# Nonlinear Automotive Actuator Analysis Based on Sum of Squares Programming

Balázs Németh, Péter Gáspár and Tamás Péni

*Abstract*— **The paper analyses the maximum Controlled Invariant Sets of vehicle actuators. In the calculation of the shape of the sets a nonlinear polynomial Sum-of-Squares (SOS) programming method is applied. The aim of the analysis is to identify the similarities and differences between the different actuator interventions and provide a theoretical basis for their coordination. The maximum Controlled Invariant Sets of the steering and the brake control systems are analyzed at various velocities and road conditions. In the analysis the nonlinear characteristics of the lateral tire force are considered with a polynomial approximation. The results of the analysis are illustrated through a simulation example.**

## I. INTRODUCTION AND MOTIVATION

Several active components are applied simultaneously in road vehicles to handle the specified performance requirements. In their simultaneous operation the integration of components must be guaranteed. The purpose of the integrated control is to take into consideration the effects of the control system on other vehicle functions, create coordination between controllers and provide priorities for actuators. In the integrated control there is a possibility to improve safety by modifying the operation of a local controller. If performance degradation or a fault has occurred in the operation of an actuator and it has been detected the degraded actuator is substituted for by another fault-free actuator which provides the same or similar control signal. Moreover, the agility and efficiency of actuators differ and the analysis provides information about the functional reconfiguration possibilities of the actuators.

The analysis and design of integrated vehicle control were in the focus of research and development, see, e.g., [1]. A vehicle control with four-wheel-distributed steering and four-wheel-distributed traction/braking systems was proposed by [2]. A yaw stability control system in which an active torque distribution and differential braking systems are used was proposed by [3]. The integration of differential braking with the front steering was proposed by [4]. An integrated control that involves both four-wheel steering and yaw moment control was proposed by [5]. Active steering and suspension controllers were also integrated to improve yaw and roll stability [6]. A global chassis control involving an active/semi-active suspension and brake was proposed by [7].

The actuator selection is usually performed by using practical considerations, see e.g. [8]. The design of lateral stability control based on set-theoretical methods was proposed by [9]. In another method the uncertain effects of the driver were also considered, see [10]. A control method in which there was a large operating region accessible by the driver and smooth interventions at the stability boundaries was proposed by [11], [12]. A reachability set-based-analysis was applied to the integration of steering and suspension controllers in [13].

In the paper a theoretical basis for the coordination of the actuators is proposed. The stability regions of the maximum control inputs are also calculated. The aim of the analysis is to identify the similarities and differences between the different actuator interventions. Although the reachability set analysis of a linear vehicle model can be a relatively fast and easily applicable technique for actuator intervention limit determination, it has some drawbacks. In the paper a nonlinear polynomial Sum-of-Squares (SOS) programming method is applied to calculate the shape of the maximum Controlled Invariant Sets of actuators.

The paper is organized as follows. In Section II the nonlinear modeling of lateral vehicle dynamics is formulated. The basics of the SOS programming method are detailed in III. The computation method of maximum Controlled Invariant Sets of lateral vehicle model is presented in Section IV. The computation results of the invariant sets at different velocities and adhesion coefficients are presented in Section V. Finally, some concluding remarks are in Section VI.

## II. NONLINEAR MODELING OF LATERAL VEHICLE DYNAMICS

In the paper actuator efficiency is analysed based on a lateral vehicle model. The nonlinear model of lateral vehicle dynamics is formulated in the following form:

$$J\ddot{\psi} = \mathcal{F}_1(\alpha_1)l_1 - \mathcal{F}_2(\alpha_2)l_2 + M_{br} \tag{1a}$$

$$mv\left(\dot{\psi} + \dot{\beta}\right) = \mathcal{F}_1(\alpha_1) + \mathcal{F}_2(\alpha_2) \tag{1b}$$

where $\mathcal{F}_1(\alpha_1)$ and $\mathcal{F}_2(\alpha_2)$ represent lateral tire forces, which depend on tire side-slip angles $\alpha_1$ and $\alpha_2$, moreover, $m$ is the mass of the vehicle, $J$ is yaw-inertia, $l_1$ and $l_2$ are geometric parameters. $\beta$ is side-slip angle of the chassis, $\dot{\psi}$ is yaw-rate. Two controlled systems are compared. In the first system the actuator is the differential braking moment $M_{br}$, while in the

B. Németh and T. Péni are with Systems and Control Laboratory, Institute for Computer Science and Control, Hungarian Academy of Sciences, Kende u. 13-17, H-1111 Budapest, Hungary. E-mail: [bnemeth;peni]@sztaki.mta.hu

P. Gáspár is with the Institute for Computer Science and Control, Hungarian Academy of Sciences, MTA-BME Control Engineering Research Group, Hungary, E-mail: gaspar.peter@sztaki.mta.hu

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second systems the actuator is the front wheel steering angle $\delta$. In the equations the tire side-slip angles are approximated in the following way:

$$\alpha_1 = \delta - \beta - \frac{\dot{\psi} l_1}{v} \tag{2a}$$
$$\alpha_2 = -\beta + \frac{\dot{\psi} l_2}{v} \tag{2b}$$

Since the lateral tire force is a crucial point of the lateral vehicle dynamics, the nonlinearities of the tire characteristics are considered. Several tire models have been published, see e.g., [14], [15], [16]. These models formulate the nonlinearity of longitudinal and lateral tire forces accurately.

In the paper a polynomial tire modeling approach is presented, by which the nonlinearities of the tire characteristics are considered in a given operation range. The nonlinear characteristics of the lateral tire force in the function of tire side-slip $\alpha$ are illustrated in Figure 1. The polynomial approximation is formulated as:

$$\mathcal{F}(\alpha) = \sum_{k=1}^{n} c_k \alpha^k = c_1 \alpha + c_2 \alpha^2 + \ldots + c_n \alpha^n \tag{3}$$

In the proposed method exponent $n$ is chosen 10. Using this approximation the tire model is valid between $\alpha = -12^\circ \ldots + 12^\circ$.

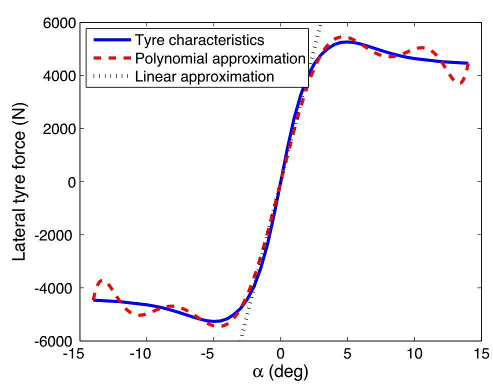


Fig. 1. Modeling of lateral tire force

Note that in several control applications the lateral forces are approximated with linear functions, such as $\mathcal{F}_i(\alpha_i) = c_i \alpha_i$, $i = [1, 2]$, where $c_i$ is cornering stiffness. The advantage of this formulation is the simple description, although the linear tire model can be used only in a narrow tire side-slip range.

In the following (2) is used to transform (1) into a polynomial state-space representation

$$\dot{x} = f(x) + gu,$$

where $x$ is the state vector, $u$ is the control input signal, $f$ and $g$ are vectors.

The yaw-rate and side-slip of the vehicle can be expressed from (2) in the following forms:

$$\dot{\psi} = v \frac{\alpha_2 - \alpha_1 + \delta}{l_1 + l_2} \tag{4a}$$
$$\beta = -\frac{\alpha_1 l_2 + \alpha_2 l_1 - l_2 \delta}{l_1 + l_2} \tag{4b}$$

(1) contains the time-derivatives of $\dot{\psi}$ and $\beta$ and they must be differentiated to obtain $\dot{\beta}$ and $\ddot{\psi}$. Then the vehicle model (1) is reformulated:

$$\dot{\alpha}_2 - \dot{\alpha}_1 = \left[ \frac{l_1 + l_2}{Jv} \left( \mathcal{F}_1(\alpha_1) l_1 - \mathcal{F}_2(\alpha_2) l_2 \right) \right] - \\ - \dot{\delta} + \frac{l_1 + l_2}{Jv} M_{br} \tag{5a}$$
$$\dot{\alpha}_1 l_2 + \dot{\alpha}_2 l_1 = v(\alpha_2 - \alpha_1) - \frac{l_1 + l_2}{mv} \left[ \mathcal{F}_1(\alpha_1) + \mathcal{F}_2(\alpha_2) \right] + \\ + v\delta + l_2 \dot{\delta} \tag{5b}$$

The states of the model are tire slip angles $\alpha_1$ and $\alpha_2$, i.e., $\dot{x} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}^T$. Thus, the nonlinearity of the lateral tire forces $\mathcal{F}_1$, $\mathcal{F}_2$ can be considered in the state equation. However, (5) includes the time-derivative of the front-wheel steering angle. Since $\delta$ is a control input, $\dot{\delta}$ is modeled as below:

$$\dot{\delta} \cong \max \left( \frac{|\dot{\delta}|}{|\delta|} \right) \cdot \delta = \nu \cdot \delta \tag{6}$$

where parameter $\nu$ represents the relationship between the maximum steering value and the variation speed of $\delta$. Since $\max \delta$ is a given fixed limit at the actuator analysis, high $\nu$ value represents a fast changing steering signal, while a slow changing steering signal is modeled with low $\nu$.

The polynomial state-space representation of the system is formulated using (5) and the substitution of (6) is as below:

$$\dot{x} = \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{f}_1(\alpha_1, \alpha_2) \\ \mathrm{f}_2(\alpha_1, \alpha_2) \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} M_{br} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \delta \tag{7}$$

where

$$\mathrm{f}_1 = \frac{l_1}{Jv} \left[ \mathcal{F}_2(\alpha_2) l_2 - \mathcal{F}_1(\alpha_1) l_1 \right] + \\ + \frac{v}{l_1 + l_2} (\alpha_2 - \alpha_1) - \frac{1}{mv} \left[ \mathcal{F}_1(\alpha_1) + \mathcal{F}_2(\alpha_2) \right],$$
$$\mathrm{f}_2 = \frac{l_2}{Jv} \left[ \mathcal{F}_1(\alpha_1) l_1 - \mathcal{F}_2(\alpha_2) l_2 \right] + \\ + \frac{v}{l_1 + l_2} (\alpha_2 - \alpha_1) - \frac{1}{mv} \left[ \mathcal{F}_1(\alpha_1) + \mathcal{F}_2(\alpha_2) \right],$$

and

$$h_1 = \frac{v}{l_1 + l_2} + \nu,$$
$$h_2 = \frac{v}{l_1 + l_2},$$
$$g_1 = -\frac{l_1}{Jv},$$
$$g_2 = \frac{l_2}{Jv}.$$

In the proposed vehicle model (7) either differential braking $M_{br}$ is applied, when $\delta = 0$, or steering angle $\delta$ is applied, when $M_{br} = 0$.

## III. FUNDAMENTALS OF SOS PROGRAMMING TECHNIQUE

In this section the concepts concerning the SOS programming method are summarized. The method is suitable to analyze and control nonlinear polynomial systems. In the following a brief survey is given before the method used for vehicle model is proposed.

Important theorems in SOS programming, such as the application of Positivstellensatz, were proposed in [17]. In this way the convex optimization methods can be used to find appropriate polynomials of the SOS problem. The approximation of nonnegative polynomial by a sequence of SOS was presented in [18]. The SOS polynomials incorporate the original nonnegative polynomials in an explicit form.

In terms of state dependent Linear Matrix Inequalities (LMIs) sufficient conditions for the solutions to nonlinear control problems were shown by [19]. In the paper the semidefinite programming relaxations based on the SOS decomposition were then used to efficiently solve such inequalities. The application of the SOS decomposition technique to non-polynomial system analysis was summarized in [20]. The application of SOS programming to several control problems was presented by [21], e.g. reachability set computation and control design algorithm. A local stability analysis of polynomial systems and an iterative computation method for their region of attraction were presented in [22]. The SOS method was applied to two non-convex problems, for example polynomial semi-definite programming and the fixed-order $\mathcal{H}_2$ synthesis problem, see [23].

The performance analysis of polynomial systems is published in [24], by which sufficient conditions were provided for bounds on the reachability set and $\mathcal{L}_2$ gain of the nonlinear system subject to norm-bounded disturbance inputs. Robust performance in polynomial control systems was analyzed in [25] and [26]. This paper considered the effects of neglected dynamics and parametric uncertainties. Numerical computation problems of convex programming based SOS method in practical applications were analyzed in [27]. As a new result the maximum controlled invariant sets of polynomial control systems were calculated in [28].

The following definitions and theorems are essential to understand SOS programming [21]. The basic elements of the method are polynomials and SOS as defined below:

*Definition 1:* *A **Polynomial** $f$ in $n$ variables is a finite linear combination of the functions $m_\alpha(x) := x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}_+^n$, $\deg m_\alpha = \sum_{i=1}^n \alpha_i$:*

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha \tag{8}$$

*with $c_\alpha \in \mathbb{R}$. Define $\mathbb{R}$ to be the set of all polynomials in $n$ variables. The degree of $f$ is defined as $f := \max_\alpha \deg m_\alpha$.*

*Definition 2:* *The **set of SOS** polynomials in $n$ variables is defined as:*

$$\Sigma_n := \left\{ p \in \mathcal{R}_n \;\middle|\; p = \sum_{i=1}^t f_i^2,\ f_i \in \mathcal{R}_n,\ i = 1, \ldots, t \right\} \tag{9}$$

*where $\mathcal{R}_n$ is the ring of polynomials in $n$ variables and of unspecified degree.*

A central theorem of SOS programming is Positivstellensatz. By the application of this theorem the set emptiness constraints of an optimization task can be transformed to SOS feasibility problems.

*Theorem 1:* ***Positivstellensatz:*** *Given polynomials $\{f_1, \ldots, f_r\}$, $\{g_1, \ldots, g_t\}$ and $\{h_1, \ldots, h_u\}$ in $\mathcal{R}_n$, the following are equivalent:*

1) *The set*

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} f_1(x) \geq 0, \ldots, f_r(x) \geq 0 \\ g_1(x) \neq 0, \ldots, g_t(x) \neq 0 \\ h_1(x) = 0, \ldots, h_u(x) = 0 \end{array} \right\} \tag{10}$$

*is empty.*

2) *There exists polynomials $f \in \mathcal{P}(f_1, \ldots, f_r)$ ($\mathcal{P}$ is a cone), $g \in \mathcal{M}(g_1, \ldots, g_t)$ ($\mathcal{M}$ is a multiplicative monoid), $h \in \mathcal{I}(h_1, \ldots, h_u)$ ($\mathcal{I}$ is an ideal) such that*

$$f + g^2 + h = 0 \tag{11}$$

There is an important connection between SOS programming and LMI problems, which was proved by [17]:

*Theorem 2:* *Given a finite set $\{p_i\}_{i=0}^m \in \mathcal{R}_n$, the existence of $\{a_i\}_{i=0}^m \in \mathbb{R}_n$ such that*

$$p_0 + \sum_{i=1}^m a_i p_i \in \Sigma_n \tag{12}$$

*is an LMI feasibility problem.*

The previous two theorems can be used to prove the generalization of the S-Procedure, which is highly significant in the forthcoming computations.

*Theorem 3:* ***Generalized S-Procedure:*** *Given symmetric matrices $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. If there exist nonnegative scalars $\{s_i\}_{i=1}^m \in \Sigma_n$ such that*

$$p_0 - \sum_{i=1}^m s_i p_i \succeq q \tag{13}$$

*with $q \in \Sigma_n$, then*

$$\bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, p_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \,|\, p_0(x) \geq 0\} \tag{14}$$

*The related set emptiness question asks if*

$$\begin{aligned} W := \{x \in \mathbb{R}^n \,|\, p_1(x) \geq 0, \ldots, p_m(x) \geq 0, \\ -p_0(x) \geq 0, p_0(x) \neq 0\} \end{aligned} \tag{15}$$

*is empty.*

## IV. COMPUTATION METHOD OF CONTROLLED INVARIANT SETS

In this section the Controlled Invariant Sets of the system [28] are computed based on the theoretical preliminaries. In the following the computation of Controlled Invariant Sets is proposed.

## A. Theoretical background

The state-space representation of the system is given in the following form (see (7)):

$$\dot{x} = f(x) + gu \tag{16}$$

where $f(x)$ is a vector, which incorporates smooth polynomial functions and $f(0) = 0$. In the next analysis one control input is considered, thus either $u = M_{br}$ or $u = \delta$. The global asymptotical stability of the system at the origin is guaranteed by the existence of the Control Lyapunov Function of the system defined as follows [29]:

*Definition 3: A smooth, proper and positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ is a Control Lyapunov Function for system (16) if*

$$\inf_{u\in\mathbb{R}} \left\{ \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g \cdot u \right\} < 0 \tag{17}$$

*for each $x \neq 0$.*

According to Definition 3 two main cases are distinguished:

1) If $\frac{\partial V}{\partial x} f(x) < 0$ then the system is stable and $u \equiv 0$. This stability scenario is contained by the next two stability criteria.
2) If $\frac{\partial V}{\partial x} f(x) > 0$ then the system is unstable. However, the system can be stabilized
   a) If
   $$\frac{\partial V}{\partial x} g < 0$$
   and
   $$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g \cdot u_{max} < 0,$$
   the upper peak-bound of control input $u$ stabilizes the system.
   b) If
   $$\frac{\partial V}{\partial x} g > 0$$
   and
   $$\frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} g \cdot u_{max} < 0,$$
   the lower peak-bound of control input $u$ stabilizes the system. Note that $u_{min} = -u_{max}$.

The Controlled Invariant Set of the system (16) is defined as the level-set of the Control Lyapunov Function at $V(x) = 1$. Thus, the fulfilment of the previous stability criterion must be guaranteed at $V(x) \leq 1$.

Above the stability criterion of the polynomial system has been formed. Based on these constraints it is necessary to find a Control Lyapunov Function $V$, which meets the following set emptiness conditions:

$$\left\{ \frac{\partial V}{\partial x} g < 0, \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g \cdot u_{max} > 0, V(x) \leq 1 \right\} = \emptyset \tag{18a}$$

$$\left\{ \frac{\partial V}{\partial x} g > 0, \frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} g \cdot u_{max} > 0, V(x) \leq 1 \right\} = \emptyset \tag{18b}$$

Note that the relations in the second inequalities are inverted to guarantee the emptiness of the sets. Since it is necessary to find the maximum Controlled Invariant Set, another set emptiness condition is also defined to improve the efficiency of the method [21]:

$$\{p(x) \leq \beta, V(x) \geq 1, V(x) \neq 1\} = \emptyset \tag{19}$$

where $p \in \Sigma_n$ is a fixed and positive definite function. $\beta$ defines a $P_\beta := \{x \in \mathbb{R}^n | \, p(x) \leq \beta\}$ level set, which is incorporated in the actual Controlled Invariant Set. Thus, the maximization of $\beta$ enlarges $P_\beta$ together with the Controlled Invariant Set.

The set emptiness conditions are reformulated to SOS conditions based on the S-procedure (see Section III). Thus, the next optimization problem is formed to find the maximum Controlled Invariant Set:

$$\max \beta \tag{20}$$

over $\quad s_1, s_2, s_3, s_4, s_5 \in \Sigma_n; \; V \in \mathcal{R}_n; \;$ V(0)=0
such that

$$-\left( \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g \cdot u_{max} \right) - s_1 \left( -\frac{\partial V}{\partial x} g \right) - \\ - s_2 (1 - V) \in \Sigma_n \tag{21a}$$

$$-\left( \frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} g \cdot u_{max} \right) - s_3 \left( \frac{\partial V}{\partial x} g \right) - \\ - s_4 (1 - V) \in \Sigma_n \tag{21b}$$

$$-\left(s_5(\beta - p) + (V - 1)\right) \in \Sigma_n \tag{21c}$$

## B. Practical computation of Controlled Invariant Sets

The optimization method of the maximum Controlled Invariant Set has been proposed in the previous parts of the section. Although (21) provides an appropriate solution to the optimization problem, it results in numerical difficulties. Note that the degree of $f(x)$ is determined by the degree of the lateral tire model, see (3).

Simultaneously $\deg V = n$, which resulted in $\deg \frac{\partial V}{\partial x} f(x) = 2n - 1$. [17] proposes that a polynomial in $n$ variables of degree $2d$ can be transformed into an LMI with $\binom{n+d}{d}$ dimensions. In the presented example the degree of tire model is $n = 10$, therefore the maximum number of degrees in (21) is $2n - 1 = 19$. The system has two variables: $\alpha_1$ and $\alpha_2$, which leads to $\binom{12}{2} = 66$ dimensions LMI. Because of the huge size of the LMI feasibility task, numerical problems may occur. Therefore the resulting Control Lyapunov Function $V$ of optimization (21) must be checked.

In the following an alternative computation method is proposed to find the maximum Controlled Invariant Set, which, in our experience, can lead to an easier calculation according. A three-step iterative method is proposed in the paper.

*Step 1*: The region of attraction of the uncontrolled system $\dot{x} = f(x)$ is determined as an initial set. In this step the maximum level set of $V_0 = 1$ is found, which is incorporated in the stable region. The SOS based computation of the region of attraction is presented in [30].

*Step 2*: An $\eta$ parameter is chosen and

$$V_\eta = V_0 \cdot \eta$$

is checked as a Local Control Lyapunov Function. The level-set $V_\eta = 1$ represents a Controlled Invariant Set $S_\eta$, in which the system can be stabilized using a finite control input $u$. Depending on parameter $\eta$ the size of the level-set can be enlarged or reduced. The SOS based computation of Local Control Lyapunov Function is proposed in [22].

*Step 3*: In the final step the acceptability and the enlarging possibility of $S_\eta$ Controlled Invariant Set must be checked. The peak-bounds of the actuation are $u_{min} = -u_{max}$ and $u_{max}$. $S_{inst} = \frac{\partial V}{\partial x} f(x) > 0$ is the unstable region of the system.

- $S_{min} = \frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} g \cdot u_{max} > 0$ is the region, which can not be stabilized by $u_{min}$.
- Similarly, $S_{max} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g \cdot u_{max} > 0$ is the region, which can not be stabilized by $u_{max}$.

If $S_\eta$ is an appropriate Controlled Invariant Set and $V_\eta$ is an appropriate Control Lyapunov Function, then

$$S_\eta \bigcap S_{inst} \bigcap S_{min} \bigcap S_{\max} = \emptyset \qquad (22)$$

The emptiness of the intersection condition defined below can be checked manually by the plot of $S_\eta$, $S_{inst}$, $S_{min}$ and $S_{\max}$. Additionally, if $S_\eta$ is appropriate then $\eta$ value can be reduced in the previous step to maximize the Controlled Invariant Set.

## V. Controlled Invariant Sets of vehicle dynamic actuators

In the following the results of the maximum Controlled Invariant Set analysis are illustrated. The actuators of the system are differential braking and front wheel steering. Two steering scenarios, such as fast intervention $\nu = 30$ and slow actuation $\nu = 1$, are also compared.

The vehicle in the analysis is a medium-size passenger car and the vehicle data and tire characteristics are derived from the CarSim vehicle dynamic software (E-Class vehicle). The lateral tire force characteristics are approximated with tenth-order polynomials using a least-squares algorithm. Since the road conditions determine vehicle dynamics, two tire-road adhesion coefficients are investigated during the analysis. $\mu = 0.8$ represents a dry, while $\mu = 0.6$ is related to a wet concrete road. The maximum actuation capabilities of each system are $|\delta_{max}| = 12^\circ$ at steering, and $|M_{br,max}| = \mu \cdot 15000 Nm$ at differential braking. Note that the consideration of $\mu$ in the braking limit is necessary, because the tire-road adhesion also determines the longitudinal dynamics.

Figure 2 presents the results of the analysis $\mu = 0.8$. At the computation of the maximum Controlled Invariant Set the following velocities are considered: $v = 36\ km/h$, $v = 72\ km/h$, $v = 108\ km/h$ and $v = 144\ km/h$. The results show that the regions are very different where the vehicle can be stabilized by the limited control inputs of the systems. The sizes of the regions depend significantly on the velocity and the steering dynamics $\nu$. The sets $\delta$, $\nu = 1$ are larger at high velocities, than the sets of $\delta$, $\nu = 30$. It means that the dynamics of steering actuation determines the regions where the vehicle can be stabilized. At fast steering intervention the sets decrease, thus fast transitions must be avoided during the actuation.

Comparing the maximum Controlled Invariant Sets of braking and steering it can be shown that $M_{br}$ actuation is especially beneficial instead of fast steering. Since there are vehicle dynamic situations where the fast intervention is unavoidable, the sudden steering maneuver can be substituted for by braking. In this case vehicle dynamics can be improved in an enlarged region. Note that the differences have significance mainly at high velocities, e.g. at $v = 36\ km/h$ the maximum Controlled Invariant Sets of the actuators are the same.

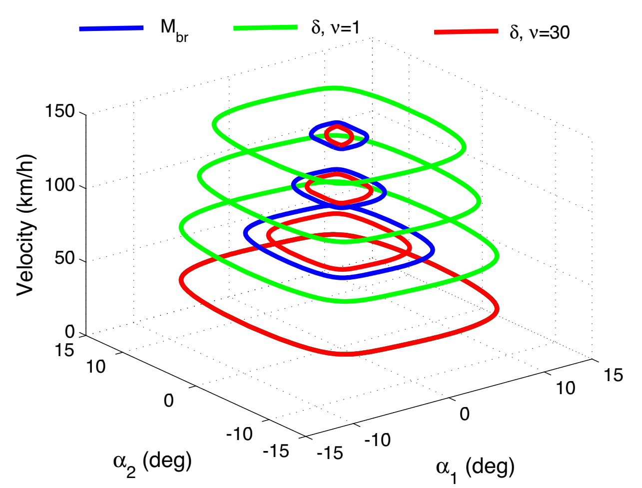


Fig. 2. Actuation limits $\mu = 0.8$

The results of the set-based analysis at $\mu = 0.6$ are illustrated in Figure 3. Comparing the sets with the previous scenario, the reduction of the sets can be observed. Since the decreased adhesion coefficient induces smaller peak values of the lateral forces, the the instable regions $S_{inst}$ are enlarged, while the maximum Controlled Invariant Sets are reduced. Moreover, at braking the actuation limit is also lower than at $\mu = 0.8$. However, the tendency in $v$ and $\nu$ dependence is the same: high velocity and fast steering intervention result in smaller regions.

The proposed analysis shows the importance of the system reconfiguration during an appropriate actuator selection. The results meet the preliminary expectations and give a theoretical basis for the control design and the actuator selection procedure.

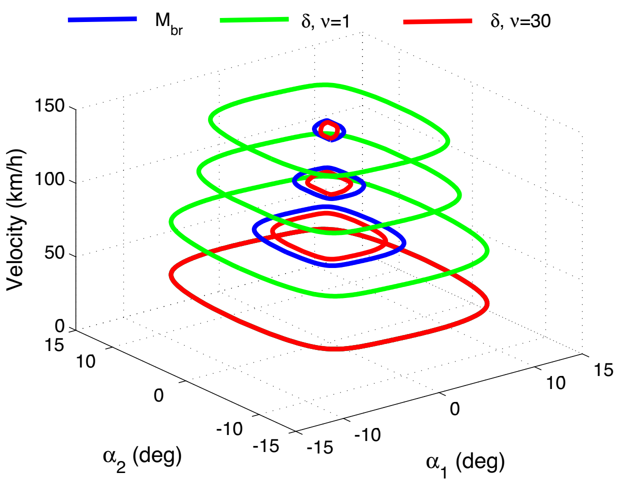


Fig. 3. Actuation limits $\mu = 0.6$

## VI. Conclusion

In the paper the maximum Controlled Invariant Sets of steering and braking systems have been examined in order to analyse their abilities for the entire vehicle system. A nonlinear polynomial SOS programming method has been applied to calculate the shape of the maximum Controlled Invariant Sets of actuators. The aim of the analysis is to provide a theoretical basis for the coordination of the actuators. The method has been illustrated through the influence of the steering and the brake control systems at various velocities and road conditions.

The analysis has shown that different vehicle dynamic regions can be reached and stabilized by differential braking and steering. The results of the presented computation can be used as a part of an actuator selection strategy.

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