divided to two main groups:

• Algorithms to estimate Laguerre coefficients as well as estimation of Takenaka–Malmquist representations. Both estimations are based on the algorithms worked out for representations in rational orthogonal bases (see, e.g., [26]), using the principle of argument function nonuniform sampled data points in the frequency scale. Both time and frequency domain realizations are available [24].

• Algorithms to estimate convergence quotient and computing the pole – two approaches has been elaborated. Direct application of Theorem 1, i.e., computing the quotient of subsequent Laguerre coefficients and estimating the limit is an obvious opportunity. Another method – that seems to be more efficient and less sensitive to noises – is the application of linear regression on the logarithm of the Laguerre coefficients, see [24] for details. Finding linear segment in the sequence of the logarithmic Laguerre coefficients indicates also the adequate selection of the Laguerre parameter with respect to identifiability of a pole.

The pole identification algorithm has been realized in Matlab ® in the form of a toolbox.

7. Illustrative example

A SISO system containing 15 poles, one real and 7 conjugated complex pairs is considered. It can be verified on Figure 1 that two conjugated complex pole–pairs are identifiable in the first identification step. By selecting the Laguerre parameter $b = 0.759e^{i\pi 0.127}$ Iteration 0 of the pole–identification procedure is performed. The Laguerre coefficients with this parameter – estimated from the time–domain impulse–response signal – can be seen in logarithmic form on Figure 2. A linear interval is searched both in the logarithmic modulus and the phase of the Laguerre coefficients by computing the linear regression with minimal residual variance within a finite sliding window. The best selection of window, called ROI (Region of Interest), can also denoted in the plot. The slopes of the regression lines are associated with the quotient q of the Laguerre sequence, hence applying an inverse Blaschke transform results in a pole position. The frequency associated with this pole is denoted by a vertical bar in the frequency diagram presented on Figure 3. The pole $0.968e^{-i\pi 0.298}$ has been identified, and its conjugated pair $0.968e^{i\pi 0.298}$ is also considered to be found simultaneously.

In Iteration 1 the poles $0.968e^{\pm i\pi 0.298}$ are considered to be known. The identifiability domains associated to this case can be seen on Figure 4. Two conjugated complex pole pairs are identifiable including a pole pair that was not identifiable in the previous step. With the selection of a Laguerre parameter $0.601e^{i\pi 0.222}$ the Laguerre coefficients seen in Figure 5 (logarithmic modulus is presented). Repeating the procedure described above the pole $0.989e^{-i\pi 0.039}$ has been identified. Figure 6 presents the actual shape of the frequency function compared to the original function (denoted by wide and thin line, respectively). It can be observed that the spectral peaks that can be associated to the already known poles are missing. The frequency associated with the pole found is denoted by a vertical bar. The identified pole and its conjugated complex pair is $0.989e^{\pm i\pi 0.039}$.

Continuing the iterations all the poles of the system can be identified.

8. Conclusion

A new identification method has been presented that starting from the representations discrete–time systems in rational orthogonal bases, and exploiting the hyperbolic geometrical properties of the representations in the unit disk is able to derive the poles of the system. Knowing the poles, by the means of ROB representations the transfer function belonging to the system can also be derived, hence this method gives a solution for nonparametric identification of linear dynamical systems.

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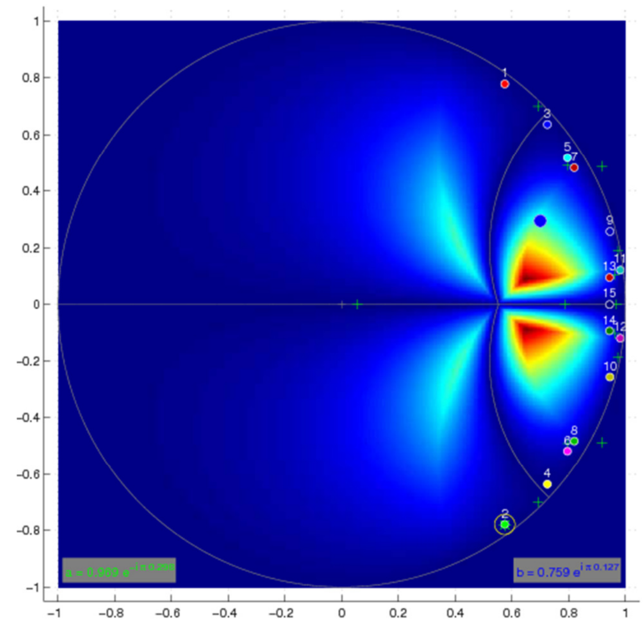

Figure 1: Iteration 0 – Selection of the Laguerre–parameter and the identified pole.

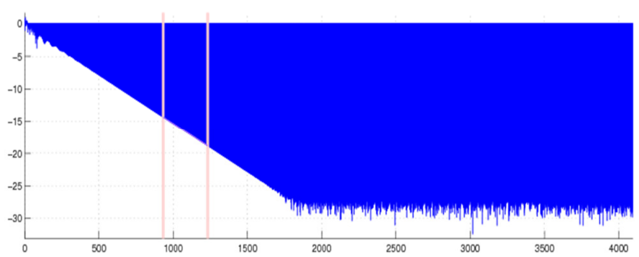

Figure 2: Iteration 0 – Logarithmic modulus of Laguerre coefficients with ROI.

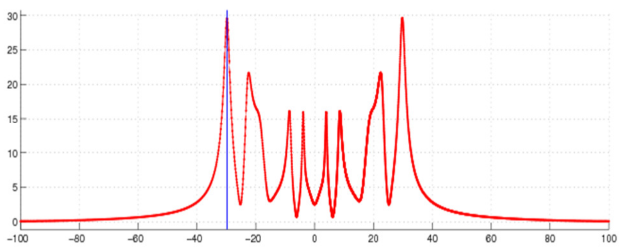

Figure 3: Iteration 0 – Modulus of the frequency domain signal of system.

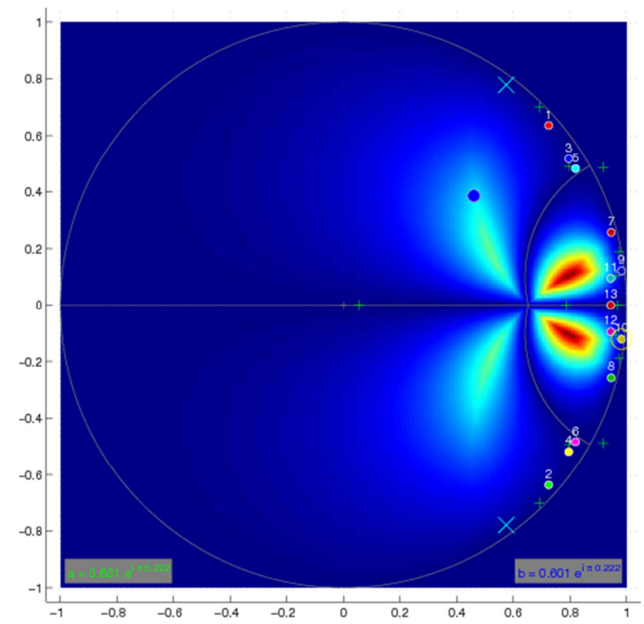

Figure 4: Iteration 1 – Selection of the Laguerre–parameter and the identified pole.

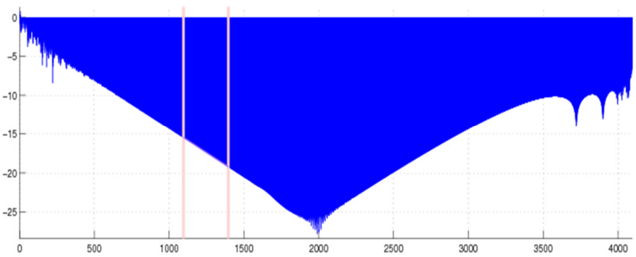

Figure 5: Iteration 1 – Logarithmic modulus of Laguerre coefficients with ROI.

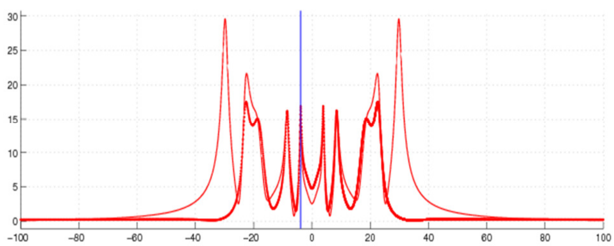

Figure 6: Iteration 1 – Modulus of the frequency domain signal.