Constructions for Nontransitive Dice Sets

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Abstract: Nontransitivity can be observed in team tournaments, voting paradoxes or independent discrete random variables. All representations are based on the same principle that majority rule may result in nontransitive relations. The aim of the paper is to realize any tournament by a dice set, and the number of faces is to be minimized. Additional properties of tournaments, e.g., rotational symmetry, make it possible to reduce the number of faces. As a special case, the Paley tournament on $p = 8k + 7$ vertices can be realized by $(p - 1)/2$ faces. The paper is closed by open questions.

Keywords: nontransitive dice, digraph, Paley tournament

1 Introduction

A nontransitive dice set is a finite set of independent discrete random variables with a finite set of values such that $prob(X > Y) > 1/2$ and $prob(Y > Z) > 1/2$ do not exclude $prob(Z > X) > 1/2$ [21, p. 69]. If for any die X in the set there exists a die Y such that $prob(Y > X) > 1/2$, then an unfair game for two players can be observed: whatever die the first player chooses, the second player will find a die in the remaining set which wins against the first player's die with probability greater than 1/2. Further dice sets are given in [9, 18, 19].

Team tournaments can also be nontransitive. Let each team have a finite number of players, each one's strength is expressed by a real number. The comparison of two teams is determined by matching each player of one team to each player of the other team. The team which has a higher number of individual wins (ties are not allowed) is said to beat the other team. In an example of Moon and Moser [15], team $\{1, 5, 9\}$ beats $\{3, 4, 8\}$, $\{3, 4, 8\}$ beats $\{2, 6, 7\}$ and $\{2, 6, 7\}$ beats $\{1, 5, 9\}$. Teams can be interpreted as dice, with equal probabilities of all their three faces.

Beating relations can be represented by a directed graph. Vertices are associated to dice and a directed edge goes from vertex i to vertex j if and only if die i beats die j. Given a tournament on n vertices, we seek for a dice set that realizes it. Let $f(n)$ denote the minimal number of faces that are necessary to realize the tournament. A construction is organized in a table as follows:

	die 1	die 2	$\dots$	die n
face 1			$\dots$	
face 2			$\dots$	
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
face m			$\dots$	

It should be noted that constructions in the literature use the assumption that all values in row (i.e., on face) i are larger than all values in row (on face) j if $i > j$. Then it is sufficient to adjust the relation between dice restricted to their faces of identical indices. The example of Moon and Moser [15] can be written in its original form (left), as well as without additive terms between rows (right) as follow:

	die 1	die 2	die 3
face 1	1	2	3
face 2	5	6	4
face 3	9	7	8

	die 1	die 2	die 3
face 1	1	2	3
face 2	2	3	1
face 3	3	1	2

Note that the table written without additive terms is identical to the one of Condorcet's paradox in voting theory [4, 22]. McGarvey [10] showed that $f(n) \leq n(n-1)$, his solution is used as a starting point (Construction 1) in our list. Stearns [20] and Erdős and Moser [8] proved that

$$c\frac{n}{\log n} < f(n) < C\frac{n}{\log n}$$

with constants c, C.

If a dice set has the property that whatever the first $k-1$ players choose, the k-th player will find a die that beats all dice of the $k-1$ players, then we speak of a nontransitive dice set for k players. van Deventer [18] proposed a dice set for three players. The Schütte property for a given k, denoted by S_k, is defined for a tournament as follows: for any choice of k vertices there exists a vertex from which directed edges go to the k ones [7, 1].

Let $p = 4m + 3$ be a prime number. The Paley tournament [17], denoted by P_p, is a simple complete directed graph of p vertices in which a directed edge goes from vertex i to vertex j if and only if $j - i$ is a quadratic residue modulo p.

The second author recently proposed a dice set that realizes P_p [2]. Each die has $p(p-1)/2$ faces with equal probability. It is known from the theorem of Graham and Spencer [13] that if $p > (k-1)^2 2^{2k-4}$, then P_p fulfills S_{k-1}, therefore, the dice set is appropriate for k players.

However, the number of dice, as well as the number of faces, which are necessary and sufficient to get a nontransitive dice set for k players is known for small values of k only [1, p. 360].

Our contribution is a list of dice set constructions. Each one realizes a given tournament, each vertex is associated to a die, and each directed edge corresponds to the binary relation between two dice 'greater with probability greater than 1/2'. The number of faces of the dice is a key factor. Constructions are listed in didactic order, starting with simpler ones but having more faces and ending with more sophisticated ones having fewer faces. We report some constructions which are not better than the best known ones, however, they might be of interest in order to use them for further sets.

In subsection 2.1 a dice set is constructed for an arbitrary tournament on n vertices, each die has $n(n-1)$ faces, followed by another dice set with $2n$ faces and by two dice sets, one for an arbitrary tournament on $2n$ vertices, each die has $4n-4$ faces and another one on $2n+1$ vertices, each die has $4n+2$ faces (being different from the previous one). Subsection 2.2 deals with the addition of a new die to a given dice set and discusses the number of new faces to be added. It is proven that any tournament on n vertices can be realized by a dice set, with at most $\lfloor \frac{6}{5}n \rfloor$ faces per die.

We focus on special tournament, too. A tournament is called to have rotational symmetry if the direction of edge $i \to j$ implies the direction of edge $i + k \to j + k$ for all values of $k = 1, 2, \ldots, n$ (if $i + k > n$, then $i + k - n$ is considered). Any tournament on n vertices with rotational symmetry can be realized by n faces (subsection 2.3). The Paley tournaments are discussed in two parts (subsection 2.4). For $p = 8k + 7$, a dice set having $(p-1)/2$ faces per die is written and shown to realize P_p. We are still working on a general construction for $p = 8k + 3$ also with $\frac{p-1}{2}$ faces, however, we have results for values 11, 19 and 59 at the moment. Section 3 concludes and summarizes possible extensions including ties and open questions.

2 Results

Given an arbitrary tournament on n vertices, the aim is to find a dice set that realizes it. Henceforward the assumption that all tables contain additive terms is kept, but we do not write them.

2.1 Dice sets with $n(n-1)$, $2n$ and $2n-2$ faces

Let us have a tournament on n vertices to be realized.

Construction 1 *For every edge of the graph we make two faces in the following way: for the $i \to j$ edge, the i-th die has two 1's, the j-th has two 0's, the k-th ($k \neq i, j$) has $-k$ and $k+1$. Each die has $n(n-1)$ faces.*

Construction 1 is given by McGarvey [10] and Bumbacea [3, 11, 12] recently followed the same idea.

Construction 2 *For every die we make two faces. The i-th die has two 0's. If die i beats die j, then $-j$ and $-n-1+j$ are written on two faces of die j. If die j beats die i, then j and $-n-j$ are written on two faces of die j. Each die has $2n$ faces.*

Construction 2 is a less efficient variant of the one proposed by Stearn [20], which realizes any tournament on n vertices with $n+2$ faces if n is even and with $n+1$ faces if n is odd.

Construction 3 *We make a round robin tournament (n rounds if n is odd and $n-1$ rounds if n is even). For every round we make two faces: in every round the winner of the i-th gets i and $-i+1$, the other die gets $i-1$ and $-i$. Each die has $2n$ faces if n is odd and $2n-2$ faces if n is even.*

Proposition 1 *Constructions 1, 2 and 3 realize the tournament.*

PROOF:

1. The result between die i and j is $1:1$ in every pair except the pair belonging to the $i \to j$ edge, where the right die wins to $2:0$.
2. The result between die i and j is $1:1$ in every pair except the pair belonging to the die i if die i beats die j or the pair belonging to the die j if die j beats die i. In this pair the right die wins to $2:0$.
3. The result between die i and j is $1:1$ in every round except in which i and j is matched. In this round the right one wins to $2:0$.

$\square$

2.2 Addition of new dice

In this subsection the idea of building a tournament, vertex by vertex, is followed.

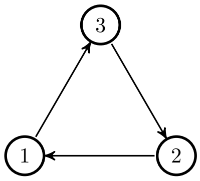
Proposition 2 *Let us have a dice set realization of a tournament on n vertices that uses $2k+1$ faces. If we add a vertex (and n directed edges) to the tournament, then the new tournament can be realized with $2k+3$ faces.*

Construction 4 *Let M and m denote the maximal and minimal values that are written on the faces of the first n dice. Write $M+1$ on k faces and $m-1$ on $k+1$ faces of the $(n+1)$-th die. Let the $(n+1)$-th die have two 0's on its $(2k+2)$-th and the $(2k+3)$-th faces and let the i-th die ($1 \leq i \leq n$) have $n+i$ and $-i$ if it is beaten by the $(n+1)$-th one, and i, $n-i$ otherwise.*

PROOF: Construction 4 realizes the augmented tournament: since the result between two original dice on the new faces is $1:1$, the result in the new construction is not changed. The $(n+1)$-th die wins against the others on the first $(2k+1)$ faces to $k+1:k$, and on the new faces the result is $1:1$ or the original die wins to $2:0$. $\square$

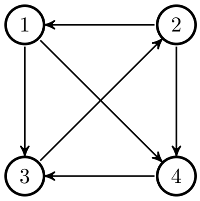
Lemma 3 *Every tournament on five or less vertices can be realized with 3 faces.*

PROOF: If the graph is transitive then one face is enough. On three vertices there is only one nontransitive tournament which is realized by following dice set:



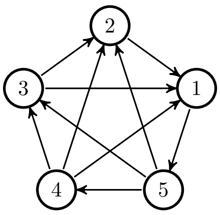
die 1	die 2	die 3
1	2	3
2	3	1
3	1	2

The reader may identify Condorcet's paradox again. There is only one nontransitive dice set on 4 vertices in which there is no vertex which beats or which is beaten by all other vertices. This tournament is realized by the following dice set:

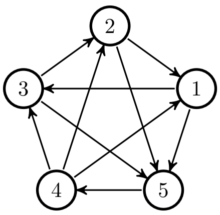


die 1	die 2	die 3	die 4
2	3	4	1
3	1	2	4
3	4	1	2

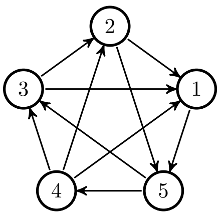
There are five nontransitive dice sets on 5 vertices in which there is no vertex which beats or which is beaten by all other vertices. These tournaments are realized by the following dice sets:



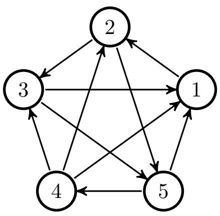
die 1	die 2	die 3	die 4	die 5
1	2	3	4	5
4	2	1	5	3
3	4	5	1	2



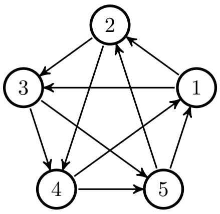
die 1	die 2	die 3	die 4	die 5
3	1	2	4	5
3	4	2	5	1
3	4	5	1	2



die 1	die 2	die 3	die 4	die 5
2	1	3	4	5
2	3	5	4	1
4	5	1	2	3



die 1	die 2	die 3	die 4	die 5
1	4	3	5	2
4	3	5	1	2
3	2	1	4	5



die 1	die 2	die 3	die 4	die 5
2	4	1	3	5
3	1	5	4	2
5	4	3	1	2

□

Corollary 4 *A tournament on $n \geq 5$ vertices can be realized with $2n - 7$ faces.*

Lemma 5 *Consider the following dice set with 5 dice and 3 faces:*

die 1	*die 2*	*die 3*	*die 4*	*die 5*
1	*2*	*3*	*4*	*5*
4	*5*	*2*	*3*	*1*
5	*1*	*4*	*2*	*3*

For every subset of these five dice there exists a die which beats exactly this subset.

Proposition 6 *Let us have a dice set realization of a tournament on n vertices that uses $2k + 1$ faces ($k \geq 1$). If we add 5 vertices (and the related edges) to the tournament, then the new tournament can be realized with $2k + 7$ faces.*

Construction 5 *Let M denote the maximal absolute values of the numbers written on the faces of the first n dice and let $M := M + 1$. On the first three new faces of the five new dice there are the five dice from Lemma 5 (A). On the other three new faces there is $-A$ and the 'large number' M is added elementwise, denoted by $-A + M$. On three of the original faces there is the 3-face construction which realizes the subtournament on the new vertices minus (elementwise) M, together denoted by $C - M$. On the half of the rest of the original faces there is an arbitrary construction plus (elementwise) M, $(D+M)$. On the other half there is $-D - M$. On the first three new faces of the original dice there is the right vector from Lemma 5 (B). On the other three faces there is $-B$.*

Original dice	*New 5 dice*
Original Construction	$-D - M$
	$D + M$
	$C - M$
B	A
$-B$	$-A + M$

Proof: Construction 5 realizes the augmented tournament. Since the result between two original dice on the new faces is 3 : 3, the result in construction 5 remains the same as in the original tournament. The result between two new faces is the same as in the construction C. The result between an original and a new one is the same as in the first three new faces. □

Corollary 7 *A tournament on n vertices can be realized with $\lfloor \frac{6}{5} n \rfloor$ faces.*

2.3 Tournament with rotational symmetry

Construction 6 *Consider a tournament with rotational symmetry. In this construction every die has n faces. On the i-th face the i-th die has a 0, on the i-th face of die k there is l if die k beats die i and $-l$ if die i beats die k where $i + l \equiv k \pmod{n}, 1 \leq l \leq n - 1$.*

Proposition 8 *Construction 6 realizes the tournament with rotational symmetry.*

Lemma 9 *If we take the sign in the construction 6, this new construction realizes the tournament with rotational symmetry.*

Proof: (of Lemma 9) We compare dice i and j. Since the number on the k-th face of die i is equal to the number written on $(k+j-i)$-th face of die j, we can make one or more cycles in the following way: take the first face of die i, then the first face of die $j(=i+j-i)$, then the first face of die $i+2(j-i)$ and so on. The number of die i's wins against die j is equal to the wins when a vertex of this cycle beats the next one. Suppose that die i beats die j. Along the cycle there are one $1:0$ change, one $0:-1$ change and one more $-1:1$ changes than $1:-1$. It means that die i beats die j in this construction. □

Proof: (of Proposition 8) We make the cycles similarly to the ones in Lemma 9. We need that along the cycle the result between two positive and between two negative numbers is draw. This is true because if there is a change $k:l$ between two positive numbers, then there is also an $l-n:k-n$ change between two negative numbers and vice versa. □

2.4 Paley tournament

Construction 7 *Consider tournament P_p for $p=8k+7$. It is known that the multiplicative group of nonzero quadratic residues is cyclic. Let q be a generator of this group. We give a construction with $\frac{p-1}{2}$ faces. On every faces of die 0 there are $\frac{p}{2}$'s. On the first face of die i there is $k \equiv \frac{p+1}{2} - i \pmod{p}$ $(1 \le k \le p)$. The j-th face of the i-th die is equal to the number written on the first face of the die $iq^{j-1} \pmod{p}$.*

Proposition 10 *Construction 7 realizes P_p.*

Lemma 11 *If $p=8k+7$, then the number of quadratic nonresidues minus the quadratic residues less then $\frac{p}{2}$ is equal to the sum of quadratic residues minus the sum of quadratic nonresidues divided by p.*

Proof: (of Lemma 11) Let d_p denote this difference. Since 2 is a quadratic residue [16, Problem 10 on p. 67], the sum of the quadratic residues divided by p is equal to number of quadratic nonresidues between 1 and $\frac{p}{2}$ and the sum of the quadratic nonresidues divided by p is equal to the number of quadratic residues between 1 and $\frac{p}{2}$. The equation in the lemma is the difference of these two equations. □

Proof: (of Proposition 10) We make the proof in three parts.

- Between two quadratic residues (or quadratic nonresidues): Suppose that i and j are quadratic residues and $j-i$ is also a quadratic residue. Note that the difference between the value on die j and i is a quadratic nonresidue on each face, and we get every nonresidue as a difference. Similarly to the proof of Proposition 8, when we compare the two dice we can make cycles. The number of die i's wins is the sum of the quadratic nonresidues divided by p. The number of die i's losses is the sum of the quadratic residues divided by p. We have that die i has d_p more wins than losses. Since Dirichlet's theorem [5, 6] states that $d_p > 0$, die i beats die j.
- Between a quadratic residue and a quadratic nonresidue: in this part first we prove that the result between a quadratic residue i and a nonresidue j depends only on the difference $j-i$ is quadratic residue or not. Then we show that a quadratic residue i has d_p more wins against quadratic nonresidue j if $j-i$ is a quadratic residue then it is a quadratic nonresidue. In the last step we show that die $p-1$ beats die 1 and the difference of the wins and losses is d_p.

- Suppose that i is a quadratic residue and j is a nonresidue. By comparing the two dice we can make cycles: 1-st face of die i, 1-st face of die j, $(k+1)$-st face of die i, $(k+1)$-st face of die j, $(2k+1)$-st face of die i, $(2k+1)$-st face of die j and so on. Now we have one (or more) cycles and every second edge of this cycle counts in the comparison of i and j, the others are from the comparison between j and iq^k. Since the total number of wins along the big cycle depends only on that if the difference between j and iq^k is a quadratic residue, the result between dice j and iq^k also depends only on that.
- In the previous step when we made the bigger cycles the total number of wins by die i against die j was the sum of the differences along the cycle divided by p. It is d_p more if the difference $j-i$ is a quadratic residue, then it is not in the bigger cycle which means it is also d_p more if we delete the added edges.
- When we compare the 1-st and the $(p-1)$-th die 1 meets p, 2 meets $p-1$, ..., s meets $p-s+1$, ..., $\frac{p-1}{2}$ meets $\frac{p+3}{2}$. If we take the absolute value of the difference of these pairs we get all the even numbers between 1 and $p-1$ and die 1 wins the comparisons if and only if the absolute value of the difference is a quadratic residue. This means that the number of die 1's wins is equal to the number of even quadratic residues between 1 and p. Since 2 is a quadratic residue, it is equal to the number of quadratic residues between 1 and $\frac{p}{2}$. The difference between the win and losses of 1 is equal to the difference between the number of quadratic residues and nonresidues less then $\frac{p}{2}$ is equal to d_p.

• Die 0 versus a quadratic residue (the nonresidue case is analogous): the number of die 0's wins against a quadratic residue is equal to the number of quadratic residues less then $\frac{p}{2}$ and the number of losses is equal to the number of quadratic nonresidues. Applying Lemma 11 and Dirichlet's theorem, we get that die 0 beats the quadratic residues.

□

3 Extensions and open questions

We proposed constructions for dice sets that realize tournaments on n vertices.

If the simple directed graph is not complete, then we can assume that the relation between two vertices without an edge between them is a *tie*. Appropriate modifications of Constructions 1, 2 and 3 make it possible to realize any simple directed graph. It is to be verified whether other constructions can be extended to the incomplete case.

The minimal number of vertices such that there exists a tournament having the Schütte property is open, as it is mentioned in the introduction. The Schütte property of the Paley tournaments is also open for the general case.

Research on finding a dice set for $p = 8k+3$ is continued, as well as to get closer to the original problem: find the most efficient realization, i.e., with minimal number of faces, of a given simple directed graph. According to Mala [14, p. 40] the question that if

$$\lim_{n\to\infty} \frac{f(n)\log n}{n}$$

exists is also open.

References

[1] M. Borowiecki, J. Grytczuk, M. Haluszczak, Z. Tuza, Schütte's tournament problem and intersecting families of sets, *Combinatorics, Probability and Computing* (2003) **12**(4), pp. 359–364

[2] S. Bozóki, Nontransitive dice sets realizing Paley tournaments for solving Schütte's tournament problem, (under review)

[3] R. Bumbacea, personal communication

[4] M. Condorcet, Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues á la Pluralité des Voix, Paris (1785)

[5] G.L. Dirichlet, Recherches sur diverses applications de l'Analyse infinitesimale à la théorie des Nombres (première partie). *Journal für die reine und angewandte Mathematik (Crelle's Journal)* (1839) **1839**(19), pp. 324–369 DOI: 10.1515/crll.1839.19.324

[6] G.L. Dirichlet, Recherches sur diverses applications de l'Analyse infinitesimale à la théorie des Nombres (seconde partie). *Journal für die reine und angewandte Mathematik (Crelle's Journal)* (1840) **1840**(21), pp. 1–12 DOI: 10.1515/crll.1840.21.1

[7] P. Erdős, On a problem in graph theory, *The Mathematical Gazette* (1963), **47** pp. 220–223

[8] P. Erdős, L. Moser, On the representation of directed graphs as unions of orderings, *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* (1964), **9** pp. 125–132

[9] M. Gardner, The Paradox of the Nontransitive Dice and the Elusive Principle of Indifference, *Scientific American* (1970), **223**(6) pp. 110–114

[10] D.C. McGarvey, A Theorem on the Construction of Voting Paradoxes, *Econometrica* (October 1953), **21**(4) pp. 608–610

[11] A.P. Goucher, Tournament dice, posted on February 15, 2013, `http://cp4space.wordpress.com/2013/02/15/tournament-dice/` Accessed on 10 April 2013

[12] A.P. Goucher, personal communication

[13] R.L. Graham, J.H. Spencer, A constructive solution to a tournament problem, *Canadian Mathematical Bulletin* (1971), **14**(1) pp. 45–48

[14] J. Mala, On λ-majority voting paradoxes, *Mathematical Social Sciences* (1999), **37**(1) pp. 39–44

[15] J.W. Moon, L. Moser, Generating oriented graphs by means of team comparisons, *Pacific Journal of Mathematics* (1967), **21**(3) pp. 531–535

[16] I.M. Niven, H.S. Zuckerman, An introduction to the theory of numbers, *Wiley* (1972) 3rd edition

[17] R.E.A.C. Paley, On orthogonal matrices, *Journal of Mathematics and Physics* (1933), **12** pp. 311–320

[18] E. Pegg, Jr., Tournament Dice, *The Mathematical Association of America, Math Games* (July 11, 2005), `http://www.maa.org/editorial/mathgames/mathgames_07_11_05.html` Accessed on 1 February 2013

[19] R.P. Savage, The Paradox of Nontransitive Dice, *The American Mathematical Monthly* (May 1994), **101**(5) pp. 429–436

[20] R. Stearns, The voting problem, *The American Mathematical Monthly* (1959), **66** pp. 761–763

[21] H. Steinhaus, S. Trybula, On a paradox in applied probabilities, *Bulletin de l'Academie Polonaise des Sciences, Serie des Sciences Mathematiques, Astronomiques et Physiques* (1959), **7** pp. 67–69

[22] A.D. Taylor, Social Choice and the Mathematics of Manipulation, *Cambridge University Press* (2005), p. 10