Discrete time minimax tracking control with state and disturbance estimation II: time-varying reference and disturbance signals

P. Bauer*, B. Kulcsar** and J. Bokor*

***Abstract*— The paper characterizes the properties of discrete time minimax tracking control problem in the case of time varying references and disturbances.**

Hereunder, a multi step tracking control synthesis is suggested for Linear Time Invariant (LTI) plants when the reference signal could be time dependent. Moreover, instead of directly rejecting the effect of the (time varying) disturbance signal, an intermediate estimation and centering step is proposed. This step eliminates the main part of the disturbance by its unbiased estimate. The solution combines the state and disturbance estimation with linear quadratic and optimal minimax tracking design. The resulted unified control solution is LQ optimal on infinite horizon with constant references and disturbances and sub-optimal on large horizons with time-varying references and disturbances.

The paper clarify the effect of the time varying signals on the stability and performance criteria. The multi step procedure is illustrated via an ascending spiral trajectory tracking simulation of a quadrotor helicopter.

***Index Terms*— LQ optimal minimax tracking, state and disturbance estimation, time-varying signals**

I. INTRODUCTION

Tracking of reference signals is important in many control applications. However, external disturbances can highly reduce the tracking performance of the systems and they are present in several systems. Here we consider discrete time (DT), LTI systems with non previewable deterministic disturbances and references (considering stochastic disturbances also). Such system models can be related to aerial vehicles having wind effects which have usually a strong deterministic component.

In case of disturbance rejection needs, minimax or equivalently $\mathcal{H}_\infty$ control techniques arise as possible solutions. However, if the disturbance lies in the low frequency range it can be difficult to provide the design trade-off between disturbance rejection and tracking performance.

In [5] a minimax tracking design for nonlinear wheeled systems (robots) is presented which applies fuzzy logic system to eliminate the uncertain dynamics and $\mathcal{H}_\infty$ control to attenuate the effect of the residue of fuzzy elimination and exogenous disturbances. The concept of this article can be (and will be) well used to solve our design problem.

The main contribution behind the proposed idea in [10] and in this paper is the extension of the state estimation problem. Under certain condition, the disturbance corrupting the plant input can be estimated jointly with the state itself. Coupled state and disturbance estimator methods are discussed in ex. [6], [7]. [8] suggests to use an augmented Kalman filter, giving the possibility of noise adaptation by weighting.

The present paper focuses on the time varying nature of the reference and disturbance signal and the properties of the resulted and augmented, discrete time minimax optimization problem. Unlike in [10], this paper uses the extension of the Kalman state estimation problem associating a dynamics to the disturbance.

The paper is organized as follows. In section II the problem is formulated and the steps of the proposed multi step solution are listed. In section III, the solution steps are detailed. In section IV, the properties for time-varying references and disturbances are stated and proven. In section V, a simulation example is published which solves the 3D trajectory tracking control of a quadrotor helicopter using constant and ramp-type references. Finally, section VI concludes the paper.

II. PROBLEM FORMULATION AND THE STEPS OF THE PROPOSED SOLUTION

Let us consider the class of DT, LTI systems with deterministic disturbances by

$$\begin{aligned} x_{k+1} =& Ax_k + B\tilde{u}_k + G(d_k + w_k) \\ y_k^r =& C_r x_k \\ y_k =& Cx_k + H(d_k + w_k) + Vv_k \end{aligned} \tag{1}$$

Where $x_k \in \mathbb{R}^n$, $\tilde{u}_k \in \mathbb{R}^m$, $d_k \in \mathbb{R}^d$, $y_k^r \in \mathbb{R}^r$, $y_k \in \mathbb{R}^p$, $w_k \in \mathbb{R}^d$, $v_k \in \mathbb{R}^v$ are the system state, input, deterministic disturbance, tracking output, measured output, stochastic disturbance and measurement noise respectively and A, B, G, C_r, C, H, V have appropriate dimensions. Assume that $n \geq m$, $n > d$, $r \leq m$, $p > d$, G is full column rank, the pair (A, B) is stabilizable and w_k and v_k are independent gaussian white noise signals, with known covariance matrices $E\{ww^T\} = Q_w$ and $E\{vv^T\} = Q_v$. Assume also that $rank(C_r B) = r$.

Let us restrict ourselves on the case when disturbance and external stochastic noise act through the same direction G in the state space.

*Systems and Control Laboratory, Computer and Automation Research Institute, Hungarian Academy of Sciences, Budapest, H-1111, Hungary bauer.peter@sztaki.hu,bokor@sztaki.hu

**Delft Center for System and Control, Faculty 3ME, Delft University of Technology, Mekelweg 2, 2628CD, Delft, the Netherlands b.a.kulcsar@tudelft.nl

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The goal is to track a prescribed constant or time-varying reference signal with maximum disturbance attenuation (minimum tracking error). The developed multi-step solution is similar to the method applied in [5]. The steps of the solution are as follows:

1) Design a stabilizing state feedback control input for system (1). This makes step 2, 4 and 5 feasible.
2) Design the optimal state and disturbance estimator for the stabilized system using an augmented Kalman filter. The stochastic noises will be considered in this step!
3) Construct the system input which cancels the disturbance effects in a least squares (LS) optimal way.
4) Design another control input, which guarantees zero steady state tracking error in case of constant reference and disturbance signals.
5) Center the original system (constructed in step 1) dynamics with the steady state equilibrium point achieved in the previous step, and design an LQ optimal minimax tracker for this centered dynamics
6) Construct the final required input signal $\tilde{u}_k$ summing up all the inputs designed in the previous steps.

In the next section the above steps will be followed to construct the final optimal (for constant references and disturbances) and sub-optimal (for time-varying references and disturbances) controllers.

III. The steps of the design procedure

Step1: Design of a stabilizing state feedback controller for (A, B)

This can be solved either with pole placement or LQ optimal regulator design. The resulting system equations are as follows (considering additional input to guarantee tracking):

$$x_{k+1} = \underbrace{(A - BK_{x1})}_{\phi} x_k + Bu_k + Gd_k \tag{2}$$

Step 2: Design an optimal state and disturbance estimator for (ϕ, C, G, H)

[8] contains a more generic description of the design, here we just show the main step of the idea. This can be solved using the following augmented system dynamics for Kalman filter design (by Q_w and Q_v):

$$\begin{aligned}\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} &= \begin{bmatrix} \phi & G \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ I \end{bmatrix} w_k \\ y_{k+1} &= \begin{bmatrix} C & H \end{bmatrix} \begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} + V v_{k+1}\end{aligned} \tag{3}$$

This approximation of the time-varying disturbance is suggested in [4] and works well also for slowly varying disturbances.

Step3: LS optimal disturbance cancellation with the control input

The task is to find a control input component which cancels most of the disturbances using their estimated value (here $()^{+}$ is the Moore-Penrose pseudo-inverse of a rectangular matrix). The equation has an exact solution if $G = B$ otherwise this solution is only LS optimal.

$$u_k = \hat{u}_k - B^{+} G \hat{d}_k \tag{4}$$

Step4: Determining the solution of the zero steady state tracking error problem considering constant reference and disturbance

The equation to be solved can be constructed considering (2) and (4) (here $d_\infty = \hat{d}_\infty$ and r_∞ denotes the constant disturbance and reference signal respectively). Here the existence of $(I - \phi)^{-1}$ is guaranteed by step 1, and the pseudo-inverse F^{+} exists because $rank(C_r B) = r$ ($r \leq m$).

$$\begin{aligned} & x_\infty = \phi x_\infty + B\hat{u}_\infty + Gd_\infty - BB^{+}G\hat{d}_\infty \\ & y_\infty = C_r x_\infty = \underbrace{C_r (I - \phi)^{-1} B}_{F} \hat{u}_\infty + \\ & + C_r (I - \phi)^{-1} (I - BB^{+}) G\hat{d}_\infty = r_\infty \\ & \hat{u}_\infty = F^{+} r_\infty - \\ & - F^{+} C_r (I - \phi)^{-1} (I - BB^{+}) G\hat{d}_\infty \end{aligned} \tag{5}$$

Step5/1: Derivation of the LQ optimal finite horizon solution for the centered output tracking minimax problem

The required steady state input to track a constant reference signal can be calculated using (5). However, the control of the transient from initial state to steady state should be considered. This can be designed together with the solution of cases with time varying references in a unified framework as follows. The centered state dynamic equation results from (2), (4) and the steady state system equation (5):

$$\begin{aligned} & x_{k+1} = \phi x_k + B\hat{u}_k + Gd_k - BB^{+}G\hat{d}_k \\ & x_\infty = \phi x_\infty + B\hat{u}_\infty + Gd_\infty - BB^{+}G\hat{d}_\infty \\ & x_{k+1} - x_\infty = \phi (x_k - x_\infty) + B(\hat{u}_k - \hat{u}_\infty) + \\ & + G(d_k - d_\infty) - BB^{+}G\left(\hat{d}_k - \hat{d}_\infty\right) \\ & \Delta\tilde{d}_k = \begin{bmatrix} \Delta d_k^T & \Delta\hat{d}_k^T \end{bmatrix}^T \quad B_d = \begin{bmatrix} G & -BB^{+}G \end{bmatrix} \\ & \Delta x_{k+1} = \phi \Delta x_k + B\Delta\hat{u}_k + B_d \Delta\tilde{d}_k \end{aligned} \tag{6}$$

The last equation in (6) gives a disturbed system dynamics around the steady state. In [10] $B_d \Delta\tilde{d}_k$ is considered together as an artificial disturbance, but the reformulation here highly improves the solvability of the resulting MDARE. The last equation together with the centered reference signal $\Delta r_k = r_k - r_\infty$ can be used to form an LQ optimal minimax tracking problem for the transient (in case of constant references) or for the case with time varying references. The formulated problem is similar to the case in [9]. At first, the finite horizon solution should be derived considering the proper functional. From this point the Lagrange multiplier method can be applied to (7) and to the last equation in (6).

$$
\begin{aligned}
& J = \frac{1}{2}\sum_{k=0}^{N-1}((\Delta x_k - \Delta\tilde{x}_k)^T Q (\Delta x_k - \Delta\tilde{x}_k) + \\
& + \Delta\hat{u}_k^T R_u \Delta\hat{u}_k - \gamma^2 \Delta\tilde{d}_k^T R_d \Delta\tilde{d}_k) + \\
& + (\Delta x_N - \Delta\tilde{x}_N)^T Q (\Delta x_N - \Delta\tilde{x}_N) \quad where: \\
& Q = C_r^T Q_2 C_r \\
& \Delta\tilde{x}_k = C_r^T \left(C_r C_r^T\right)^{-1} \Delta r_k = H\Delta r_k
\end{aligned} \tag{7}
$$

The costate update equation, optimal control, worst case disturbance and the structure of the costate variable results as follows.

$$
\begin{aligned}
& \lambda_k = Q(\Delta x_k - \Delta\tilde{x}_k) + \phi^T \lambda_{k+1} \\
& \Delta\hat{u}_k = -R_u{}^{-1} B^T \lambda_{k+1} \\
& \Delta\tilde{d}_k^* = \frac{1}{\gamma^2} R_d{}^{-1} B_d^T \lambda_{k+1} \\
& \lambda_k = P_k \Delta x_k + S_k \Delta\tilde{x}_{k+1} - Q\Delta\tilde{x}_k \\
& \lambda_N = Q\Delta x_N - Q\Delta\tilde{x}_N \quad \to P_N = Q, \quad S_N = 0
\end{aligned} \tag{8}
$$

Finally, the Modified Riccati Difference Equation (MRDE) and an additional recursive equation results. The last expression in (9) is the expanded form of the costate variable. The optimal control and worst case disturbance can be calculated using this and (8).

$$
\begin{aligned}
& P_k = Q + \phi^T P_{k+1}[I + \underbrace{\overline{B}R^{-1}\overline{B}^T}_{M} P_{k+1}]^{-1}\phi \\
& SR_k = Q\Delta\tilde{x}_k + \phi^T \left[I + P_{k+1}\overline{B}R^{-1}\overline{B}^T\right]^{-1} SR_{k+1} \\
& SR_{k+1} = Q\Delta\tilde{x}_{k+1} - S_{k+1}\Delta\tilde{x}_{k+2} \\
& \overline{B} = \begin{bmatrix} B & B_d \end{bmatrix} \quad R = \begin{bmatrix} R_u & 0 \\ 0 & -\gamma^2 R_d \end{bmatrix} \\
& \lambda_{k+1} = P_{k+1}\left[I + MP_{k+1}\right]^{-1}\phi\Delta x_k - \\
& - \left[I + P_{k+1}M\right]^{-1}(QH\Delta r_{k+1} - S_{k+1}H\Delta r_{k+2})
\end{aligned} \tag{9}
$$

This completes the derivation of the minimax tracking controller for finite horizon problems. All the calculation expressions in (9) are recursive, so they need the knowledge of the reference signal on the whole horizon in advance. This difficulty should be solved considering the infinite horizon solution.

Step 5/2: Derivation of LQ optimal and LQ sub-optimal infinite horizon solutions

For infinite horizon the MDARE can be easily constructed from (9). Denote its solution by P_∞. Now the generalized form of the costate variable can be written as:

$$
\begin{aligned}
& \lambda_{k+1} = P_\infty \left[I + MP_\infty\right]^{-1}\phi\Delta x_k - \\
& - \left[I + P_\infty M\right]^{-1}(S_1\Delta r_{k+1} - S_2\Delta r_{k+2})
\end{aligned} \tag{10}
$$

This way $u_k = -R_u^{-1}B^T\lambda_{k+1}$ and $\Delta\tilde{d}_k^* = \frac{1}{\gamma^2}R_d{}^{-1}B_d^T\lambda_{k+1}$ are satisfied if one writes back λ_{k+1} into them. To get an LQ optimal solution S_1 and S_2 should be selected to satisfy the other requirement $\lambda_k = Q\Delta x_k - QH\Delta r_k + \phi^T\lambda_{k+1}$. Substituting the general expression for λ (10) into this last requirement and doing some manipulations considering the last equation in (9) and assuming ϕ is invertible (this can be guaranteed with pole placement design in Step 1) results in a system of equations. In (11) the MDARE is written which is satisfied for all Δx_k. For constant references $\Delta r_k = \Delta r_{k+1} = \Delta r_{k+2} = 0$, (12), (13) and (14) are also satisfied and so, *the solution is optimal*. However, unfortunately it is impossible to satisfy the last two equations for nonzero Δr_{k+2} reference values.

$$
P_\infty\Delta x_k = Q\Delta x_k + \phi^T P_\infty\left[I + MP_\infty\right]^{-1}\phi\Delta x_k \tag{11}
$$

$$
-S_1\Delta r_k = -QH\Delta r_k \tag{12}
$$

$$
S_2\Delta r_{k+1} = -\phi^T\left[I + P_\infty M\right]^{-1}S_1\Delta r_{k+1} \tag{13}
$$

$$
0 = \phi^T\left[I + P_\infty M\right]^{-1}S_2\Delta r_{k+2} \tag{14}
$$

So, the general LQ optimal solution of the problem is impossible. However, in real applications at time instant k Δr_{k+2} usually should be considered with linear extrapolation because it is not known (see [9]). Considering this fact a sub-optimal selection of S_1 and S_2 is possible (defining $M_2 = \left[I + P_\infty M\right]^{-1}$):

$$
\begin{aligned}
& \Delta r_{k+2} = 2\Delta r_{k+1} - \Delta r_k \\
& - S_1\Delta r_k = -QH\Delta r_k - \phi^T M_2 S_2\Delta r_k \\
& S_2\Delta r_{k+1} = -\phi^T M_2 S_1\Delta r_{k+1} + 2\phi^T M_2 S_2\Delta r_{k+1} \\
& \underbrace{\begin{bmatrix} I & -\phi^T M_2 \\ \phi^T M_2 & I - 2\phi^T M_2 \end{bmatrix}}_{Z} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} QH \\ 0 \end{bmatrix}
\end{aligned} \tag{15}
$$

In (15) Z is an invertible matrix so, the system of equations can be solved for S_1 and S_2. Finally the control input for the centralized problem (the worst case disturbance has an analogous form):

$$
\begin{aligned}
& \Delta\hat{u}_k = -K_{x2}\Delta x_k + \overline{K}_{S_1}\Delta r_{k+1} + K_{S_2}\Delta r_k \\
& \overline{K}_{S_1} = K_{S_1} - 2K_{S_2} \\
& K_{x2} = R_u^{-1}B^T P_\infty \left[I + \overline{B}R^{-1}\overline{B}^T P_\infty\right]^{-1}\phi \\
& K_{S_1} = R_u^{-1}B^T\left[I + P_\infty\overline{B}R^{-1}\overline{B}^T\right]^{-1}S_1 \\
& K_{S_2} = R_u^{-1}B^T\left[I + P_\infty\overline{B}R^{-1}\overline{B}^T\right]^{-1}S_2
\end{aligned} \tag{16}
$$

Step 6: The construction of the final control input signal

The final control input signal can be constructed considering (2), (4), (5), (6) and (16). The final result is:

$$
\begin{aligned}
& \tilde{u}_k = -K_x\hat{x}_k - K_{S_2}(r_{k+1} - r_k) + K_{r_\infty}r_{k+1} + \\
& + K_{d_\infty}\hat{d}_k \quad where \quad K_x = K_{x1} + K_{x2} \\
& K_{r_\infty} = \left(K_{x2}(I - \phi)^{-1}B + I\right)F^{+} \\
& M_3 = (I - BB^{+}) \\
& K_{d_\infty} = [K_{x2}(I - \phi)^{-1}M_3 - \\
& - K_{x2}(I - \phi)^{-1}BF^{+}C_r(I - \phi)^{-1}M_3 - \\
& - F^{\mathcal{R}}C_r(I - \phi)^{-1}M_3 - B^{+}]G
\end{aligned} \tag{17}
$$

Note that the estimated state is used instead of the real system state, the r_{k+1} reference is used with K_{r_∞} instead of r_∞ and $\hat{d}_k$ is used instead of $\hat{d}_\infty$ and this provides the applicability both for constant and time-varying references and disturbances. The control input of the state and disturbance estimator u_k should be calculated using K_{x2} instead of K_x (and ϕ should be used instead of A!). In the next section the statement and proof of properties for time-varying references and disturbances will be done.

IV. Properties for time-varying references and disturbances

We assume that the disturbances and their estimates are l_∞ signals with the following l_∞ norms (including w_k and noise effects on disturbance estimate also):
$\|d+w\|_\infty = D < \infty, \quad \|\hat{d}\|_\infty = \hat{D} < \infty$

Theorem 1 (BIBO stability): The derived control solution guarantees BIBO stability for l_∞ reference signals.

Proof: The boundedness of the states and outputs should be proven. The l_∞ norm of a bounded r_k reference signal is: $\|r_k\|_\infty = \max_k |r_k| = R_m < \infty$
Notice that x_k is the system state, meanwhile the estimated $\hat{x}_k$ state is used in the control input (see (17)). This generates the need to characterize their difference and construct a scheme where the stable system matrix $\phi_2 = \phi - BK_{x2}$ can be used instead of the possibly unstable A. This characterization can be easily done using the state estimation error:

$$\hat{x}_k = x_k + x_k^e \tag{18}$$

Using expression (18) the control input can be redefined. Because the state estimator is stable the state estimation error has a finite l_∞ norm also defined in the following equations:

$$\begin{aligned}\tilde{u}_k = &-K_x x_k - K_x x_k^e - K_{S_2}(r_{k+1} - r_k) + K_{r_\infty} r_{k+1} + \\ &+ K_{d_\infty}\hat{d}_k \quad \& \quad \|x_k^e\|_\infty = E\end{aligned} \tag{19}$$

Using (1) and (19) the state at time step n of the controlled system (with initial state x_0) is as follows:

$$\begin{aligned}x_n = &\phi_2^n x_0 - \\ &- \left[\sum_{k=0}^{n-1} \phi_2^k BK_x x_{n-1-k}^e\right] + \left[\sum_{k=0}^{n-1} \phi_2^k BK_{S_2} r_{n-1-k}\right] + \\ &+ \left[\sum_{k=0}^{n-1} \phi_2^k B\left(K_{r_\infty} - K_{S_2}\right) r_{n-k}\right] + \left[\sum_{k=0}^{n-1} \phi_2^k G d_{n-1-k}\right] + \\ &+ \left[\sum_{k=0}^{n-1} \phi_2^k BK_{d_\infty}\hat{d}_{n-1-k}\right]\end{aligned} \tag{20}$$

To have an upper bound for the length of x_n take the euclidean (l_2) norm of both sides in (20) and consider (19) and $\|\phi_2^k\| \le KR_\sigma^k \;\; where\ R_\sigma \in \mathbb{R},\ R_\sigma < 1\ and\ K \in \mathbb{R}$ for a stable ϕ_2 matrix (this can be proven and $\|.\|$ denotes the induced l_2 norm of a matrix). This way the l_∞ norm of x_k is as shown at the end of (21). This is a finite value so, the input to state stability is satisfied. The boundedness of the outputs and output tracking errors can be proven in a similar way. The effect of v_{k+1} in $\|y_k\|_\infty$ and $\|e_k\|_\infty$ can be considered with its l_∞ norm as an additional term.

$$\begin{aligned}|x_n| \le &KR_\sigma^n|x_0| + K\left[\sum_{k=0}^{n-1} R_\sigma^k\right](\|BK_x\|E + \|BK_{S_2}\|R_m + \\ &+ \|B\left(K_{r_\infty} - K_{S_2}\right)\|R_m + \|BK_{d_\infty}\|\hat{D} + \|G\|D) \\ \|x_k\|_\infty = &\max_n |x_n| < K|x_0| + \\ &+ \frac{K}{1-R_\sigma}(\|BK_x\|E + \|B\left(K_{r_\infty} - K_{S_2}\right)\|R_m + \\ &+ \|BK_{S_2}\|R_m + \|BK_{d_\infty}\|\hat{D} + \|G\|D) < \infty\end{aligned} \tag{21}$$

■

Theorem 2 (finite error for ramp references): The derived control solution guarantees finite tracking error in all time steps for ramp-type references

Proof: A ramp-type reference signal can always be represented with its starting value and increment (or decrement): $r_{k+1} = r_k + \Delta r^r = r_0 + (k+1)\Delta r^r$

The state of the controlled system in the nth time step with ramp-type reference signal can be written as follows:

$$\begin{aligned}x_n = &\phi_2^n x_0 - \left[\sum_{k=0}^{n-1}\phi_2^k\right] BK_{S_2}\Delta r^r + \left[\sum_{k=0}^{n-1}\phi_2^k\right] BK_{r_\infty} r_0 - \\ &- \left[\sum_{k=0}^{n-1} \phi_2^k BK_x x_{n-1-k}^e\right] + \left[\sum_{k=0}^{n-1}\phi_2^k BK_{d_\infty}\hat{d}_{n-1-k}\right] + \\ &+ \left[\sum_{k=1}^{n} k\phi_2^{n-k}\right] BK_{r_\infty}\Delta r^r + \left[\sum_{k=0}^{n-1}\phi_2^k G d_{n-1-k}\right]\end{aligned} \tag{22}$$

The tracking error in the nth step and its upper bound (after tedious manipulations) can be formulated as:

$$\begin{aligned}e_n = &y_n - r_n = C_r x_n - r_0 - n\Delta r^r \\ |e_n| \le &\|C_r\|KR_\sigma^n|x_0| + \|C_r\|\frac{K}{(1-R_\sigma)}\|BK_x\|E + \\ &+ \|C_r\|\frac{K}{(1-R_\sigma)}|BK_{S_2}\Delta r^r| + \|C_r\|\frac{K}{(1-R_\sigma)}\|BK_{d_\infty}\|\hat{D} + \\ &+ \|C_r\|\frac{K}{(1-R_\sigma)}|BK_{r_\infty} r_0| + \|C_r\|\frac{K}{(1-R_\sigma)}\|G\|D + \\ &+ |r_0| + \|C_r\|\frac{KR_\sigma}{(1-R_\sigma)^2}|BK_{r_\infty}\Delta r^r| + \\ &+ \|C_r\| n R_\sigma^n \frac{KR_\sigma}{(1-R_\sigma)}|BK_{r_\infty}\Delta r^r| < \infty\end{aligned} \tag{23}$$

■

V. The simulation example

The usefulness of the developed method is proven with a discrete time (DT) equivalent of a continuous time (CT)

quadrotor dynamical model. The model has the following state, input and measured output variables:
States:

- $n_1 \quad n_2 \quad n_3 \quad n_4$ rotational speeds of the four electric motors
- $u \quad v \quad w$ velocity components in *body coord. sys.*
- $P \quad Q \quad R$ angular velocity components in *body coord. sys.*
- $\varphi \quad \theta \quad \psi$ Euler angels
- Z vertical position in *earth coord. sys.*

Inputs:

- δ_{pitch} pitching command
- δ_{roll} rolling command
- δ_{yaw} yawing command
- $\delta_{asc/desc}$ ascending / descending command

Measured outputs:

- $n_1 \quad n_2 \quad n_3 \quad n_4$ rotational speeds of the four ele motors
- $\dot{u} \quad \dot{v} \quad \dot{w}$ accelerations in *body coord. sys.*
- $P \quad Q \quad R$ angular velocity components in *body c sys.*
- $\varphi \quad \theta \quad \psi$ Euler angels
- $h = -Z$ flight altitude in *earth coord. sys.* (assu flat ground)

The sample time was selected to be $T_s = 0.0125sec$ bec the open loop bandwidth is $\omega_{oB} = 20rad/sec$ and so, $1/(4\omega_{oB})$. The structure of the DT dynamical equatio the same as in (1). d is wind disturbance, which h significant constant (in *earth coord. sys.*) and much sma time varying (w) component. In *body coord. sys.* the constant part becomes time varying because of the rotation of *body coord. sys.* relative to *earth*. The measurement noises were determined using real sensor data.

The goal was to track an ascending spiral trajectory whi can be achieved by tracking four given signals: u=const a v=0 velocity components (constant signals), ψ continuou increasing azimuth angle and Z continuously decreasi position in *earth coord. sys.* (this means increasing altitud The latter two are ramp-type references. The determin tic wind disturbance in *earth coord. sys.* is considered $d = [0.15\ 0.05\ -0.05]$. The stochastic wind compon has ± 0.02 extremal values. The weighting of control desi contains the following matrices:

$$R_u = < 10\ 10\ 10\ 10 > \quad input\ weight$$
$$R_d = < 1e3\ 1e3\ 1e3\ 1e3\ 1e3\ 1e3 > \quad disturbance\ weig.$$
$$Q_2 = < 1e5\ 1e4\ 1e4\ 1e6 > tracking\ error\ weight$$

The controller was designed following the proposed o steps (see section II). During the design the MDARE should be solved with γ iteration using the so called bisection algorithm as in the continuous time (CT) case. But the MDARE should be solved using the augmented input matrix $\overline{B}$ (see (9)) and this way it considers also the worst case disturbance as a useful input applicable to stabilize the system. This can result in an unstable system at the achieved minimum γ value if one does not generate also the worst case disturbance as a control input. But in real applications the generation of worst case disturbance as an input is usually impossible (such as here). This problem is pointed out also in [11] for CT minimax control. The solution similar to the one proposed in [11] is to do γ iteration also for the stability or instability of $\phi - BK_{x2}$ besides the solvability or unsolvability of the MDARE. This way larger final gamma value results, but the controlled system will be stable purely with the control input (the worst case disturbance is not needed). Here the achieved gamma value is 3.2825 which is acceptable for the attenuation of the disturbance residual. The results are shown in Figures 1, 2 and 3.

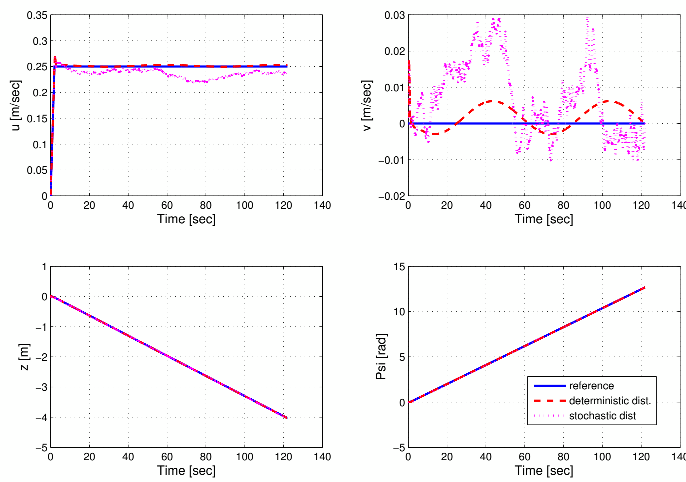

Fig. 1. The tracking of references

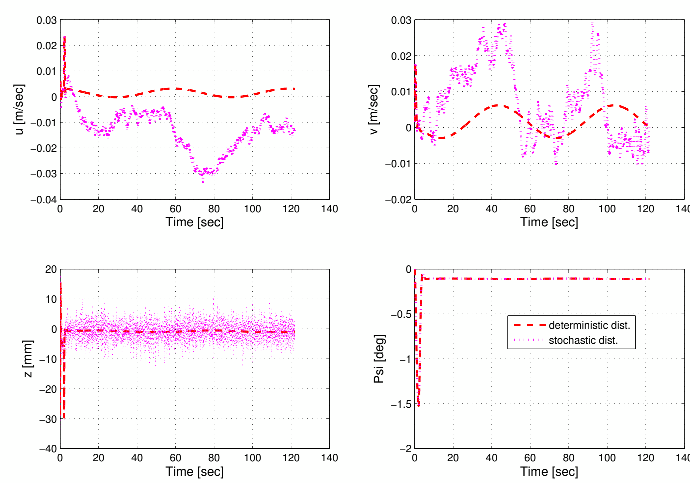

Fig. 2. Reference tracking errors

Two cases were tested, one without stochastic disturbances (w_k) (red lines) and one with them (magenta line). As can be seen in the Figures the tracking results are satisfactory for all four references without w_k and they are not satisfactory for velocity components with w_k. This can be seen also in Figure 3 where the noisy trajectory has larger distance from

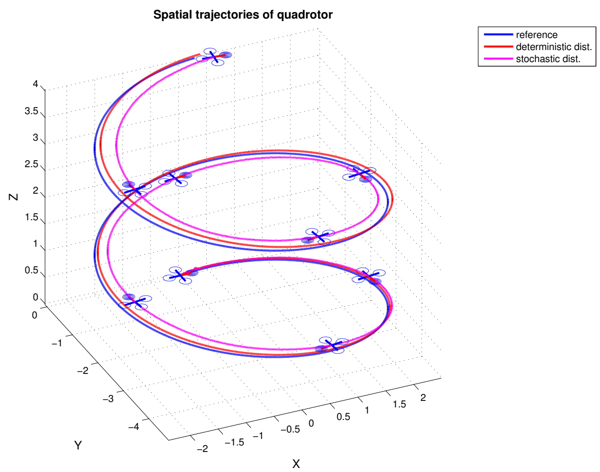

Fig. 3. Spatial trajectories of the quadrotor

the reference (blue) one. The tracking of azimuth angle and altitude is satisfactory both without and with w_k. The steady state azimuth angle error is about $-0.11deg$ which is very good the altitude error is between $\pm 1cm$ which is acceptable. The more decreased performance of the control with w_k is originated in the decreased performance of coupled state and disturbance estimator in this noisy case. This results in higher errors for the velocity components because they are not measured just estimated and this estimated value is used in the formulation of the tracking error during the control. The estimator does not have zero estimation error even in the case without w_k and v_k because it has unstable (1) poles. But these are required to achieve satisfactory disturbance estimates. This means that a false estimate can result in false tracking so, the improvement and refinement of the state and disturbance estimator is required to improve the quality of control. Another aspect can be to avoid tracking of non measured outputs. But as a conclusion it can be stated that the algorithm can work well even for time-varying reference signals so, its further development can be fruitful.

VI. CONCLUSIONS

The paper presents an LQ optimal minimax tracking solution for DT, LTI systems with deterministic and stochastic disturbances. The solution can be achieved through multi step design method, where one of the most important step is to jointly estimate the disturbance with the state vector.

The controller's properties for time-varying references and disturbances are examined (guarantee of BIBO stability and finite tracking error in all time steps for ramp-type references).

The performance of the proposed solution was tested with a DT quadrotor model. The goal was to track an ascending spiral trajectory containing constant and ramp-type references. Deterministic and stochastic wind disturbances were considered. The tracking performance is acceptable with purely deterministic disturbances, but it is not satisfactory with additional stochastic disturbances.

The work can be improved by robustifying the coupled disturbance-state estimator.

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