Feedback stabilization, a geometric view

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Abstract – *The intention of the paper is to demonstrate the beauty of geometric interpretations in robust control. We emphasize Klein's approach, i.e., the view in which geometry should be defined as the study of transformations and of the objects that transformations leave unchanged, or invariant. We demonstrate through the example of the basic control task of feedback stabilization that a natural framework to formulate different control problems is the projective world that contains as points the equivalence classes determined by the stabilizable plants and whose natural motions are the Möbius transforms. In this context the controller blending problem is placed in a more general setting: an operation is given under which well-posedness is a group while stability is a semigroup. Moreover, an operation is given that makes controllers with strongly stable property a group.*

Keywords: Stabilizability, YOULA parameterization, Controller group, Projective geometry

1. Introduction and motivation

Geometry is one of the richest areas for mathematical exploration. The visual aspects of the subject make exploration and experimentation natural and intuitive. At the same time, the abstractions developed to explain geometric patterns and connections make the approach extremely powerful and applicable to a wide variety of situations.

In the nineteenth century development of the Bolyai-Lobachevsky geometry, as the first instance of non-euclidean geometries, had a great impact on the evolution of mathematical thinking. Non-Euclidean geometry has turned out to be more than just a logical curiosity, and many of its basic features continue to play important roles in several branches of mathematics and its applications.

In many of Euclid's theorems, he moves parts of figures on top of other figures. Felix Klein, in the late 1800s, developed an axiomatic basis for Euclidean geometry that started with the notion of an existing set of transformations and he proposed that geometry should be defined as the study of transformations (symmetries) and of the objects that transformations leave unchanged, or invariant.

This view has come to be known as the Erlanger Program. The set of symmetries of an object has a very nice algebraic structure: they form a group. In [10] the authors emphasize Klein's approach to geometry, i.e., to relate geometric properties to different groups, and demonstrate that a natural framework to formulate specific control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms.

By studying this algebraic structure, we can gain deeper insight into the geometry of the figures under consideration. Another advantage of Klein's approach is that it allows us to relate different geometries. In this paper we put an emphasize on this concept of the geometry and its direct applicability to control problems related to feedback stabilization.

Section 2 gives the basic definition of feedback stability. Then Section 3 presents the Klein view through the projective matrix space. The material is based on [5], [6]. The relevance of this model to control problems is detailed in Section 4. Despite the fact that the stable plants does not form a subgroup of this group, the operation is suitable for controller blending, since preserves stability. Section 5 presents the surprising fact that for strongly stable controllers it is possible to define a group structure (provided that the set is not empty). We conclude the paper by an illustrative example.

2. Basic settings

A central concept of control theory is that of the feedback and the stability of the feedback loop. In this context causality plays a definite role. For practical reasons our basic objects, the systems, i.e., plants and controllers, are causal. As a consequence continuity is formulated as a property of boundedness and causality of the corresponding map. Boundedness here involves some topology. In what follows we consider linear systems, i.e., the signals are elements of some normed linear spaces and an operator means a linear map that acts between signals. Thus, boundedness of the systems is regarded as boundedness in the induced operator norm.

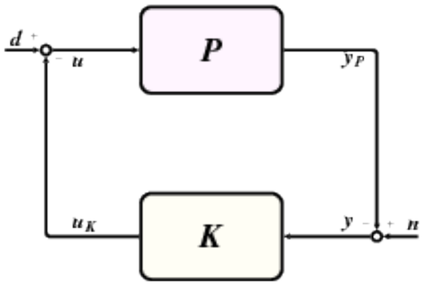

Figure 1: Feedback connection

To fix the ideas let us consider the feedback-connection depicted on Figure 1. Suppose that the signals are elements of the Hilbert space $\mathcal{H}_1, \mathcal{H}_2$ (e.g., $\mathcal{H}_i = \mathcal{L}^{n_i}[0,\infty)$) endowed by a resolution structure defined by a nest algebra which determines the causality concept on these spaces. For more details on nest algebras and causality, see [4].

According to the feedback connection of Figure 1 we can introduce an ambient signal space (the Hilbert space $\mathcal{H}$) with a natural splitting $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. It is convenient to consider the signals

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, p = \begin{pmatrix} u \\ y_P \end{pmatrix},$$

$$k = \begin{pmatrix} u_K \\ y \end{pmatrix}, z = \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{H}.$$

When $y_p = Pu$ and $u_k = Ky$, where P and K are linear operators, then $\mathcal{P}$ is the graph subspace $\begin{pmatrix} u \\ Pu \end{pmatrix}$ of P while $\mathcal{K}^{-1}$ is the inverse graph subspace $\begin{pmatrix} Ky \\ y \end{pmatrix}$ corresponding to K, respectively.

The feedback connection is called well-posed if for every $w \in \mathcal{H}$ there is a unique $p \in \mathcal{P}$ and $k \in \mathcal{K}^{-1}$ such that $w = p + k$ (causal invertibility) and it is called stable if the map $w \to z$ is a bounded causal map.

Basically two questions are related to feedback connections: stabilizability, i.e., whether there is any controller that makes the feedback loop stable and, if it the case, to provide a characterization of the stabilizing controllers.

Unbounded operators on a given space do not form an algebra, nor even a linear space, because each one is defined on its own domain. At this point the association of the operator with a linear space – its graph subspace – turns to be fruitful. Observe that stability of the feedback loops implies that the operator matrix

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

should be stable, i.e., all the block elements are stable. It follows that
$P = -GE^{-1} = -H^{-1}G$ and $K = -FH^{-1} = -E^{-1}F$.

Thus, even the system P is unbounded, through its factorization – we will call it stable factorization – the associated graph is formulated in terms of bounded operators.

It is known that the existence of a double coprime factorization implies feedback stabilizability. In most of the usual model classes actually there is an equivalence. Given a double coprime factorization the set of the stabilizing controllers is provided through the well-known Youla parametrization:

$$\mathcal{K}_{stab} = \{K \mid K = (U + MQ)(V + NQ)^{-1},$$
$$Q \text{ stable}, (V + NQ)^{-1} \text{exists}\}.$$

In what follows we provide a geometric view of this result by showing how elementary considerations leads to an exhaustive characterization of the problem. As a starting point of Euclidean and non-Euclidean worlds the most fundamental geometries are the projective and affine-ones.

Perhaps it is not very surprising that feedback stability is related to such geometries. Following the Kleinian project we have to identify the proper mathematical objects and the groups associated to these objects that are related to the concept of stability and stabilizing controllers.

3. Projective Matrix Space

Projective geometry become a fundamental area of modern mathematics with far reaching applications both in the mathematical theory, as algebraic geometry, and also in different applications fields, such as art, computer vision or even control theory, see, e.g., [3]. For a thorough treatment of the subject the interested reader might consult [2] or [1]. The extension of the projective ideas used in this context is based on [5], [6].

Considering the block operator matrices, we can define the equivalence relation in the following way:
$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \cong \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ if and only if there exists an invertible R such that $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$.

Then the equivalence classes $\underline{P}$ are considered as the points of the projective space $\mathbb{P}$. Introduce the map $\underline{\mathrm{i}}$ such that

$$\underline{P} = \underline{\mathrm{i}}\left\{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\right\} \quad \text{and} \quad \left\{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\right\} = \underline{\mathrm{i}}^{-1}(\underline{P}).$$

Then, $\underline{P}$ is called finite if, for any $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \in \underline{\mathrm{i}}^{-1}(\underline{P})$ the operator P_1 is invertible and $\mathbb{P}_f$ denotes the set of these finite points. Finite points are related to graph subspaces $\begin{pmatrix} I \\ P \end{pmatrix}$ of the linear operators P.

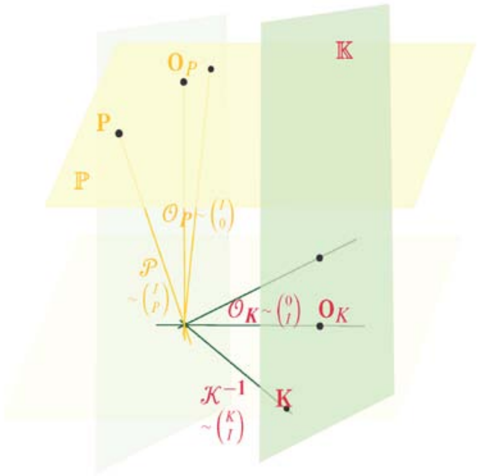

Figure 2: Finite points

The action (projectivity) of invertible block matrices

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

on $\underline{P} \in \mathbb{P}$ are defined through $\underline{P}^S = \underline{\mathrm{i}}(P^S)$ where $P^S = S\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$. The projectivities of $\mathbb{P}$ form a group under composition.

Then the Möbius transformation

$$\mathfrak{M}_S(\underline{P}) = (C + DP)(A + BP)^{-1} \tag{1}$$

is the restriction of the projectivity S to the finite points of the space, i.e., $\mathfrak{M}_S$ operates on $\mathbb{P}_f$.

Analogously one can define the Möbius transformation that corresponds to the inverse graph subspaces as

$$\mathfrak{M}_S(\overline{P}) = (A + BP)(C + DP)^{-1}. \tag{2}$$

Note that while the linear transformation S acts on the geometric, projective level and it is defined everywhere, the Möbius transformation acts on a coordinate level, it is nonlinear (rational) and it is defined only on the domain $A + BP$ invertible.

4. Feedback stabilization

The common tool in formulating robust feedback control problems is to use system interconnections that can be described as linear fractional transforms (LFTs), as a general framework to include the rational dependencies that occur. It is apparent that Möbius transformations are special LFTs.

The set of stabilizing controllers for a given plant and the set of all suboptimal $\mathcal{H}_\infty$ can be expressed by using certain Möbius transformations. In what follows the common background of these transformations and their relations with the hyperbolic geometries will be highlighted.

Even the system P is unbounded, through its coprime factorization $P = NM^{-1}$, with M, N suitable bounded causal operators, the associated graph $\mathcal{P} = \Pi\xi = \begin{pmatrix} M \\ N \end{pmatrix}\xi$ is formulated in terms of bounded with $\xi \in \mathcal{H}_p$ (a suitable Hilbert space) operators.

We are not very restrictive if it is assumed that there exists a double coprime factorization, i.e., $P = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$ and causal bounded $U, V, \widetilde{U}$ and $\widetilde{V}$ such that

$$\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix}\begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad , \tag{3}$$

an assumption which is often made when setting the stabilization problem, see, e.g., [8]. Recall that $\begin{pmatrix} M \\ N \end{pmatrix}$ and $\begin{pmatrix} U \\ V \end{pmatrix}$ are determined only up to outer, i.e., stable with stable inverse, factors S and S'.

It is not hard to make the connection with the construction presented in the previous section: plants P, represented by $\underline{P} = \{\begin{pmatrix} M \\ N \end{pmatrix}\}$, are the finite points $\underline{\mathbb{P}}_f$ while the controllers $K = UV^{-1}$ in a feedback connection are described by the inverse relation (inverse graph), i.e., $\overline{K} = \{\begin{pmatrix} U \\ V \end{pmatrix}\}$.

Well-posedness (feedback stability) of the pair (P, K) means that the matrix $\Xi_P = \begin{pmatrix} M & U \\ N & V \end{pmatrix}$ has a bounded causal inverse. Stable plants Q form a subset of the finite points, i.e., those for which $\begin{pmatrix} I \\ Q \end{pmatrix} \in \underline{\mathrm{i}}^{-1}(\underline{Q})$.

It is obvious that the zero plant is stabilized by the entire stable set, and only by that set. Since the image of $\underline{0}$ under the projectivity defined by Ξ_P is the plant P, it follows that the stabilizing controllers are described by by $\Pi_P\begin{pmatrix} Q \\ I \end{pmatrix} \sim \begin{pmatrix} K \\ I \end{pmatrix}$, i.e.,

$$\mathcal{K}_{stab} = \{K \mid K = U + MQ)(V + NQ)^{-1}, Q \text{ stable}\}$$

which is the well-known Youla parametrization

To emphasize this point we reformulate the standard stability result as follows:

Proposition 4.1 *The plant P has a double coprime factorization if and only if there is a projectivity defined by an outer Ξ with Ξ_{11} invertible such that $\mathfrak{M}_\Xi(P) = 0$.*
Then all stabilizing controllers are given by
$K = \mathfrak{M}_{L^*\Xi^{-1}L}(Q)$, *$Q$ stable , where* $L = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

The proof is straightforward and it is omitted for brevity.

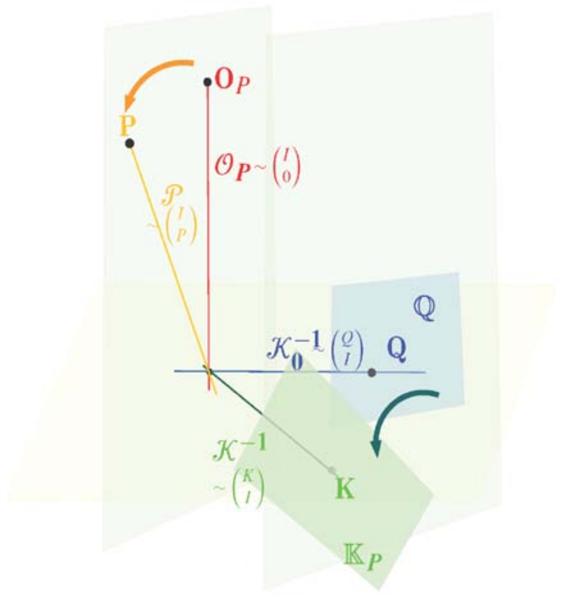

Figure 3: Projectivity vs. controller parametrization

5. Group of stabilizing controllers

In order to design efficient algorithms that operate on the set of controllers that fulfil a given property, e.g., stability or a prescribed norm bound, it is important to have an operation that preserves that property, i.e., a suitable blending method.

Available approaches use the Youla parameters in order to define this operation for stability in a trivial way. As these approaches ignore the well-posedness problem by assuming strictly proper plants, they do not provide a general answer to the problem.

This paper places the blending problem in a more general context: an operation is given under which well-posedness is a group while stability is a semigroup. Moreover, an operation is given that makes controllers with strongly stable property a group. This operation will be used in the next section to construct the sensor group.

Considering the zero plant, i.e., $P = 0_P$, one has that mere addition preserves well-posedness and stability. Moreover, the set of these controllers form a group. In the general case, however, addition of controllers does not ensure well-posedness.

As a remedy, let us start from the observation that

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I - PK \end{pmatrix},$$

which suggests to define the controller blending through

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & I - PK_1 \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - PK_2 \end{pmatrix}.$$

It turns out that on controllers the corresponding operation is

$$K = K_1(I - PK_2) + K_2 = K_1 *_P K_2. \tag{4}$$

Under this operation well-posed controllers form a group $(\Gamma_P, *_P)$.

The unit of this group is the zero controller $K = 0_K$ and the corresponding inverse elements are given by

$$K^{-P} = -K(I - PK)^{-1}.$$

While clearly not all elements of Γ_P are stabilizing, e.g., 0_K is not stabilizing for an unstable plant, $(\Sigma_P, *_P)$ is still a semigroup, where we have denoted by $\Sigma_P \subset \Gamma_P$ the set of stabilizing controllers.

Indeed if $K = K_1 *_P K_2$ with $K_1, K_2 \in \Sigma_P$, then

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & K_1 \\ P & I \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - PK_2 \end{pmatrix}$$

and we have

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -K_2(I - PK_2)^{-1} \\ 0 & (I - PK_2)^{-1} \end{pmatrix} \begin{pmatrix} I & K_1 \\ P & I \end{pmatrix}^{-1}.$$

Thus K is a stabilizing controller.

This semigroup does not have a unit, in general. However, if there is a stabilizing controller K_0 such that

$$K_0^{-P} = -K_0(I - PK_0)^{-1}$$

is also a stabilizing controller, then $(\Sigma_P, \boxdot_P)$ with

$$K_1 \boxdot_P K_2 = K_1 *_P K_0^{-P} *_P K_2$$

is a semigroup with a unit (K_0). This may happen only if the plant is strongly stabilizable.

If a plant is stabilizable in general it is not obvious whether there exists a stable controller as a stabilizing one. If such a controller exists, then we call it a strongly stabilizing controller.

For a general characterization of strong stability one can observe that strongly stabilizing controller exists if and only if there is a stable Q such that $V + NQ$ is unimodular, or, equivalently, if there is a stable K such that

$$-\tilde{N}K + \tilde{M} = \tilde{M}(I - PK)$$

is unimodular. Here we considered the double coprime factorization of P, i.e., $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$.

If we denote by $\Sigma_{P,S}$ the set of strongly stabilising controllers, then if this set is not empty, then:

Theorem 5.1 $(\Sigma_{P,s}, \boxdot_P)$ *with the operation (blending) defined as*

$$K = K_1 \boxdot_P K_2 = K_1 *_P K_0^{-P} *_P K_2 =$$

$$= K_2 + (K_1 - K_0)(I - PK_0)^{-1}(I - PK_2) \qquad (5)$$

is the group of strongly stable controllers, where $K_0 \in \Sigma_{P,s}$ is arbitrary. The corresponding inverse is given by

$$K^{\boxdot_P^{-1}} = K_0 - (K - K_0)(I - PK)^{-1}(I - PK_0). \qquad (6)$$

Observe that the operation (motion) introduced above defines a certain symmetry around the fixed element K_0 of the set $\Sigma_{P,s}$.

We end this paper by illustrating the effect of the group action on a simple configuration, when the plant can be stabilized through state feedback. In this case all the facts derived in this section can be easily verified, see also the formulae listed in the Appendix.

6. Example: state feedback

As an example let us consider the plant

$$P = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

parametrized by its state space description, with two stabilizing (state feedback) controllers given by:

$$K_1 = \begin{bmatrix} 0 & 0 \\ 0 & F_1 \end{bmatrix}, \text{ and } K_2 = \begin{bmatrix} 0 & 0 \\ 0 & F_2 \end{bmatrix},$$

respectively.

It is easy to check that applying operation (4) leads to the dynamic controller

$$K = \begin{bmatrix} A & BF_2 \\ -F_1 & F_1 + F_2 \end{bmatrix}.$$

The corresponding closed loop matrix in the usual basis ($[x \ \ x - x_c]$) is

$$A_{cl} = \begin{pmatrix} A + BF_2 & BF_1 \\ 0 & A + BF_1 \end{pmatrix},$$

i.e., K is a stabilizing controller, as expected.

Taking a stabilizing feedback

$$K_0 = \begin{bmatrix} 0 & 0 \\ 0 & F_0 \end{bmatrix}$$

and computing K_0^{-P} gives

$$K_0^{-P} = \begin{bmatrix} A + BF_0 & BF_0 \\ -F_0 & -F_0 \end{bmatrix}.$$

The corresponding closed loop matrix taking the basis $[x \ \ x + x_c]$ is

$$A_{cl} = \begin{pmatrix} A & BF_0 \\ 0 & A \end{pmatrix},$$

i.e., the closed loop system is not stable, in general.

Computing a controller leads to

$$K = \begin{bmatrix} A + BF_0 & B(F_2 - F_0) \\ -(F_1 - F_0) & F_1 + F_2 - F_0 \end{bmatrix},$$

which is clearly stable.

Note that the degree of the controller (n) is less than the expected one $(2n)$. This is due to the elimination of the unobservable and uncontrollable modes, for details see the Appendix.

The matrix of the closed loop in the usual basis ($[x \ \ x - x_c]$) is

$$A_{cl} = \begin{pmatrix} A + BF_2 & B(F_1 - F_0) \\ 0 & A + BF_1 \end{pmatrix},$$

i.e., K is a stabilizing controller, as expected.

Taking a stabilizing feedback

$$K = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$$

and computing $K^{\boxdot_P^{-1}}$ according to (6) leads to

$$K^{\boxdot_P^{-1}} = \begin{bmatrix} A + BF & B(F - F_0) \\ -(F - F_0) & -F + 2F_0 \end{bmatrix},$$

which is clearly stable.

The corresponding closed loop matrix taking the basis $[x \ \ x + x_c]$ is

$$A_{cl} = \begin{pmatrix} A + BF_0 & -B(F - F_0) \\ 0 & A + BF_0 \end{pmatrix},$$

i.e., the closed loop system is stable, as we have already expected.

7. Conclusions

The paper emphasizes Klein's approach to geometry and demonstrates that a natural framework to formulate different control problems is the projective world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms.

In order to solve the controller blending problem an operation was defined under which well-posedness is a group while stability is a semigroup. Moreover, an operation was given that makes controllers with strongly stable property a group.

The proposed framework provides a common background of robust control design techniques and suggests a unified strategy for problem solutions. Similar techniques can be used to handle robust control problems, too. The corresponding group action in that case is given by the hyperbolic group while the relevant geometry is the non-euclidean geometry.

Besides the educative value a merit of the presentation for control engineers might be a unified view on the robust control problems that reveals the main structure of the problem at hand and give a skeleton for the algorithmic development.

8. References

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9. Appendix

If a state space realization of the plant and of the controller is given, then one can easily derive an expression for the resultant of the operation defined by the product rule.

The computations are based on the following observations: the state space realization for the sum of systems is given by

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} + \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} =$$

$$= \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_1 + D_2 \end{bmatrix},$$

while the product of the systems can be expressed as:

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} =$$

$$= \begin{bmatrix} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ C_1 & D_1 C_2 & D_1 D_2 \end{bmatrix}.$$

Note that these realizations are not necessarily minimal. As an example, for the state feedback case shown in the paper one can eliminate the unobservable/uncontrollable modes and obtain the reduced order expressions.

If the matrix D is invertible then the system is invertible, and a realization of the inverse system is provided by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A - BD^{-1}C & -BD^{-1} \\ D^{-1}C & D^{-1} \end{bmatrix}.$$